Adaptive Stabilization of Fractional-order Nonlinear System Considering Input Saturation Phenomenon

Xiaomin Tian, Zhong Yang

Abstract—**This paper investigates the adaptive control of fractional-order nonlinear system. The system is fluctuated by unmodeled dynamics and external disturbances, and the bounds of these uncertainties are unknown in advance. The effects of input saturation and unknown system parameters are taken into account in this paper. To deal with these unknown parameters, a robust adaptive control strategy is proposed. Then a fractional-order version of Lyapunov function is given to verify the stability of uncertain fractional-order nonlinear system. Simulation results are given to prove the feasibility and effectiveness of the given control strategy.**

Index Terms—**fractional-order nonlinear system, adaptive stabilization, input saturation, robust control.**

I. Introduction

ALTHOUGH fractional calculus is a mathematical topic with long history, its application in physics and engineering fields has attracted more and more researchers' attention only in the recent two decades [1-3]. It is has been proven that fractional calculus can supply a superb instrument for describing the processes with memory and hereditary properties.

Fractional-order nonlinear system has more significant advantages, for example, fractional-order system has more wider stability region compared to the corresponding integer-order system, and fractional-order mathematical model can reveal more dynamic attributes of a system. In fractional-order nonlinear system, fractional-order chaotic system is a distinguished phenomenon that is characterized with some special features.

Recently, studying fractional-order system has become an active research area. In particular, control and synchronization of the fractional-order chaotic systems have attracted much attention from various scientific fields. Some methods have been proposed to achieve chaos synchronization in fractional-order chaotic systems. Such as sliding mode control [4-6], nonlinear feedback control [7], a nonlinear state observer [8], active control [9], adaptive control [10-12], etc.

Nevertheless, in most of the above mentioned methods, parameters of the fractional-order chaotic systems are assumed to be known. In fact, the effects of unknown parameters maybe destroy the system's behavior and even cause unbounded outputs, so it is urgent to design a control engineer to deal with this problem. Meanwhile, many controllers are designed based on full information about the systems' dynamics, but due to the complexity of the practical condition, most physical systems need to be described by additional unmodeled dynamics and external disturbances, even when the bounds of these system uncertainties are unknown beforehand, the researcher should pay much attention to the controller design.

This work is supported by the Foundation of Jinling Institute of Technology (Grant No: jit-fhxm-201607 and jit-b-201706), the Natural Science Foundation of Jiangsu Province University (Grant No: 17KJB120003).
Xiaomin Tian and Zhong Yang are with College of Intelligent Science and Control Engineering, Jinling Institute of Technology, Nanjing 211169, China. Email: tianxiaomin100@163.com

In the other hand, all control methods in the abovementioned works are derived based on the ideal assumption of the control inputs, actually, because of the limited operation of control actuators, most of the actual systems are subjected to input constraint. The saturation nonlinearity is often encountered in various engineering systems, which can cause unpredictable and undesirable motions in systems. Thus it is imperative to consider the effects of the input saturation. Recently, Aghababa proposed some control strategy for integer-order switched system with input saturation [13-15]. While, the stabilization of fractional-order nonlinear system with unknown parameters and saturation nonlinear phenomenon in control input are not considered simultaneously. Furthermore, almost all control scheme in existing literature are focus on the stability analysis of fractional-order systems based on traditional Lyapunov theory, the application of fractional-order Lyapunov stability theory is still an challenging problem and very few articles are dedicated to this problem.

Motivated by the above discussion, the main goal of this paper is to propose a new adaptive control strategy to realize the stabilization of fractional-order nonlinear system with unknown system parameters, unknown bounded uncertainties and saturation nonlinear inputs. The structure of this paper is organized as follows. In section 2, relevant definitions, lemmas are given. Main results are presented in section 3. Simulation results are shown in section 4. Finally, conclusions are included in section 5.

II. Preliminaries

The Riemann-Liouville, Caputo definition are main definitions of fractional calculus.

Definition 1 The α th-order Riemann-Liouville fractional integration of function $f(t)$ is given by

$$ {}_{t_0}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \tag{1} $$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2 For $n-1 < \alpha \le n$, $n \in R$, the Riemann-Liouville fractional derivative of order α of the function $f(t)$

is defined as

$$
\begin{aligned}
{}_{t_0}D_t^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \\
&= \frac{d^n}{dt^n} I^{n-\alpha} f(t) \qquad (2)
\end{aligned}
$$

Definition 3 The Caputo fractional derivative of order α of the function $f(t)$ is defined as

$$
{}_{t_0}D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, & m-1<\alpha<m \\ \frac{d^m}{dt^m} f(t), & \alpha = m \end{cases} \qquad (3)
$$

where m is the smallest integer number, larger than α.
Lemma 1 (see [16, 17]) Consider the autonomous system

$$
D^{\alpha} x = Ax \qquad or \qquad D^{\alpha} x = f(x) \qquad (4)
$$

where $\alpha \in (0, 1]$ is the fractional order and $x = (x_1, x_2, ..., x_n)^T$ is the state variable. $A \in R^{n\times n}$ is a constant matrix. If there is a real symmetric positive definite matrix P such that the inequation $J = x^T P D^{\alpha} x \leq 0$ always holds for any states, then system (4) is asymptotically stable.

