Unconditionally Stable High Accuracy Alternating Difference Parallel Method for the Fourth-order Heat Equation

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***Abstract*—In this paper, we present a highly accurate alternating parallel difference method which solves the fourth-order heat equation subject to specific initial and boundary conditions. Based on a group of new Saul'yev type asymmetric difference schemes and the Crank-Nicolson scheme for the fourth-order heat equation, we derive a high-order, unconditionally stable and intrinsic parallel difference method. We also give the existence and uniqueness, the stability and the error estimate of numerical solution for the alternating difference parallel method. Theoretical analysis demonstrates that this method have obvious parallelism, unconditional stability and fourth-order convergence in space. Numerical experimentations are also conducted to compare the new method with the existing method.**

***Index Terms*—fourth-order heat equation, alternating difference method, unconditional stability, high accuracy, parallel computation.**

I. Introduction

CONSIDER the following fourth-order heat equation

$$Lu = \frac{\partial u}{\partial t} + \alpha \frac{\partial^4 u}{\partial x^4} = 0, \quad x \in [0, l], t \in [0, T], \tag{1}$$

with initial and boundary conditions given by

$$\begin{aligned} &u(x, 0) = f(x), \quad x \in [0, l], \\ &u(0, t) = g_1(t), \quad t \in [0, T], \\ &u(l, t) = g_2(t), \quad t \in [0, T], \\ &u_{xx}(0, t) = g_3(t), \quad t \in [0, T], \\ &u_{xx}(l, t) = g_4(t), \quad t \in [0, T]. \end{aligned}$$

where $f(x), g_1(t), g_2(t), g_3(t), g_4(t)$ are given functions, α is a constant.

This equation is relevant in modeling several problems in physics and biology. Various numerical schemes have been developed for solving one and two dimensional fourth-order heat equation in recent years [13,14,27,28,29,30]. over the past decade, the alternating parallel difference methods for parabolic partial equations have been studied extensively $[1-19]$. Evans and Abdullah first developed the alternating group explicit (AGE) scheme($[1, 2]$) for parabolic equation. The AGE scheme uses the explicit scheme and the implicit scheme alternately in the time and space direction, which can implement the parallel computation and is unconditionally stable. Then Zhang et al. proposed the alternating segment explicit-implicit (ASE-I) scheme($[3]$) and the alternating segment Crank-Nicolson (ASC-N) scheme($[4]$). Then, the alternating parallel difference schemes have been extended to one and two dimensional diffusion systems($[5, 8]$), dispersive equation($[6, 7, 9, 11, 12, 15]$), nonlinear three-order KdV equation($[10]$), one and two dimensional fourth-order heat equation($[13, 14, 16]$), nonlinear Leland equation($[17]$), quanto option pricing model($[18]$), some fractional equations ($[19, 20]$), respectively. The results of numerical examples show that these schemes have unconditional stability and intrinsic parallelism. Meanwhile, the introduction of the alternating schemes leads to the rapid development of the domain decomposition parallel methods($[21-26]$). However, the majority of the literature have focused their attentions on the parallelism, the major problem in the above algorithms is that the truncation error is only near second order. In view of the limited information available of highly accurate parallel difference method, this paper undertakes a study of the construction of high-order accurate algorithm for the fourth-order heat equation. Inspired by literatures $[8, 11, 12]$, in this paper, we present a new highly accurate alternating parallel difference method by a group of new high-order accurate asymmetric difference schemes for the solution of Eq.(1) together with its truncation error analysis to confirm the superiority of this new method over the existing method.

In the next section, we give the fourth-order accurate alternating difference method based on a group of new asymmetric difference schemes and the Crank-Nicolson scheme for Eq.(1). In Section 3, the existence and uniqueness of the solution are discussed. Section 4 , the truncation errors and the unconditional stability are proved. In Section 5, we present numerical experiments which were performed to test the fourth-order accuracy and unconditional stability. Finally, a brief conclusion is given.

