Global Finite-Time Stabilization for a Class of Upper-Triangular Systems Subject to Input Saturation

Yanling Shang, Jiacai Huang, Xinxin Shi and Xiaochun Zhu

***Abstract*—This paper addresses the problem of saturated global finite-time stabilization by state feedback for a class of nonlinear systems with upper-triangular structure. By using the homogeneous domination approach and the nested saturation technique, a saturated state feedback controller is successfully constructed to guarantee that states of the closed-loop system are globally finite-time regulated to zero without violation of the input constraint. A simulation example is provided to demonstrate the effectiveness of the proposed method.**

***Index Terms*—upper-triangular systems, input saturation, homogeneous domination approach, Nested saturation, finite-time stabilization.**

I. Introduction

During the past years, upper-triangular systems have received widely attention because they can be used to model many practical systems, such as the ball and beam system, the cart-pendulum system, the TORA system, and so on. However, the design of globally stabilizing controller for a upper-triangular system has proven to constitute a challenging task due to the fact that such system is neither feedback linearizable nor stabilized by applying the frequently-used backstepping approach. To give this difficulty a solution, a number of intelligent approaches have been developed such as the nested-saturation method [1-6] and the forwarding technique [7, 8]. Based on these effective methods, the problem global stabilization of upper-triangular systems has been well-studied recently [9-16]. However, the effect of the input constraint is omitted in the above-mentioned results.

As we all know, the actuator saturation is a common phenomenon in practical systems due to the inherent physical limitations of devices. Its existence often severely limits system performance, giving rise to undesirable inaccuracy or leading to instability [17-20]. For example, consider the following simple system:

$$\dot{x} = u + x \tag{1}$$

where $x \in R$ is a state variable, and $u \in R$ denotes the plant input. Obviously, system (1) is globally controllable, but when $|u| \leq u^{max}$ is required, there does not exist any saturated control to globally stabilize this system with initial value $x(0) > u^{max}$. Thus, it is of great significance to study the problem of saturated global stabilization of upper-triangular systems. Nevertheless, to the best of our knowledge, this issue on upper-triangular systems has not been well-addressed in the literature.

Based on the above observations and considering that the finite-time stable systems have better convergence and disturbance rejection properties [21-23], this paper focuses on solving the problem of global finite-time stabilization for a class upper-triangular systems by saturated state feedback. The major obstacle to tackle this problem lies in that the common assumptions and finite-time control techniques mainly for unsaturated upper-triangular systems are infeasible here. Until now it still remains unanswered that under what conditions the upper-triangular systems may exist saturated finite-time controller. To overcome the aforementioned difficulty, we first place a general homogeneous growth condition and design an unsaturated finite-time state feedback controller for the considered system by employing the homogeneous domination approach. Then, we impose a series of nested saturations to the developed controller and obtain a saturated state feedback controller, which renders that the states of the closed-loop system globally finite-time convergence to zero.

Notations. Throughout this paper, the following notations are adopted. R^+ denotes the set of all nonnegative real numbers and R^n denotes the real n-dimensional space. For a given vector X, X^T denotes its transpose, and $|X|$ denotes its Euclidean norm. C^i denotes the set of all functions with continuous ith partial derivatives. Besides, let $\sum_j^i(\cdot) = 0$ if $j > i$ and the arguments of the functions will be omitted or simplified, whenever no confusion can arise from the context. For instance, we sometimes denote a function $f(x(t))$ by simply $f(x)$, $f(\cdot)$ or f.

II. Problem formulation and preliminaries

In this paper, we consider a class of upper-triangular systems systems described by

$$\begin{aligned} &\dot{x}_i = x_{i+1} + \phi_i(t, x_{i+2}, \cdots, x_n, u), \quad i = 1, \cdots, n-2 \\ &\dot{x}_{n-1} = x_n + \phi_{n-1}(t, u) \\ &\dot{x}_n = u \end{aligned} \tag{2}$$

where $x = (x_1, \cdots, x_n)^T \in R^n$, $u \in R$ are the system state and control input, respectively. The continuous functions $\phi_i : R \times R^{n-i} \to R$, $i = 1, \cdots, n-1$ represent unknown nonlinear perturbations.