For the detailed application of the above lemma in fractional-order chaotic systems, the reader can refer to Refs. [16-20].

III. Main Results

In this section, a robust adaptive controller and unknown parameter update laws are designed to achieve the stabilization of fractional-order nonlinear system with saturation nonlinearity in control input.

The dynamic mathematical model of fractional-order nonlinear system with unknown parameters, unknown bounded unmodeled dynamics and external disturbances can be described by

$$
\begin{aligned}
D^{\alpha} x_1 &= F_1(x)\delta_1 + f_1(x) + \Delta f_1(x) + d_1(t) + sat(u_1(t)) \\
D^{\alpha} x_2 &= F_2(x)\delta_2 + f_2(x) + \Delta f_2(x) + d_2(t) + sat(u_2(t)) \\
&\vdots \\
D^{\alpha} x_n &= F_n(x)\delta_n + f_n(x) + \Delta f_n(x) + d_n(t) + sat(u_n(t)) \quad (5)
\end{aligned}
$$

where $\alpha \in (0, 1)$ is fractional order of the system, $x = (x_1, x_2, \cdots, x_n)^T$ is the state vector of the system, $F_i(x)$ is the row vector of system, δ_i is the column vector of unknown system parameters, $f_i(x)$ is the nonlinear section of system, $\Delta f_i(x)$ and $d_i(t)$ are unmodeled dynamics and external disturbances, respectively. $Sat(u_i(t))$ is the nonlinear saturation input, i=1, 2, ..., n.
Assumption 1 The nonlinear saturation function is defined as follows:

$$
Sat(u(t)) = \begin{cases} u_H, & if u(t) \geq u^h \\ \theta u(t), & if u^l \leq u(t) \leq u^h \\ u_L, & if u(t) \leq u^l \end{cases} \qquad (6)
$$

where u_H, $u^h \in R^+$, and u_L, $u^l \in R^-$ are bounds of the saturation function and $\theta \in R$ is the saturation slope. Subsequently, the above saturation function can be rewritten as

$$
Sat(u(t)) = u(t) + \Delta u(t) \qquad (7)
$$

and $\Delta u(t)$ is satisfied as

$$
\Delta u(t) = \begin{cases} u_H - u(t), & if\ u(t) \geq u^h \\ (\theta-1)u(t), & if\ u^l \leq u(t) \leq u^h \\ u_L - u(t), & if\ u(t) \leq u^l \end{cases} \qquad (8)
$$

A typical nonlinear saturation function is described in Fig.1.

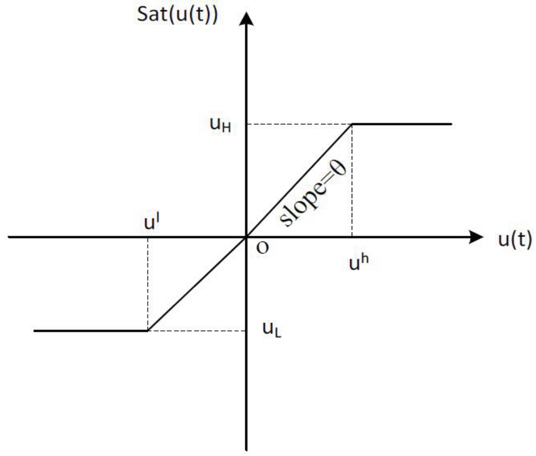

Fig. 1: A typical nonlinear saturation function

Assumption 2 It is assumed that the system uncertainties $\Delta f_i(x) + d_i(t)$, i=1, 2, 3, are bounded by:

$$
|\Delta f_i(x) + d_i(t)| \leq \gamma_i \qquad (9)
$$

where γ_i is an unknown positive constant.
Assumption 3 It is reasonable that to assume the uncertainties $\Delta u_i(t)$, i=1, 2, 3, are bounded as follows:

$$
|\Delta u_i(t)| \leq \phi_i \qquad (10)
$$

where ϕ_i is an unknown positive constant.