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II. The Alternating Difference Parallel Method

A. The Finite Difference Approximations

In solving problem (1), we discretize the domain of definition $[0, l] \times [0, T]$ by parallel lines $x = x_j = jh(j = 0, 1, 2 \cdots, J)$, $t = t^n = n\tau(n = 0, 1, 2 \cdots, N)$, where $h = l/J$ is space mesh length, $\tau = T/N$ is time mesh length. J and N are positive integers. Let u_j^n represents the exact solution of Eq.(1) and U_j^n be the approximate solution at the grid point (x_j, t^n). We first give the Crank-Nicolson scheme (2)

$$-rU_{j+3}^{n+1}+12rU_{j+2}^{n+1}-39rU_{j+1}^{n+1}+(1+56r)U_j^{n+1}-39rU_{j-1}^{n+1}+12rU_{j-2}^{n+1}-rU_{j-3}^{n+1}=rU_{j+3}^n-12rU_{j+2}^n+39rU_{j+1}^n+(1-56r)U_j^n+39rU_{j-1}^n-12rU_{j-2}^n+rU_{j-3}^n, \quad (2)$$

and twelve new asymmetric schemes (3) – (14)

$$-rU_{j+3}^{n+1}+6rU_{j+2}^{n+1}-6rU_{j+1}^{n+1}+(1+r)U_j^{n+1}=rU_{j+3}^n-18rU_{j+2}^n+72rU_{j+1}^n+(1-111r)U_j^n+78rU_{j-1}^n-24rU_{j-2}^n+2rU_{j-3}^n, \quad (3)$$

$$(1+r)U_j^{n+1}-6rU_{j-1}^{n+1}+6rU_{j-2}^{n+1}-rU_{j-3}^{n+1}=rU_{j-3}^n-18rU_{j-2}^n+72rU_{j-1}^n+(1-111r)U_j^n+78rU_{j+1}^n-24rU_{j+2}^n+2rU_{j+3}^n, \quad (4)$$

$$-rU_{j+3}^{n+1}+18rU_{j+2}^{n+1}-72rU_{j+1}^{n+1}+(1+111r)U_j^{n+1}-78rU_{j-1}^{n+1}+24rU_{j-2}^{n+1}-2rU_{j-3}^{n+1}=rU_{j+3}^n-6rU_{j+2}^n+6rU_{j+1}^n+(1-r)U_j^n, \quad (5)$$

$$-2rU_{j+3}^{n+1}+24rU_{j+2}^{n+1}-78rU_{j+1}^{n+1}+(1+111r)U_j^{n+1}-72rU_{j-1}^{n+1}+18rU_{j-2}^{n+1}-rU_{j-3}^{n+1}=(1-r)U_j^n+6rU_{j-1}^n-6rU_{j-2}^n+rU_{j-3}^n, \quad (6)$$

$$-rU_{j+3}^{n+1}+12rU_{j+2}^{n+1}-33rU_{j+1}^{n+1}+(1+28r)U_j^{n+1}-6rU_{j-1}^{n+1}=rU_{j+3}^n-12rU_{j+2}^n+45rU_{j+1}^n+(1-84r)U_j^n+72ru_{j-1}^n-24rU_{j-2}^n+2rU_{j-3}^n, \quad (7)$$

$$-6rU_{j+1}^{n+1}+(1+28r)U_j^{n+1}-33rU_{j-1}^{n+1}+12rU_{j-2}^{n+1}-rU_{j-3}^{n+1}=rU_{j-3}^n-12rU_{j-2}^n+45rU_{j-1}^n+(1-84r)U_j^n+72rU_{j+1}^n-24rU_{j+2}^n+2rU_{j+3}^n, \quad (8)$$

$$-rU_{j+3}^{n+1}+12rU_{j+2}^{n+1}-45rU_{j+1}^{n+1}+(1+84r)U_j^{n+1}-72rU_{j-1}^{n+1}+24rU_{j-2}^{n+1}-2rU_{j-3}^{n+1}=rU_{j+3}^n-12rU_{j+2}^n+33rU_{j+1}^n+(1-28r)U_j^n+6rU_{j-1}^n, \quad (9)$$

$$-2rU_{j+3}^{n+1}+24rU_{j+2}^{n+1}-72rU_{j+1}^{n+1}+(1+84r)U_j^{n+1}-45rU_{j-1}^{n+1}+12rU_{j-2}^{n+1}-rU_{j-3}^{n+1}=6rU_{j+1}^n+(1-28r)U_j^n+33rU_{j-1}^n-12rU_{j-2}^n+rU_{j-3}^n, \quad (10)$$