The objective of this paper is to find a state feedback control design strategy which globally finite-time stabilizes the system (2) under the following saturation constraint:

$$-u^{max} \leq u \leq u^{max} \tag{3}$$

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where u^{max} is a priori known positive real number.

To this end, the following assumption regarding system (2) is imposed.

Assumption 1. For $i = 1, \cdots, n-1$, there are constants $b > 0$ and $\tau \in (-1/n, 0)$ such that

$$|\phi_i(\cdot)| \le b \sum_{j=i+2}^{n+1} |x_j|^{(r_i+\tau)/r_j}$$

where $x_{n+1} = u$, $r_1 = 1$, $r_{i+1} = r_i + \tau > 0$, $i = 1, \cdots, n$ and $\sum_{l=1}^{n} p_1 \cdots p_{l-1} = 1$ for the case of $l = 1$.

In what follows, we review some useful definitions and lemmas which will serve as the basis of the coming control design and performance analysis.

Definition 1[21]**.** Consider a system

$$\dot{x} = f(x) \text{ with } f(0) = 0, \quad x \in R^n \tag{4}$$

where $f : U_0 \to R^n$ is continuous with respect to x on an open neighborhood U_0 of the origin $x = 0$. The equilibrium $x = 0$ of the system is (locally) finite-time stable if it is Lyapunov stable and finite-time convergent in a neighborhood $U \in U_0$ of the origin. By "finite-time convergence," we mean: If, for any initial condition $x(0) \in U$, there is a settling time $T > 0$, such that every solution $x(t)$ with $x(0)$ as its initial condition of (4) is well defined with $x(0) \in U \setminus \{0\}$ for $t \in [0, T)$ and satisfies $\lim_{t\to T} x(t) = 0$ and $x(t) = 0$ for any $t \ge T$. If $U = U_0 = R^n$, the origin is a globally finite-time stable equilibrium.

Lemma 1[21]**.** Consider the nonlinear system (4). Suppose there is a C^1 function $V(x)$ defined in a neighborhood $\hat{U} \in R^n$ of the origin, real numbers $c > 0$ and $0 < \alpha < 1$, such that

(i) $V(x)$ is positive definite on $\hat{U}$;

(ii) $\dot{V}(x) + cV^{\alpha}(x) \le 0, \quad \forall x \in \hat{U}$.

Then, the origin of system (4) is locally finite-time stable with $T \le \frac{V^{1-\alpha}(x(0))}{c(1-\alpha)}$ for initial condition $x(0)$ in some open neighborhood $U \in \hat{U}$ of the origin. If $U = R^n$ and $V(x)$ is also radially unbounded (i.e., $V(x) \to +\infty$ as $x \to +\infty$), the origin of system (4) is globally finite-time stable.

Definition 2[9]**.** Weighted Homogeneity: For fixed coordinates $(x_1, \cdots, x_n) \in R^n$ and real numbers $r_i > 0$, $i = 1, \cdots, n$,

- the dilation $\Delta_\varepsilon(x)$ is defined by $\Delta_\varepsilon(x) = (\varepsilon^{r_1}x_1, \cdots, \varepsilon^{r_n}x_n)$ for any $\varepsilon > 0$, where r_i is called the weights of the coordinates. For simplicity, we define dilation weight $\Delta = (r_1, \cdots, r_n)$.
- a function $V \in (R^n, R)$ is said to be homogeneous of degree τ if there is a real number $\tau \in R$ such that $V(\Delta_\varepsilon(x)) = \varepsilon^{\tau} V(x_1, \cdots, x_n)$ for any $x \in R^n \setminus \{0\}, \varepsilon > 0$.
- a vector field $f \in (R^n, R^n)$ is said to be homogeneous of degree τ if there is a real number $\tau \in R$ such that $f_i(\Delta_\varepsilon(x)) = \varepsilon^{\tau+r_i} f_i(x)$, for any $x \in R^n \setminus \{0\}, \varepsilon > 0$, $i = 1, \cdots, n$.
- a homogeneous p-norm is defined as $\|x\|_{\triangle,p} = (\sum_{i=1}^{n} |x_i|^{p/r_i})^{1/p}$ for all $x \in R^n$, for a constant $p \ge 1$. For simplicity, in this paper, we choose $p = 2$ and write $\|x\|_{\triangle}$ for $\|x\|_{\triangle,2}$.