Our goal in this paper is to design a robust controller to realize the adaptive stabilization of fractiona-order nonlinear system (5) with unknown system parameters, unknown bounded uncertainties and nonlinear saturation inputs, then use the fractional-order version of Lyapunov theory to prove the controlled system's stability.
Theorem 1 Consider the fractional-order nonlinear system (5), if the bounds of model uncertainties and external disturbances are unknown in advance, then the controller can designed as

$$
u_i(t) = -(|F_i(x)||\hat{\delta}_i| + |f_i(x)| + \hat{\phi}_i + \hat{\gamma}_i + k_i)sgn(x_i) \qquad (11)
$$

where $i = 1, 2, ..., n$. $\hat{\delta}_i$, $\hat{\phi}_i$ and $\hat{\gamma}_i$ are estimation of δ_i, ϕ_i, γ_i, respectively. k_i is positive constant, $sgn(\cdot)$ is a sign function, for avoiding chattering, $tanh(\cdot)$ can be used to substitute the sign function. To deal with these unknown parameters, the following estimation update rules are proposed.

$$
D^{\alpha} \hat{\delta}_i = F_i^T(x) x_i \qquad (12)
$$

$$
D^{\alpha} \hat{\phi}_i = \rho_i |x_i| \qquad (13)
$$

$$
D^{\alpha} \hat{\gamma}_i = \eta_i |x_i| \qquad (14)
$$

in which, ρ_i and η_i are positive adaptive constants, then the adaptive stabilization of fractional-order nonlinear system (5) can be realized.

Proof On the basis of Lemma 1, the positive definite matrix P can be select as

$$P = diag\Big(I_m, \frac{1}{\rho_1}, \cdots, \frac{1}{\rho_n}, \frac{1}{\eta_1}, \cdots, \frac{1}{\eta_n}\Big) \tag{15}$$

where I_m represents m-dimensional identity matrix, and m-n is the dimension of ($\hat{\delta}_1^T$, ..., $\hat{\delta}_n^T$). Denote $X^T = (x_1, x_2, ..., x_n, \widetilde{\delta}_1^T, ..., \widetilde{\delta}_n^T, \widetilde{\phi}_1, ..., \widetilde{\phi}_n, \widetilde{\gamma}_1, ..., \widetilde{\gamma}_n)$, then a function can be established to prove the stability of the closed-loop system, that is

$$J = X^T P D^\alpha X \tag{16}$$

according to the above denotation, we have

$$\begin{aligned} J &= x_1 D^\alpha x_1 + x_2 D^\alpha x_2 + \cdots + x_n D^\alpha x_n + \widetilde{\delta}_1^T D^\alpha \widetilde{\delta}_1 \\ &\quad + \cdots + \widetilde{\delta}_n^T D^\alpha \widetilde{\delta}_n + \frac{1}{\rho_1}\widetilde{\phi}_1 D^\alpha \widetilde{\phi}_1 + \cdots + \frac{1}{\rho_n}\widetilde{\phi}_n D^\alpha \widetilde{\phi}_n \\ &\quad + \frac{1}{\eta_1}\widetilde{\gamma}_1 D^\alpha \widetilde{\gamma}_1 + \cdots + \frac{1}{\eta_n}\widetilde{\gamma}_n D^\alpha \widetilde{\gamma}_n \\ &= x_1[F_1(x)\delta_1 + f_1(x) + \Delta f_1(x) + d_1(t) + sat(u_1(t))] \\ &\quad + \cdots + x_n[F_n(x)\delta_n + f_n(x) + \Delta f_n(x) + d_n(t) \\ &\quad + sat(u_n(t))] + \widetilde{\delta}_1^T D^\alpha \widetilde{\delta}_1 + \cdots + \widetilde{\delta}_n^T D^\alpha \widetilde{\delta}_n \\ &\quad + \frac{1}{\rho_1}\widetilde{\phi}_1 D^\alpha \widetilde{\phi}_1 + \cdots + \frac{1}{\rho_n}\widetilde{\phi}_n D^\alpha \widetilde{\phi}_n + \frac{1}{\eta_1}\widetilde{\gamma}_1 D^\alpha \widetilde{\gamma}_1 \\ &\quad + \cdots + \frac{1}{\eta_n}\widetilde{\gamma}_n D^\alpha \widetilde{\gamma}_n \end{aligned} \tag{17}$$