$$-rU_{j+3}^{n+1}+12rU_{j+2}^{n+1}-39rU_{j+1}^{n+1}+(1+55r)U_j^{n+1}-33rU_{j-1}^{n+1}+6rU_{j-2}^{n+1}=rU_{j+3}^n-12rU_{j+2}^n+39rU_{j+1}^n+(1-57r)U_j^n+45rU_{j-1}^n-18rU_{j-2}^n+2rU_{j-3}^n, \quad (11)$$

$$6rU_{j+2}^{n+1}-33rU_{j+1}^{n+1}+(1+55r)U_j^{n+1}-39rU_{j-1}^{n+1}+12rU_{j-2}^{n+1}-rU_{j-3}^{n+1}=rU_{j-3}^n-12rU_{j-2}^n+39rU_{j-1}^n+(1-57r)U_j^n+45rU_{j+1}^n-18rU_{j+2}^n+2rU_{j+3}^n, \quad (12)$$

$$-rU_{j+3}^{n+1}+12rU_{j+2}^{n+1}-39rU_{j+1}^{n+1}+(1+57r)U_j^{n+1}-45rU_{j-1}^{n+1}+18rU_{j-2}^{n+1}-2rU_{j-3}^{n+1}=rU_{j+3}^n-12rU_{j+2}^n+39rU_{j+1}^n+(1-55r)U_j^n+33rU_{j-1}^n-6rU_{j-2}^n, \quad (13)$$

$$-2rU_{j+3}^{n+1}+18rU_{j+2}^{n+1}-45rU_{j+1}^{n+1}+(1+57r)U_j^{n+1}-39rU_{j-1}^{n+1}+12rU_{j-2}^{n+1}-rU_{j-3}^{n+1}=-6rU_{j+2}^n+33rU_{j+1}^n+(1-55r)U_j^n+39rU_{j-1}^n-12rU_{j-2}^n+rU_{j-3}^n, \quad (14)$$

where $r=\alpha\tau/12h^4$.

Let $L_h^{(2)}$, $L_h^{(3)}$, $L_h^{(4)}$, $L_h^{(5)}$, $L_h^{(6)}$, $L_h^{(7)}$, $L_h^{(8)}$, $L_h^{(9)}$, $L_h^{(10)}$, $L_h^{(11)}$, $L_h^{(12)}$, $L_h^{(13)}$, $L_h^{(14)}$ be the discretized operators for L based on schemes (2) – (14). From the Taylor series expansion at (x_j,t^n), we obtain the following truncation error expressions (15) – (27) for formulaes (2) – (14):

$$L_h^{(2)}u_j^n-[Lu]_j^n=-6rh^4[\frac{\partial^5u}{\partial t\partial x^4}]_j^n+O(\tau^2+h^4), \quad (15)$$

$$L_h^{(3)}u_j^n-[Lu]_j^n=3rh[\frac{\partial^2u}{\partial t\partial x}]_j^n+\frac{9}{2}rh^2[\frac{\partial^3u}{\partial t\partial x^2}]_j^n+\frac{5}{2}rh^3[\frac{\partial^4u}{\partial t\partial x^3}]_j^n+3r\tau^2h[\frac{\partial^4u}{\partial t^3\partial x}]_j^n+O(\tau+h^4), \quad (16)$$

$$L_h^{(4)}u_j^n-[Lu]_j^n=-3rh[\frac{\partial^2u}{\partial t\partial x}]_j^n+\frac{9}{2}rh^2[\frac{\partial^3u}{\partial t\partial x^2}]_j^n-\frac{5}{2}rh^3[\frac{\partial^4u}{\partial t\partial x^3}]_j^n-3r\tau^2h[\frac{\partial^4u}{\partial t^3\partial x}]_j^n+O(\tau+h^4), \quad (17)$$