Lemma 2[9]**.** Suppose $V : R^n \to R$ is a homogeneous function of degree τ with respect to the dilation weight Δ. Then the following holds:

(i) $\partial V/\partial x_i$ is homogeneous of degree $\tau - r_i$ with r_i being the homogeneous weight of x_i.

(ii) There is a constant c such that $V(x) \le c\|x\|_{\triangle}^{\tau}$. Moreover, if $V(x)$ is positive definite, then $\underline{c}\|x\|_{\triangle}^{\tau} \le V(x)$, where $\underline{c}$ is a constant.

III. The Design of Saturated State Feedback Controller

In this section, we give a constructive procedure for the globally finite-time stabilizer of system (2) by saturated state feedback. Before designing the controller, we first introduce the following coordinate transformation:

$$z_1 = x_1, \quad z_i = \frac{x_i}{L^{\kappa_i}}, \quad i = 2, \cdots, n, \quad v = \frac{u}{L^{\kappa_n+1}} \tag{5}$$

where $\kappa_i = n - 1$, $i = 1, \cdots, n-$ and $0 < L < 1$ is a constant to be determined later.

Then, under the new coordinates z_i's, system (2) is transformed into:

$$\begin{aligned} \dot{z}_i &= Lz_{i+1} + \frac{\phi_i}{L^{\kappa_i}}, \quad i = 1, \cdots, n-1 \\ \dot{z}_n &= Lv \end{aligned} \tag{6}$$

Noting that the transformation (5) is invertible, thus in the next, we turn to designing a saturated state feedback controller for system (6). The design idea is as follows: First, an unsaturated state feedback controller is constructed for nonlinear system (6) by using the homogeneous domination approach. Then, by imposing a series of nested saturations to the developed controller and obtain a saturated global state feedback controller for system (6).

A. Unsaturated state feedback controller design

Step 1. Choose the Lyapunov function $V_1 = W_1 = \int_0^{z_1} (s^{1/r_1} - 0)^{2-\tau-r_1} ds$. Clearly, the first virtual controller

$$z_2^* = -\beta_1^* \xi_1^{r_2} \tag{7}$$

with $\xi_1 = z_1$ and $\beta_1^* \ge n$ being a constant, renders

$$\dot{V}_1 \le -nL\xi_1^2 + L\xi_1^{2-\tau-r_1}(z_2 - z_2^*) + \frac{\partial V_1}{\partial z_1}\phi_1 \tag{8}$$

Step i $(i = 2, \cdots, n)$**.** In this step, we can obtain the following property, whose similar proof can be found in [9] and hence is omitted here.

Proposition 1. Assume that at step $i-1$, there is a C^1, proper and positive definite Lyapunov function V_{i-1}, and a set of virtual controllers $z_1^*, \cdots, z_i^*$ defined by

$$\begin{array}{rclrcl} z_1^* &=& 0, & \xi_1 &=& z_1^{1/r_1} - z_1^{*1/r_1} \\ z_2^* &=& -\beta_1^*\xi_1^{r_2}, & \xi_2 &=& z_2^{1/r_2} - z_2^{*1/r_2} \\ \vdots & & & \vdots & & \\ z_i^* &=& -\beta_{i-1}^*\xi_{i-1}^{r_i} & \xi_i &=& z_i^{1/r_i} - z_i^{*1/r_i} \end{array} \tag{9}$$

with $\beta_j^* > 0$, $j = 1, \cdots, i-1$, being constants, such that

$$\begin{aligned} \dot{V}_{i-1} \le{}& -(n-i+2)L\sum_{j=1}^{i-1}\xi_j^2 + \sum_{j=1}^{i-1}\frac{\partial V_i}{\partial z_j}\frac{\phi_j}{L^{\kappa_j}} \\ &+L\xi_i^{2-\tau-r_{i-1}}(z_i - z_i^*) \end{aligned} \tag{10}$$

Then the ith Lyapunov function defined by

$$V_i = V_{i-1} + \int_{z_i^*}^{z_i} (s^{1/r_i} - z_i^{*1/r_i})^{2-\tau-r_i} ds \tag{11}$$

is C^1, proper and positive definite, and there exists the C^0 virtual controller $z_{i+1}^* = -\beta_i^* \xi_i^{r_{i+1}/\rho}$ such that

$$\begin{aligned}\dot{V}_i \le & -(n-i+1)L\sum_{j=1}^{i}\xi_j^2 + \sum_{j=1}^{i}\frac{\partial V_i}{\partial z_j}\frac{\phi_j}{L^{k_j}} \\ & +L\xi_i^{2-\tau-r_i}(z_{i+1}-z_{i+1}^*)\end{aligned} \quad (12)$$

where $\beta_i > 0$ is a constant.