according to Eq.(7), we obtain

$$\begin{aligned} J &= x_1[F_1(x)\delta_1 + f_1(x) + \Delta f_1(x) + d_1(t) + u_1(t) \\ &\quad + \Delta u_1(t)] + \cdots + x_n[F_n(x)\delta_n + f_n(x) + \Delta f_n(x) \\ &\quad + d_n(t) + u_n(t) + \Delta u_n(t)] + \sum_{i=1}^n \widetilde{\delta}_i^T D^\alpha \widetilde{\delta}_i \\ &\quad + \sum_{i=1}^n \frac{1}{\rho_i}\widetilde{\phi}_i D^\alpha \widetilde{\phi}_i + \sum_{i=1}^n \frac{1}{\eta_i}\widetilde{\gamma}_i D^\alpha \widetilde{\gamma}_i \\ &= x_1 F_1(x)\delta_1 + x_1 f_1(x) + x_1 \Delta f_1(x) + x_1 d_1(t) \\ &\quad + x_1 u_1(t) + x_1 \Delta u_1(t) + \cdots + x_n F_n(x)\delta_n \\ &\quad + x_n f_n(x) + x_n \Delta f_n(x) + x_n d_n(t) + x_n u_n(t) \\ &\quad + x_n \Delta u_n(t) + \sum_{i=1}^n \widetilde{\delta}_i^T D^\alpha \widetilde{\delta}_i + \sum_{i=1}^n \frac{1}{\rho_i}\widetilde{\phi}_i D^\alpha \widetilde{\phi}_i \\ &\quad + \sum_{i=1}^n \frac{1}{\eta_i}\widetilde{\gamma}_i D^\alpha \widetilde{\gamma}_i \\ &\le x_1 F_1(x)\delta_1 + |x_1||f_1(x)| + |x_1|(|\Delta f_1(x) + d_1(t)|) \\ &\quad + x_1 u_1(t) + |x_1||\Delta u_1(t)| + \cdots + x_n F_n(x)\delta_n \\ &\quad + |x_n||f_n(x)| + |x_n|(|\Delta f_n(x) + d_n(t)|) + x_n u_n(t) \\ &\quad + |x_n||\Delta u_n(t)| + \sum_{i=1}^n \widetilde{\delta}_i^T D^\alpha \widetilde{\delta}_i + \sum_{i=1}^n \frac{1}{\rho_i}\widetilde{\phi}_i D^\alpha \widetilde{\phi}_i \\ &\quad + \sum_{i=1}^n \frac{1}{\eta_i}\widetilde{\gamma}_i D^\alpha \widetilde{\gamma}_i \end{aligned} \tag{18}$$

according to Eq.(12), it yields

$$\begin{aligned} \sum_{i=1}^n \widetilde{\delta}_i^T D^\alpha \widetilde{\delta}_i &= \sum_{i=1}^n (\hat{\delta}_i - \delta_i)^T D^\alpha \hat{\delta}_i \\ &= \sum_{i=1}^n \hat{\delta}_i^T F_i^T(x) x_i - \sum_{i=1}^n \delta_i^T F_i^T(x) x_i \\ &= \sum_{i=1}^n x_i F_i(x)\hat{\delta}_i - \sum_{i=1}^n x_i F_i(x)\delta_i \end{aligned} \tag{19}$$

substituting Eq.(19) into (18), and according to Eqs.(9), (10), (13), (14), we get

$$\begin{aligned} J &\le |x_1||f_1(x)| + |x_1|(|\Delta f_1(x) + d_1(t)|) + x_1 u_1(t) \\ &\quad + |x_1||\Delta u_1(t)| + \cdots + |x_n||f_n(x)| + |x_n|(|\Delta f_n(x) \\ &\quad + d_n(t)|) + x_n u_n(t) + |x_n||\Delta u_n(t)| + \sum_{i=1}^n x_i F_i(x)\hat{\delta}_i \\ &\quad + \sum_{i=1}^n \frac{1}{\rho_i}\widetilde{\phi}_i D^\alpha \widetilde{\phi}_i + \sum_{i=1}^n \frac{1}{\eta_i}\widetilde{\gamma}_i D^\alpha \widetilde{\gamma}_i \\ &\le |x_1||f_1(x)| + \gamma_1|x_1| + x_1 u_1(t) + \phi_1|x_1| + \cdots \\ &\quad + |x_n||f_n(x)| + \gamma_n|x_n| + x_n u_n(t) + \phi_n|x_n| \\ &\quad + \sum_{i=1}^n x_i F_i(x)\hat{\delta}_i + \sum_{i=1}^n (\hat{\phi}_i - \phi_i)|x_i| + \sum_{i=1}^n (\hat{\gamma}_i - \gamma_i)|x_i| \\ &= |x_1||f_1(x)| + x_1 u_1(t) + \cdots + |x_n||f_n(x)| \\ &\quad + x_n u_n(t) + \sum_{i=1}^n x_i F_i(x)\hat{\delta}_i + \sum_{i=1}^n \hat{\phi}_i|x_i| + \sum_{i=1}^n \hat{\gamma}_i|x_i| \end{aligned} \tag{20}$$