$$L_h^{(5)}u_j^n-[Lu]_j^n=-3rh[\frac{\partial^2u}{\partial t\partial x}]_j^n-\frac{9}{2}rh^2[\frac{\partial^3u}{\partial t\partial x^2}]_j^n-\frac{5}{2}rh^3[\frac{\partial^4u}{\partial t\partial x^3}]_j^n-3r\tau^2h[\frac{\partial^4u}{\partial t^3\partial x}]_j^n+O(\tau+h^4), \quad (18)$$

$$L_h^{(6)}u_j^n-[Lu]_j^n=3rh[\frac{\partial^2u}{\partial t\partial x}]_j^n-\frac{9}{2}rh^2[\frac{\partial^3u}{\partial t\partial x^2}]_j^n+\frac{5}{2}rh^3[\frac{\partial^4u}{\partial t\partial x^3}]_j^n+3r\tau^2h[\frac{\partial^4u}{\partial t^3\partial x}]_j^n+O(\tau+h^4), \quad (19)$$

$$L_h^{(7)}u_j^n-[Lu]_j^n=-6rh[\frac{\partial^2u}{\partial t\partial x}]_j^n+7rh^3[\frac{\partial^4u}{\partial t\partial x^3}]_j^n-6r\tau^2h[\frac{\partial^4u}{\partial t^3\partial x}]_j^n-3r\tau h[\frac{\partial^3u}{\partial t^2\partial x}]_j^n+O(\tau+h^4), \quad (20)$$

$$L_h^{(8)}u_j^n-[Lu]_j^n=6rh[\frac{\partial^2u}{\partial t\partial x}]_j^n-7rh^3[\frac{\partial^4u}{\partial t\partial x^3}]_j^n+6r\tau^2h[\frac{\partial^4u}{\partial t^3\partial x}]_j^n+3r\tau h[\frac{\partial^3u}{\partial t^2\partial x}]_j^n+O(\tau+h^4), \quad (21)$$

$$L_h^{(9)}u_j^n-[Lu]_j^n=6rh[\frac{\partial^2u}{\partial t\partial x}]_j^n-7rh^3[\frac{\partial^4u}{\partial t\partial x^3}]_j^n+6r\tau^2h[\frac{\partial^4u}{\partial t^3\partial x}]_j^n+3r\tau h[\frac{\partial^3u}{\partial t^2\partial x}]_j^n+O(\tau+h^4), \quad (22)$$

$$L_h^{(10)}u_j^n-[Lu]_j^n=-6rh[\frac{\partial^2u}{\partial t\partial x}]_j^n+7rh^3[\frac{\partial^4u}{\partial t\partial x^3}]_j^n-6r\tau^2h[\frac{\partial^4u}{\partial t^3\partial x}]_j^n-3r\tau h[\frac{\partial^3u}{\partial t^2\partial x}]_j^n+O(\tau+h^4), \quad (23)$$

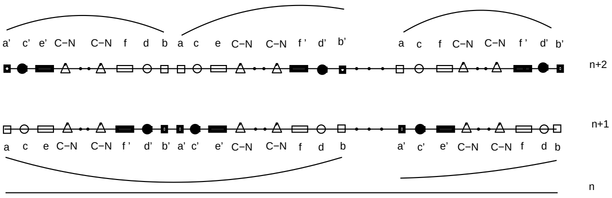

Fig. 1. The Diagram of the Alternating Difference Parallel Method

$$L_h^{(6)}u_j^n-[Lu]_j^n=3rh[\frac{\partial^2 u}{\partial t\partial x}]_j^n-\frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]_j^n$$

$$+\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^n+3r\tau^2h[\frac{\partial^4 u}{\partial t^3\partial x}]_j^n+O(\tau+h^4), \qquad (24)$$

$$L_h^{(6)}u_j^n-[Lu]_j^n=-3rh[\frac{\partial^2 u}{\partial t\partial x}]_j^n-\frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]_j^n$$

$$-\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^n-3r\tau^2h[\frac{\partial^4 u}{\partial t^3\partial x}]_j^n+O(\tau+h^4), \qquad (25)$$

$$L_h^{(6)}u_j^n-[Lu]_j^n=-3rh[\frac{\partial^2 u}{\partial t\partial x}]_j^n+\frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]_j^n$$

$$-\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^n-3r\tau^2h[\frac{\partial^4 u}{\partial t^3\partial x}]_j^n+O(\tau+h^4), \qquad (26)$$

$$L_h^{(6)}u_j^n-[Lu]_j^n=3rh[\frac{\partial^2 u}{\partial t\partial x}]_j^n+\frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]_j^n$$

$$+\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^n+3r\tau^2h[\frac{\partial^4 u}{\partial t^3\partial x}]_j^n+O(\tau+h^4), \qquad (27)$$