Hence at step n, choosing

$$V_n = \sum_{i=1}^{n}\int_{z_i^*}^{z_i}\left(s^{1/r_i} - z_i^{*1/r_i}\right)^{2-\tau-r_i} ds \quad (13)$$

and

$$\begin{aligned} z_{n+1}^* &= -\beta_n^* \xi_n^{r_{n+1}} \\ &= -\beta_n^*\Big(z_n^{1/r_n} + \beta_{n-1}^{*1/r_n}\Big(z_{n-1}^{1/r_{n-1}} + \cdots \\ &\qquad +\beta_2^{*1/r_3}\Big(z_2^{1/r_2} + \beta_1^{*1/r_2}z_1\Big)\Big)\Big)^{r_{n+1}} \\ &= -\beta_n^*\Big(\bar{\beta}_n^* z_n^{1/r_n} + \bar{\beta}_{n-1}^* z_{n-1}^{1/r_{n-1}} \\ &\qquad + \cdots + \bar{\beta}_1^* z_1^{1/r_1}\Big)^{r_{n+1}} \end{aligned} \quad (14)$$

where

$$\bar{\beta}_i^* = \begin{cases} \bar{\beta}_{n-1}^{*1/r_n}\cdots\bar{\beta}_i^{*1/r_{i+1}}, & i = 1,\cdots,n-1 \\ 1, & i = n \end{cases} \quad (15)$$

from Proposition 1, we arrive at

$$\begin{aligned}\dot{V}_n \le & -L\sum_{j=1}^{n}\xi_j^2 + \sum_{j=1}^{n-1}\frac{\partial V_n}{\partial z_j}\frac{\phi_j}{L^{\kappa_j}} \\ & +L\xi_n^{2-\tau-r_n}(v - z_{n+1}^*)\end{aligned} \quad (16)$$

Consequently, the following result is obtained.

Lemma 3. For the nonlinear system (6) under Assumption 1, the unsaturated state feedback controller $v = z_{n+1}^*$ in (14) renders the origin of the closed-loop system is globally finite-time stable.

Proof. Since V_n is positive definite and proper with respect to $z = (z_1, \cdots, z_n)^T$, by introducing the dilation weight $\Delta = (r_1, \cdots, r_n)$, from Definition 2, it can be shown that V_n is homogeneous of degree $2-\tau$ with respect to Δ. By Lemma 2, there is a constant $\bar{c}_1$, such that

$$V_n \le \bar{c}_1 \|z\|_\Delta^{2-\tau} \quad (17)$$

where $\bar{c}_1 > 0$ and $\|z\|_\Delta = \sqrt{(\sum_{i=1}^n |z_i|^{2/r_i})}$. Similarly, since the $\sum_{j=1}^n \xi_j^2$ is homogeneous of degree 2, by Lemma 2.2 there is a constant $\bar{c}_2$ such that

$$\dot{V}_n \le -L\bar{c}_2\|z\|_\Delta^2 + \sum_{j=1}^{n-1}\frac{\partial V_n}{\partial z_j}\frac{\phi_j}{L^{\kappa_j}} \quad (18)$$

From Assumption 1, (5),(14) and $0 < L < 1$, we can find constants $\delta_j > 0$ and $\alpha_j > 0$ such that

$$\begin{aligned}\left|\frac{\phi_j}{L^{\kappa_j}}\right| &\le b\sum_{h=j+2}^{n+1} L^{\kappa_h(r_j+\tau)/r_h - \kappa_j}|z_h|^{(r_j+\tau)/r_h} \\ &\le \delta_j L^{1+\alpha_j}\|z\|_\Delta^{r_j+\tau}\end{aligned} \quad (19)$$

where $z_{n+1} = z_{n+1}^*$.