bring Eq.(11) into (20), it has

$$\begin{aligned} J &\le |x_1||f_1(x)| - x_1\big(|F_1(x)||\hat{\delta}_1| + |f_1(x)| + \hat{\phi}_1 + \hat{\gamma}_1 \\ &\quad + k_1\big)sgn(x_1) + \cdots + |x_n||f_n(x)| - x_n\big(|F_n(x)||\hat{\delta}_n| \\ &\quad + |f_n(x)| + \hat{\phi}_n + \hat{\gamma}_n + k_n\big)sgn(x_n) + \sum_{i=1}^n x_i F_i(x)\hat{\delta}_i \\ &\quad + \sum_{i=1}^n \hat{\phi}_i|x_i| + \sum_{i=1}^n \hat{\gamma}_i|x_i| \end{aligned} \tag{21}$$

owning to $x_i sgn(x_i) = |x_i|$, then

$$\begin{aligned} J &\le |x_1||f_1(x)| - x_1\big(|F_1(x)||\hat{\delta}_1| + |f_1(x)| + \hat{\phi}_1 + \hat{\gamma}_1 \\ &\quad + k_1\big)sgn(x_1) + \cdots + |x_n||f_n(x)| - x_n\big(|F_n(x)||\hat{\delta}_n| \\ &\quad + |f_n(x)| + \hat{\phi}_n + \hat{\gamma}_n + k_n\big)sgn(x_n) \\ &\quad + \sum_{i=1}^n |x_i||F_i(x)||\hat{\delta}_i| + \sum_{i=1}^n \hat{\phi}_i|x_i| + \sum_{i=1}^n \hat{\gamma}_i|x_i| \\ J &\le -k_1|x_1| - k_2|x_2| - \cdots - k_n|x_n| \le -k||x|| < 0 \end{aligned} \tag{22}$$

where $k = min\{k_i\} > 0, i = 1, 2, ..., n$, that is, system (5) is asymptotic stability. Consequently, the adaptive stabilization of fractional-order nonlinear system with unknown parameters and saturation nonlinear inputs is achieved. Therefore, the proof is completed.

IV. Simulation Results

In this section, a simulation example is given to verify the feasibility and effectiveness of the proposed control strategy. The fractional-order Chen system with unknown parameters and saturation inputs is described as

$$\begin{aligned} D^\alpha x_1 &= a(x_2 - x_1) + \Delta f_1(x) + d_1(t) + sat(u_1(t)) \\ D^\alpha x_2 &= bx_1 + cx_2 - x_1x_3 + \Delta f_2(x) + d_2(t) + sat(u_2(t)) \\ D^\alpha x_3 &= -dx_3 + x_1x_2 + \Delta f_3(x) + d_3(t) + sat(u_3(t)) \end{aligned} \tag{23}$$

letting $\alpha = 0.92$, $k_1 = k_2 = k_3 = 5$, $\rho_1 = \rho_2 = \rho_3 = 2$, $\eta_1 = \eta_2 = \eta_3 = 4$. In this example, $F_1(x) = x_2 - x_1$,

$F_2(x) = (x_1, x_2)$, $F_3(x) = -x_3$, $\delta_1 = a$, $\delta_2 = (b, c)^T$, $\delta_3 = d$, the initial values are set as $x(0) = (5, 8, -4)^T$, $\hat{\delta}_1(0) = 0$, $\hat{\delta}_2(0) = (0, 0)^T$, $\hat{\delta}_3(0) = 0$, $\hat{\phi}_1(0) = \hat{\phi}_2(0) = \hat{\phi}_3(0) = 0$, $\hat{\gamma}_1(0) = \hat{\gamma}_2(0) = \hat{\gamma}_3(0) = 0$. the unmodeled dynamics and external disturbances are selected as

$$\begin{aligned} \Delta f_1(x) + d_1(t) &= 0.025cos(2t)x_1 + 0.015sin(t) \\ \Delta f_2(x) + d_2(t) &= -0.02cos(6t)x_2 + 0.01sin(2t) \\ \Delta f_3(x) + d_3(t) &= 0.015cos(3t)x_3 + 0.02sin(3t)(24) \end{aligned}$$

the saturation nonlinear inputs in this example are

$$Sat(u_1(t)) = \begin{cases} 5, & if u_1(t) \geq 1 \\ 5u_1(t), & if -1 \leq u_1(t) \leq 1 \\ -5, & if u_1(t) \leq -1 \end{cases} \quad (25)$$

$$Sat(u_2(t)) = \begin{cases} 8, & if u_2(t) \geq 2 \\ 4u_2(t), & if -2 \leq u_2(t) \leq 2 \\ -8, & if u_2(t) \leq -2 \end{cases} \quad (26)$$

$$Sat(u_3(t)) = \begin{cases} 6, & if u_3(t) \geq 1.5 \\ 4u_3(t), & if -1.5 \leq u_3(t) \leq 1.5 \\ -6, & if u_3(t) \leq -1.5 \end{cases} \quad (27)$$

The simulation results are depicted in Fig.2-6. Fig.2 shows the time response of state trajectories in system (23) with the controller actived.