B. The Alternating Difference Parallel Method

The new highly accurate alternating parallel difference method is constructed as follow. Assuming $J-1=12+2l$, $l\geq 1$ is a positive integer. we consider the model of the segment at the $(n+1)$st and the $(n+2)$nd time levels, where n is an even number. We divide the nodes of the $(n+1)$st time level into k segments, each segment contains $12+2l$ nodes in x direction. Based on the alternating technique, we divide the nodes of the $(n+2)$nd time level into $k+1$ segments, the first and the $(k+1)$st segments contain $6+l$ nodes in x direction. The other segments contain $12+2l$ nodes in x direction. Let C-N represents difference scheme (2) and (a,b a',b',c,d,c',d',e,f,e',f') represent difference schemes $(3-14)$, respectively. The nodes in every segment can be computed by the asymmetric difference schemes according to the rule (a,c,e,C-N,f',d',b',a',c',e',C-N,f,d,b) displayed in Figure 1.

The highly accurate alternating parallel difference method can be expressed as

$$(I+rG_1)U^{n+1}=(I-rG_2)U^n, \qquad (28)$$

$$(I+rG_2)U^{n+2}=(I-rG_1)U^{n+1}, \qquad (29)$$

$$n=0,2,4,6,\cdots.$$

where $U^n=(u_1^n,u_2^n,\cdots,u_{J-1}^n)^T$, and the matrices G_1 and G_2 are given by

$$G_1=\begin{pmatrix} Q & & & & \\ & Q & & & \\ & & \ddots & & \\ & & & Q & \\ & & & & Q \end{pmatrix},$$

$$G_2=\begin{pmatrix} P^r & & & & P \\ & Q & & & \\ & & \ddots & & \\ & & & Q & \\ P^T & & & & P^l \end{pmatrix}, Q=\begin{pmatrix} P^l & P^T \\ P & P^r \end{pmatrix},$$

where Q is $(12+2l)\times(12+2l)$, P^r, P^l are $(6+l)\times(6+l)$, and Q, P, P^r, P^l are given by

$$P=\begin{pmatrix} & & & -2 & 24 & -78 \\ & & & & -2 & 24 \\ & & & & & -2 \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix},$$

$$P^l=\begin{pmatrix} 1 & -6 & 6 & -1 & & & & \\ -6 & 28 & -33 & 12 & -1 & & & \\ 6 & -33 & 55 & -39 & 12 & -1 & & \\ -1 & 12 & -39 & 56 & -39 & 12 & -1 & \\ & & & \ddots & \ddots & \ddots & & \\ & & -1 & 12 & -39 & 57 & -45 & 18 \\ & & & -1 & 12 & -45 & 84 & -72 \\ & & & & -1 & 18 & -72 & 111 \end{pmatrix}$$

$$P^r=\begin{pmatrix} 111 & -72 & 18 & -1 & & & & \\ -72 & 84 & -45 & 12 & -1 & & & \\ 18 & -45 & 57 & -39 & 12 & -1 & & \\ -1 & 12 & -39 & 56 & -39 & 12 & -1 & \\ & & & \ddots & \ddots & \ddots & & \\ & & -1 & 12 & -39 & 55 & -33 & 6 \\ & & & -1 & 12 & -33 & 28 & -6 \\ & & & & -1 & 6 & -6 & 1 \end{pmatrix}$$

III. The Existence and Uniqueness of the Solution

To prove the the existence, uniqueness and the stability, we have to introduce the following Lemmas [31].