Noting that for $j = 1, \cdots, n$, $\partial V_n/\partial z_j$ is homogeneous of degree $2-\tau-r_j$, with aid of (19), we can find a positive constant c_j such that

$$\left|\frac{\partial V_n}{\partial z_j}\right|\left|\frac{\phi_j}{L^{\kappa_j}}\right| \le c_j L^{1+\alpha_j}\|z\|_\Delta^2 \quad (20)$$

Substituting (20) into (18) yields

$$\begin{aligned}\dot{V}_n &\le -L(\bar{c}_2 - \sum_{j=1}^{n} c_j L^{\alpha_j})\|z\|_\Delta^2 \\ &\le -L(\bar{c}_2 - \sum_{j=1}^{n} c_j L^{\alpha_{min}})\|z\|_\Delta^2\end{aligned} \quad (21)$$

where $0 < \alpha_{min} = min_{1\le j\le n}\{\alpha_j\}$. Apparently, by choosing a small enough L, the right-hand side of (21) is negative definite. Furthermore, it can be deduced from (17) and (21) that there is a constant $\bar{c}_3 > 0$ such that

$$\dot{V}_n \le -\bar{c}_3 V_n^{2/(2-\tau)} \quad (22)$$

Thus by Lemma 1, it is concluded that the closed-loop system consisting of (6) and $v = z_{n+1}^*$ in (14) is globally finite-time stable.

B. Saturated state feedback controller design

In this subsection, a saturated state feedback controller is designed to solve the global finite-time stabilization problem for system (6). By the combined saturation technique, we impose a series of nested saturations to the finite-time controller $v = z_{n+1}^*$ in (14) and obtain a saturated finite-time controller as following form

$$v_{ssf} = v_n(Z_n) = -\beta_n\sigma^{r_{n+1}}\Big(Z_n^{1/r_n} - v_{n-1}^{1/r_n}(Z_{n-1})\Big) \quad (23)$$

where $v_0 = 0$, $v_i(Z_i) = -\beta_i\sigma^{r_{i+1}}(Z_i^{1/r_i} - v_{i-1}^{1/r_i}(Z_{i-1}))$, $Z_i = (z_1, \cdots, z_i)$, $i = 1, \cdots, n$,

$$\sigma(x) = \begin{cases} \varepsilon sign(x), & |x| > \varepsilon \\ x, & |x| \le \varepsilon \end{cases}$$

for a small constant $\varepsilon > 0$ to be determined later, and the gains β_i's are selected as

$$\begin{aligned} &\beta_1 > max\Big\{\beta_1^*, 2^{r_2+1}\Big\} \\ &\beta_i > max\Big\{\beta_i^*, 2^{r_{i+1}}\Big(4(1+\beta_{i-1})\alpha_{i-1}(\cdot) + 2\Big)\Big\} \\ &\quad i = 2, \cdots, n \end{aligned} \quad (24)$$

with

$$\begin{aligned} &\alpha_1(\beta_1) = \beta_1^{1/r_2}(1+\beta_1) \\ &\alpha_j(\beta_1, \cdots, \beta_j) = \frac{\beta_j^{1/r_{j+1}}}{r_j}(1+\beta_{j-1})^{1/r_j - 1}(1+\beta_j) \\ &\qquad +\beta_j^{1/r_{j+1}}\alpha_{j-1}(\cdot), \quad j = 2, \cdots, n-1 \end{aligned} \quad (25)$$

Remark 1. From (23) and the definition of saturation function $\sigma(\cdot)$, it can clearly be seen that the controller $v_{ssf} = v_n(Z_n)$ is bounded by a constant $\beta_n\varepsilon^{r_{n+1}}$, which means that the bound of controller (23) can be arbitrarily small by choosing appropriate design constant ε.

We begin our the main result of this subsection by introducing an important lemma, whose similar proof can be found in [2].