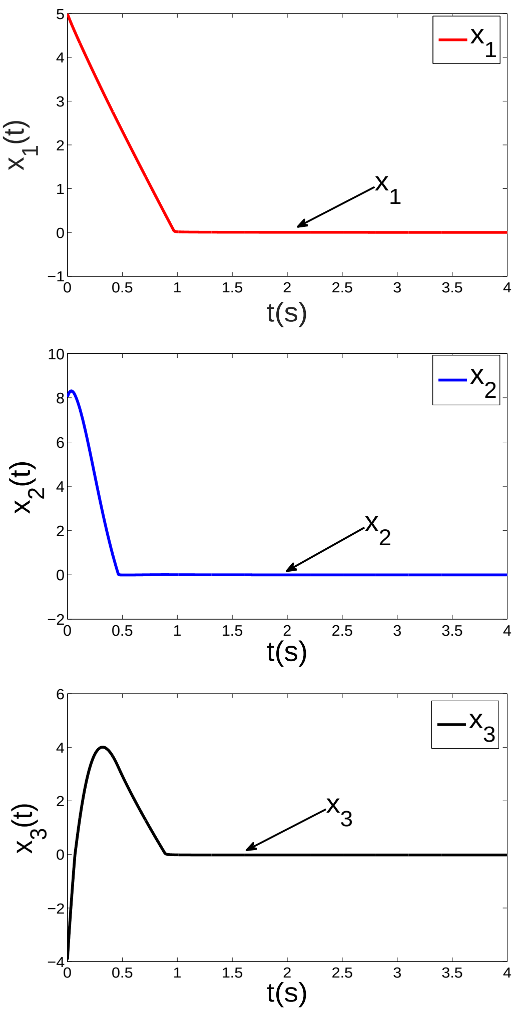

Fig. 2: Time response of fractional-order Chen system (23) with control

It is obviously that on the control of robust controller (11), all state trajectories converge to zero asymptotically. Fig.3 depicts the time evolution of estimate parameters, It is not hard to see all unknown parameters gradually converge to actual values.

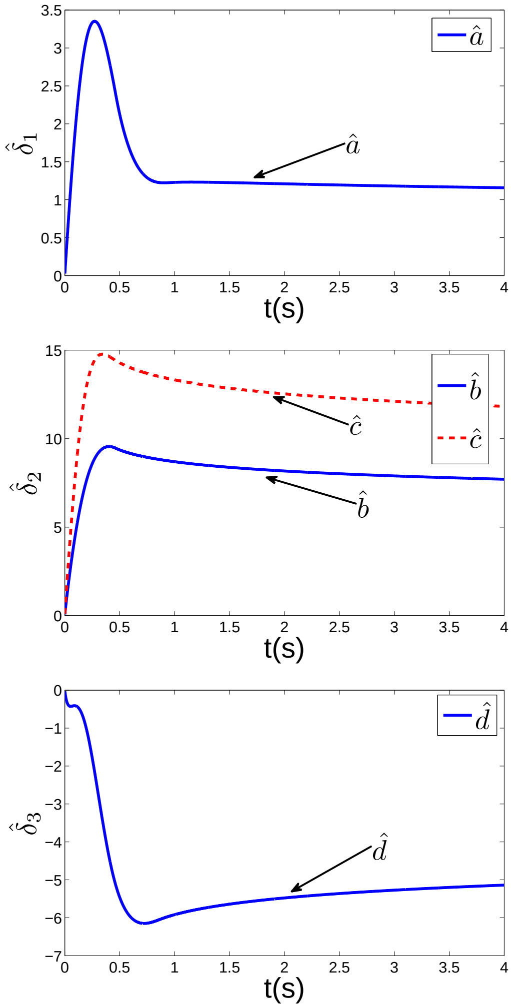

Fig. 3: Time evolution of estimate parameters of system (23)

The estimations of the unmodeled dynamics, external disturbances and input uncertainties are shown in Figs. 4 and 5, respectively. It is not hard to see all adaptation parameters are approach to some fixed values.