Lemma 1 *If $\rho>0$, $C+C^T$ is nonnegative definite, then $(I+\rho C)^{-1}$ exists and there holds*

$$||(I+\rho C)^{-1}||_2\leq 1.$$

Lemma 2 *Under the conditions of Lemma 1, the following inequality holds*

$$||(I-\rho C)(I+\rho C)^{-1}||_2\leq 1.$$

Lemma 3 *For any real number r, and the symmetric non-negative matrices G_1 and G_2, matrices rG_1 and rG_2 in the* *alternating difference method* (28)–(29) *are both symmetric and non-negative definite.*

From the initial conditions and the boundary conditions of the equation, we know the initial solution of the first time layer. Assuming the value U^{2n} is known, the value U^{2n+1} can be computed by

$$(I + rG_1)U^{2n+1} = (I - rG_2)U^{2n}$$

By Lemma 3 and Lemma 1, $(I + rG_1)^{-1}$ exists, then the above equation has a unique solution. In the same way, we can compute the value U^{2n+2} by the equation

$$(I + rG_2)U^{2n+2} = (I - rG_1)U^{2n+1}$$

which has a unique solution.

Theorem 1 *The solution of the alternating difference method* (28)–(29) *exists and is unique.*

IV. The Truncation Errors and the Stability

A. The Analysis of the Truncation Errors

Let us give out the error analysis for the new alternating parallel difference method. In order to give the truncation errors analysis, for schemes $(2) - (14)$, we give the Taylor series expansions $(30) - (42)$ at (x_j, t^{n+1}), respectively.

$$L_h^{(2)}u_j^n - [Lu]_j^{n+1} = 6rh^4[\frac{\partial^5 u}{\partial t\partial x^4}]_j^{n+1} + O(\tau^2 + h^4), \quad (30)$$

$$L_h^{(3)}u_j^n - [Lu]_j^{n+1} = 3rh[\frac{\partial^2 u}{\partial t\partial x}]_j^{n+1} + \frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]_j^{n+1}$$

$$+\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^{n+1} + 3r\tau^2h[\frac{\partial^4 u}{\partial t^3\partial x}]_j^{n+1} + O(\tau + h^4), \quad (31)$$

$$L_h^{(6)}u_j^n - [Lu]_j^{n+1} = -3rh[\frac{\partial^2 u}{\partial t\partial x}]_j^{n+1} + \frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]_j^{n+1}$$

$$-\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^{n+1} - 3r\tau^2h[\frac{\partial^4 u}{\partial t^3\partial x}]_j^{n+1} + O(\tau + h^4), \quad (32)$$

$$L_h^{(7)}u_j^n - [Lu]_j^{n+1} = -3rh[\frac{\partial^2 u}{\partial t\partial x}]_j^{n+1} - \frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]_j^{n+1}$$

$$-\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^{n+1} - 3r\tau^2h[\frac{\partial^4 u}{\partial t^3\partial x}]_j^{n} + O(\tau + h^4), \quad (33)$$

$$L_h^{(8)}u_j^n - [Lu]_j^{n+1} = 3rh[\frac{\partial^2 u}{\partial t\partial x}]_j^{n+1} - \frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]_j^{n+1}$$

$$+\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^{n+1} + 3r\tau^2h[\frac{\partial^4 u}{\partial t^3\partial x}]_j^{n+1} + O(\tau + h^4), \quad (34)$$

$$L_h^{(9)}u_j^n - [Lu]_j^{n+1} = -6rh[\frac{\partial^2 u}{\partial t\partial x}]_j^{n+1} + 7rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^{n+1}$$

$$-6r\tau^2h[\frac{\partial^4 u}{\partial t^3\partial x}]_j^{n+1} + O(\tau + h^4), \quad (35)$$

$$L_h^{(10)}u_j^n - [Lu]_j^{n+1} = 6rh[\frac{\partial^2 u}{\partial t\partial x}]_j^{n+1} - 7rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^{n+1}$$

$$+6r\tau^2h[\frac{\partial^4 u}{\partial t^3\partial x}]_j^{n+1} + O(\tau + h^4), \quad (36)$$

$$L_h^{(11)}u_j^n - [Lu]_j^{n+1} = 6rh[\frac{\partial^2 u}{\partial t\partial x}]_j^{n+1} - 7rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^{n+1}$$