Lemma 4. Consider the system (6) with saturated controller (23). For $i = 1, \cdots, n-1$, under the condition $|z_j| \le \varepsilon^{r_j}(1+\beta_{j-1})$, $j = i+1, \cdots n+1$, there exist a series of functions $\alpha_i(\beta_1, \cdots, \beta_i)$ defined as (25) and a constant $0 < \varepsilon_1 < 1$ such that for any $0 < \varepsilon \le \varepsilon_1$, $\bar{t} \ge \underline{t}$, the following inequalities hold:

$$\left|\frac{\phi_i}{L^{\kappa_i}}\right| \le L\varepsilon^{r_{i+1}} \quad (26)$$

$$\left|v_i^{1/r_{i+1}}(Z_i(\bar{t}))-v_i^{1/r_{i+1}}(Z_i(\underline{t}))\right|\leq L\alpha_i(\cdot)\varepsilon^{1+\tau}(\bar{t}-\underline{t}) \tag{27}$$

With the help of Lemmas 3 and 4, we are ready to state the main result of this paper.

Theorem 1. For the upper-triangular system (2) under Assumption 1, the saturated state feedback controller $u=L^{\kappa_n+1}v_{ssf}$ in (5) and (23), renders that the origin of the closed-loop system is globally finite-time stable.

Proof. The proof is proceeded in two steps. In the first step, it is proved that the control law with coefficients β_i's preset in (24) ensures that all states will converge to a region determined by the saturation function $\sigma(\cdot)$. Then, the saturated controller (23) reduces to the unsaturated controller (14). As a result, the global finite-time stability for the closed-loop system (6) with (23) can be guaranteed by appropriately choosing the gain L. Since the proof is quite similar to that of Theorem 4.1 in [2], we only briefly present the first step of the proof.

Step 1. At this stage we will find a time instance t_1 in such a way that for $t\geq t_1$

$$Z_n(t)\in Q_n=\left\{Z_n:|z_n^{1/r_n}(t)-v_{n-1}^{1/r_n}(Z_{n-1}(t))|<\varepsilon\right\} \tag{28}$$

By contradiction, it can be shown that there is a time instant t_1 such that

$$|z_n^{1/r_n}(t_1)-v_{n-1}^{1/r_n}(Z_{n-1}(t_1))|\leq\frac{\varepsilon}{2} \tag{29}$$

Otherwise, we assume that $|z_n^{1/r_n}(t)-v_{n-1}^{1/r_n}(Z_{n-1}(t))|>\varepsilon/2$ for all $t\geq 0$. Now, the case when

$$z_n^{1/r_n}(t)-v_{n-1}^{1/r_n}(Z_{n-1}(t))>\frac{\varepsilon}{2},\quad\forall t\geq 0 \tag{30}$$

is considered. In this case, for all $t\geq 0$, by (6) and (23), we have

$$\begin{aligned}\dot{z}_n(t)&=-L\beta_n\sigma^{r_{n+1}}\Big(z_n^{1/r_n}(t)-v_{n-1}^{1/r_n}(Z_{n-1}(t))\Big)\\&\leq -L\beta_n(\varepsilon/2)^{r_{n+1}}\\&:=-\mu_n\varepsilon^{r_{n+1}}\end{aligned} \tag{31}$$

with $\mu_n=L\beta_n(1/2)^{r_{n+1}}>0$, which implies that $z_n(t)<z_n(0)-\mu_n\varepsilon^{r_{n+1}}t,\ \ \forall t\geq 0$. Consequently, as time goes to infinity, $z_n(t)\to-\infty$, which leads to a contradiction disavowing (30) by noticing the fact $|v_{n-1}^{1/r_n}(Z_{n-1}(t))|\leq\beta_{n-1}^{1/r_n}\varepsilon$. Similarly, we can show the case $z_n^{1/r_n}(t)-v_{n-1}^{1/r_n}(Z_{n-1}(t))<-\varepsilon/2,\ \ \forall t\geq 0$, is also impossible. In conclusion, there must exist a time instance t_1 such that (29) holds.