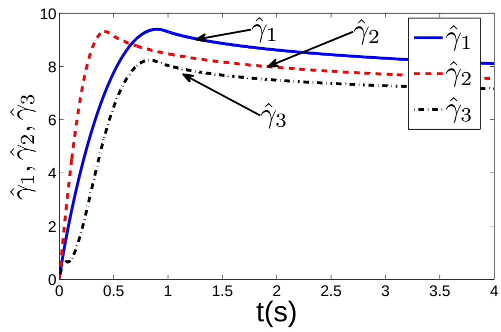

Fig. 4: Time evolution of unmodeled dynamics and external disturbances

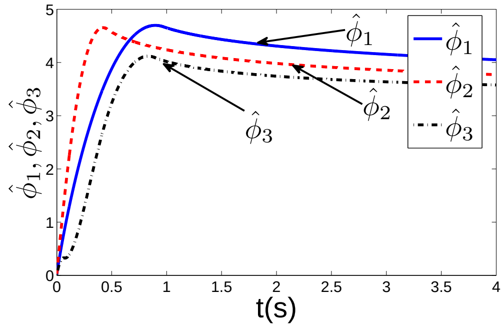

Fig. 5: Time response of input uncertainties

The time histories of the applied control inputs are plotted in Fig. 6. One can see that the control inputs are feasible in practical applications.

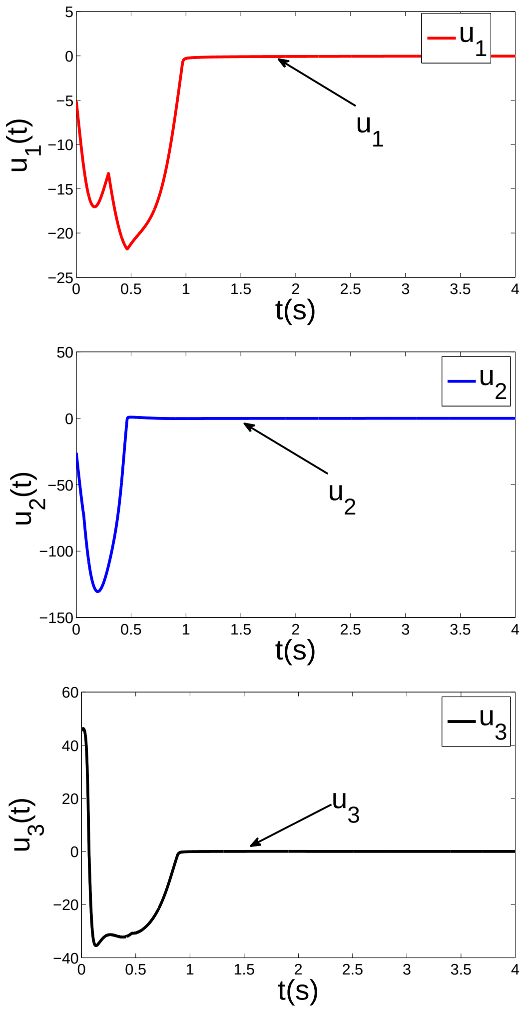

Fig. 6: Time histories of the applied control inputs

All above simulation results sufficiently demonstrate that the proposed control scheme is effective in stabilizing this kinds of uncertain fractional-order nonlinear systems with unknown parameters and saturation nonlinear inputs.

V. Conclusions

This paper researched the problem of stabilizing an uncertain fractional-order nonlinear systems with unknown parameters and saturation nonlinear inputs. The system is perturbed by uncertain unmodeled dynamics and external disturbances, and the bounds of both unmodeled dynamics and external disturbances are assumed to be unknown in advance. To handle these unknown parameters, some appropriate update laws are proposed. Fractional-order version of Lyapunov theory is used to demonstrate the stability of the closed-loop system. Simulation results verified that the proposed control strategy is effective and feasible.

Acknowledgement

This work is supported by the Foundation of Jinling Institute of Technology (Grant No: jit-fhxm-201607 and jit-b-201706), the Natural Science Foundation of Jiangsu Province University (Grant No: 17KJB120003).

References

[1] R. L. Bagley, "Calico R.A. Fractional order state equations for the control of viscoelastically damped structure," *Journal of Guidance, Control, Dynamics*, vol. 14, no. 2, pp. 304-311, 1991.

[2] H. H. Sun, A. A. Abdelwahad, B. Oharal, "Linear approximation of transfer function with a pole of fractional power," *IEEE Transaction on Automation Control*, vol. 29, no. 5, pp. 441-444, 1984.

[3] M. Ichise, Y. Nagayanagi, T. Kojima, "An analog simualtion of non-integer order trasfer functions for analysis of electrode process," *Journal of Electroanalytical Chemistry and Interfacial Electrochemistry*, vol. 33, pp. 253-265, 1971.

[4] D. Y. Chen, R. F. Zhang, J. C. Sprott, "Synchronization between integer-order chaotic systems and a class of fractional order chaotic systems via sliding mode control," *Chaos*, vol. 22, pp. 023130, 2012.