Next, we will prove that the following holds after the time instant t_1

$$|z_n^{1/r_n}(t)-v_{n-1}^{1/r_n}(Z_{n-1}(t))|<\varepsilon,\quad\forall t\geq t_1 \tag{32}$$

If (32) is not true, there exists at least one time instant t_1^* such that $|z_n^{1/r_n}(t_1^*)-v_{n-1}^{1/r_n}(Z_{n-1}(t_1^*))|=\varepsilon$. Specifically, there are $t_1^{'}<\infty$ and $t_1^*<\infty$ such that either

$$z_n^{1/r_n}(t_1^{'})-v_{n-1}^{1/r_n}(Z_{n-1}(t_1^{'}))=\varepsilon/2 \tag{33}$$

$$z_n^{1/r_n}(t_1^*)-v_{n-1}^{1/r_n}(Z_{n-1}(t_1^*))=\varepsilon \tag{34}$$

$$\frac{\varepsilon}{2}\leq z_n^{1/r_n}(t)-v_{n-1}^{1/r_n}(Z_{n-1}(t))\leq\varepsilon,\quad t\in[t_1^{'},t_1^*] \tag{35}$$

in the positive region, or $z_n^{1/r_n}(t_1^{'})-v_{n-1}^{1/r_n}(Z_{n-1}(t_1^{'}))=-\varepsilon/2$, $z_n^{1/r_n}(t_1^*)-v_{n-1}^{1/r_n}(Z_{n-1}(t_1^*))=-\varepsilon$, $-\varepsilon\leq z_n^{1/r_n}(t)-v_{n-1}^{1/r_n}(Z_{n-1}(t))\leq-\varepsilon/2$, $\forall t\in[t_1^{'},t_1^*]$ in the negative case.

In what follows, it will be claimed that the positive case (33)–(35) is impossible. By (35) and (31), we have

$$\dot{z}_n(t)\leq-\mu_n\varepsilon^{p_nr_{n+1}},\quad t\in[t_1^{'},t_1^*] \tag{36}$$

which leads to

$$\mu_n\varepsilon^{p_nr_{n+1}}(t_1^*-t_1^{'})\leq z_n(t_1^{'})-z_n(t_1^*) \tag{37}$$

By (33), (34) and the fact that $|v_{n-1}^{1/r_n}(Z_{n-1})|\leq\beta_{n-1}^{1/r_n}\varepsilon$, we obtain

$$\begin{aligned}z_n(t_1^{'})&=\Big(\frac{\varepsilon}{2}+v_{n-1}^{1/r_n}(Z_{n-1}(t_1^{'}))\Big)^{r_n}\\&\leq(1+\beta_{n-1}^{1/r_n})^{r_n}\varepsilon^{r_n}\\&\leq(1+\beta_{n-1})\varepsilon^{r_n}\end{aligned} \tag{38}$$

and

$$\begin{aligned}z_n(t_1^*)&=\Big(\varepsilon+v_{n-1}^{1/r_n}(Z_{n-1}(t_1^*))\Big)^{r_n}\\&\geq-(1+\beta_{n-1}^{1/r_n})^{r_n}\varepsilon^{r_n}\\&\geq-(1+\beta_{n-1})\varepsilon^{r_n}\end{aligned} \tag{39}$$

Combining (38) with (39), from (37), the following time estimate is obtained:

$$t_1^*-t_1^{'}\leq\frac{2}{\mu_n}(1+\beta_{n-1})\varepsilon^{r_n-p_nr_{n+1}}=\frac{2}{\mu_n}(1+\beta_{n-1})\varepsilon^{-\tau} \tag{40}$$

In light of (37),a direct substitution gives

$$\varepsilon/2\leq|v_{n-1}^{1/r_n}(Z_{n-1}(t_1^*))-v_{n-1}^{1/r_n}(Z_{n-1}(t_1^{'}))| \tag{41}$$

By considering (35) and the fact that $|v_{n-1}^{1/r_n}(Z_{n-1})|\leq\beta_{n-1}^{1/r_n}\varepsilon$, one has

$$|z_n(t)|\leq(1+\beta_{n-1}^{1/r_n})^{r_n}\varepsilon^{r_n}\leq(1+\beta_{n-1})\varepsilon^{r_n},\ t\in[t_1^{'},t_1^*] \tag{42}$$

Further,by the definition of μ_n and the choice of (17), we have

$$\begin{aligned}\mu_n&=L\beta_n(1/2)^{r_{n+1}}\\&\geq L(1/2)^{r_{n+1}}2^{r_{n+1}}\Big(4(1+\beta_{n-1})\alpha_{n-1}(\cdot)+2\Big)\\&>4L(1+\beta_{n-1})\alpha_{n-1}(\cdot)\end{aligned} \tag{43}$$

This together with (41) renders that

$$\varepsilon/2\leq|v_{n-1}^{1/r_n}(Z_{n-1}(t_1^*))-v_{n-1}^{1/r_n}(Z_{n-1}(t_1^{'}))|<\varepsilon/2 \tag{44}$$

which obviously is a contradiction. Therefore the case of (33)–(35) will never happen. Similarly, it can be shown, using an almost same argument as the positive case, that $z_n^{1/r_n}(t)-v_{n-1}^{1/r_n}(Z_{n-1}(t))$ will never cross $-\varepsilon$. Hence for $t\leq t_1$ we have

$$|z_n^{1/r_n}(t)-v_{n-1}^{1/r_n}(Z_{n-1}(t))|<\varepsilon \tag{45}$$

Following the same line shown in the first step, at the final step, we can obtain that there exists a time instance t_n, such that when $t\geq t_n$, $Z_n(t)$ will enter and stay in the set

$$\begin{aligned}Q=\Big\{Z_n:|z_1^{1/r_1}|<\varepsilon,|z_2^{1/r_2}-v_1^{1/r_2}(z_1)|<\varepsilon,\cdots,\\|z_n^{1/r_n}-v_{n-1}^{1/r_i}(Z_{n-1})|<\varepsilon\Big\}\end{aligned} \tag{46}$$

Therefore, when $t\geq t_n$, the saturated controller (23) becomes the unsaturated controller (14). Therefore by Lemma 3, there is a constant $0<L<1$ such that the controller (14) globally finite-time stabilizes system (6). As a result, under the saturated controller (23), the global finite-time stability for system (2) is proved.

IV. Simulation Example

Consider the following upper-triangular system

$$\begin{aligned}\dot{x}_1 &= x_2 + u^3 \\ \dot{x}_2 &= u\end{aligned} \tag{47}$$

with the requirement of $|u| \leq u^{max} = 1$. Choosing $\tau = -2/5 \in (-1, 0)$, we have $r_1 = 1$, $r_2 = 3/5$ and $r_3 = 1/5$. It is obvious that Assumption 1 holds with $b = 1$. Therefore, by Theorem 1, we can explicitly construct a saturated state feedback controller for this example. Specifically, we can choose

$$u = -L^3\beta_2\sigma^{1/5}\left(z_2^{5/3} + \beta_1^{5/3}\sigma(z)\right) \tag{48}$$

with appropriate positive constants β_1, β_2, ε and a small enough gain L such that the state feedback controller (48) renders the system (47) globally finite-time stable.

In the simulation, by choosing the design parameters as $\beta_1 = 1.2$, $\beta_2 = 1.4$, $\varepsilon = 0.6$ and $L = 0.8$ and the initial condition as $(x_1(0), x_2(0))= (1, -2)$, Figure 1 is gained to exhibit the responses of the closed-loop system, from which the validity of the proposed method is demonstrated.

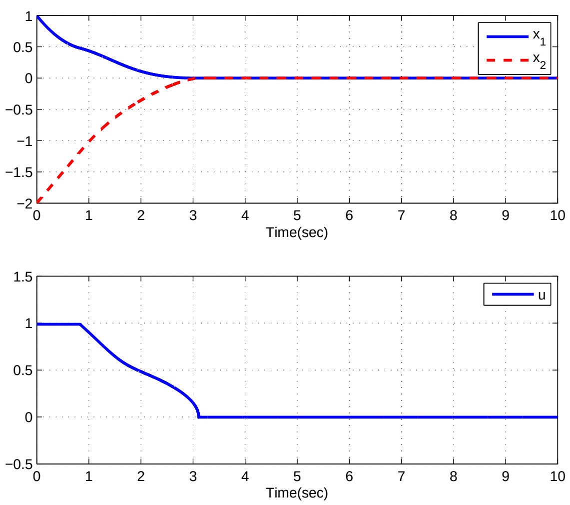

Fig. 1. The responses of the closed-loop system (47)–(48).

V. Conclusion

This paper has solved the problem of saturated stabilization by state feedback for a class of upper-triangular systems. With the help of the homogeneous domination approach and the nested saturation technique, a constructive design procedure for state feedback control is given, which can guarantee that the closed-loop system states are globally finite-time regulated to zero and the amplitude of the control signal is bounded.

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