[5] D. Y. Chen, R. F. Zhang, X. Y. Ma, "Synchronization between a novel class of fractional-order and integer-order chaotic system via a sliding mode controller," *Chinese Physical B*, vol. 21, no. 12, pp. 120507, 2012.

[6] D. Y. Chen, Y. X. Liu, X. Y. Ma, R. F. Zhang, "Control of a class of fractional-order Chaotic systems via sliding mode," *Nonlinear Dynamics*, vol. 67, pp. 893-901, 2012.

[7] R. X. Zhang, S. P. Yang, "Stabilization of fractional-order chaotic system via a sinlge state adaptive-feedback controller," *Nonlinear Dynamics*, vol. 68, pp. 45-51, 2012.

[8] J. G. Lu, "Nonlinear observer design to synchronize fractional-order chaotic system via a scalar transmitted signal," *Physica A*, vol. 359, pp. 107-118, 2006.

[9] H. Targhvafard, "Erjaee G.H. Phase and anti-phase synchronization of fractional order chaotic via active control," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, pp. 4079-4088, 2011.

[10] Z. M. Odibat, "Adaptive feedback control and synchronization of non-identical chaotic fractional order systems," *Nonlinear Dynamics*, vol. 60, pp. 479-487, 2010.

[11] R. X. Zhang, S. P. Yang, "Robust synchronization of two different fractional-order chaotic systems with unknown parameters using adaptive sliding mode approach," *Nonlinear Dynamics*, vol. 71, pp. 269-278, 2013.

[12] C. L. Li, Y. N. Tong, "Adaptive control and synchronization of a fractional-order chaotic system," *Pramana Journal of Physics*, vol. 80, pp. 583-592, 2013.

[13] M. P. Aghababa, "Adaptive control of complex systems with unknown dynamics and input constraint: Applied to a chaotic elastic beam," *International Journal of Adaptive Control and Signal Processing*, vol. 32, pp. 213-228, 2018.

[14] M. P. Aghababa, "Finite-tiem control of a class of nonlinear switched systems in spite of unknown parameters and input saturation," *Nonlinear Analysis: Hybrid Systems*, vol. 31, pp. 220-232, 2019.

[15] M. P. Aghababa, "Controlling fluctuated chaotic power systems with compensation of input saturation: Application to electric direct current machines," *Journal of Dynamic Systems, Measurement, and Control*, vol. 141, pp. 011012, 2019.

[16] J. B. Hu, Y. Han, L. D. Zhao, "A novel stability theorem for fractional systems and its applying in synchronizing fractional chaotic system based on backstepping approach," *Acta Physica Sinica*, vol. 58, pp. 2235-2239, 2009.

[17] C. L. Li, Y. N. Tong, "Adaptive control and synchronization of a fractional-order chaotic system," *Pramana Journal of Physics*, vol. 80, pp. 583-592, 2013.

[18] J. B. Hu, Y. Han, L. D. Zhao, "Synchronizing fractional chaotic systems based on Lyapunov equation," *Acta Physica Sinica*, vol. 57, pp. 7522-7526, 2008.

[19] G. Q. Si, Z. Y. Sun, H. Y. Zhang, Y. B. Zhang, "Parameter estimation and topology identification of uncertain fractional order complex networks," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, pp. 5158-5171, 2012.

[20] L. X. Yang, J. Jiang, "Adaptive synchronization of drive-response fractional-order complex dynamical networks with uncertain parameters," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, pp. 1496-1506, 2014.

[21] G. E. Fatma, K. Songul, G. R. Leo, "A Numerical Investigation of VMS-POD Model for Darcy-Brinkman Equations," *Proceedings of The World Congress on Engineering*, London, pp. 7-11, 2018.

[22] W. Gao, C. Shi, "Sufficient Conditions for The Existence of Fractional Factors in Different Settings," *IAENG International Journal of Applied Mathematics*, vol. 49, no. 2, pp. 145-154, 2019.

[23] G. Mogos, "Ciphertext-Policy Attribute-Based Encryption using Quantum Multilevel Secret Sharing Scheme," *IAENG International Journal of Computer Science*, vol. 45, no. 4, pp. 500-504, 2018.

Xiaomin Tian (1987-) gained her master degree from Anhui University of Science and Technology in 2012, and gained doctor degree from Southeast University in 2015. Now, she is a teacher in Jinling Institute of Technology, and her main research direction is about the control of fractional-order system.

Zhong Yang (1968-) gained his doctor degree from Nanjing University of Aeronautics and Astronautics in 1996. Now, he is a professor in Jinling Institute of Technology, and his main research direction is about the development of new energy resources.