$$+6r\tau^2h[\frac{\partial^4 u}{\partial t^3\partial x}]_j^{n+1} + O(\tau + h^4), \quad (37)$$

$$L_h^{(12)}u_j^n - [Lu]_j^{n+1} = -6rh[\frac{\partial^2 u}{\partial t\partial x}]_j^{n+1} + 7rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^{n+1}$$

$$-6r\tau^2h[\frac{\partial^4 u}{\partial t^3\partial x}]_j^{n+1} + O(\tau + h^4), \quad (38)$$

$$L_h^{(13)}u_j^n - [Lu]_j^{n+1} = 3rh[\frac{\partial^2 u}{\partial t\partial x}]_j^{n+1} - \frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]_j^{n+1}$$

$$+\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^{n+1} + 3r\tau^2h[\frac{\partial^4 u}{\partial t^3\partial x}]_j^{n+1} + O(\tau + h^4), \quad (39)$$

$$L_h^{(14)}u_j^n - [Lu]_j^{n+1} = -3rh[\frac{\partial^2 u}{\partial t\partial x}]_j^{n+1} + \frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]_j^{n+1}$$

$$-\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^{n+1} - 3r\tau^2h[\frac{\partial^4 u}{\partial t^3\partial x}]_j^{n+1} + O(\tau + h^4), \quad (40)$$

$$L_h^{(15)}u_j^n - [Lu]_j^{n+1} = -3rh[\frac{\partial^2 u}{\partial t\partial x}]_j^{n+1} + \frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]_j^{n+1}$$

$$-\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^{n+1} - 3r\tau^2h[\frac{\partial^4 u}{\partial t^3\partial x}]_j^{n+1} + O(\tau + h^4), \quad (41)$$

$$L_h^{(16)}u_j^n - [Lu]_j^{n+1} = 3rh[\frac{\partial^2 u}{\partial t\partial x}]_j^{n+1} - \frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]_j^{n+1}$$

$$+\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^{n+1} + 3r\tau^2h[\frac{\partial^4 u}{\partial t^3\partial x}]_j^{n+1} + O(\tau + h^4), \quad (42)$$

In this method, there are seven pairs of schemes (2) with (2), (3) with (5), (4) with (6), (7) with (9), (8) with (10), (11) with (13), (12) with (14) which are alternatingly used between two adjacent times levels. There are six pairs of schemes (3) with (5), (4) with (6), (7) with (9), (8) with (10), (11) with (13), (12) with (14) which are symmetrically used in every independent segment at the same time level.

For the symmetrical C-N schemes (2) with (2), from the Taylor series expansions (15) at (x_j, t^n) and (30) at (x_j, t^{n+1}), we can see its truncation error's leading parts' signs are opposite and can be canceled out. So we can obtain that its truncation error is of order $O(\tau^2 + h^4)$.

For the antisymmetrical schemes (3) with (5), (4) with (6), (7) with (9), (8) with (10), (11) with (13), (12) with (14). By comparing the error results (16) with (33), (17) with (34), (18) with (31), (19) with (32), (20) with (37), (21) with (38), (22) with (35), (23) with (36), (24) with (41), (25) with (42), (26) with (39), (27) with (40), we find that the first four terms have opposite signs, it is obvious that they can be canceled out. The truncation errors at these points are of the order $O(h^4)$ in space.

B. The Analysis of the Stability

Theorem 2 *For any real number r, the parallel alternating difference method (28)–(29) is unconditionally stable.*

Proof. By eliminating U^{n+1} from (28)–(29), we obtain $U^{n+2} = GU^n$. where G is the growth matrix

$$G = (I + rG_2)^{-1}(I - rG_1)(I + rG_1)^{-1}(I - rG_2).$$

For any even number n, there holds

$$G^n = (I + rG_2)^{-1}(I - rG_1)(I + rG_1)^{-1} \cdot [(I - rG_2) (I + rG_2)^{-1}(I - rG_1)(I + rG_1)^{-1}]^{n-1}(I - rG_2).$$

Since G_1 and G_2 are all symmetric, for any real number r, we can obtain the following inequality from the Kellogg Lemma

$$||G^n||_2 \le ||(I + rG_2)^{-1}||_2 \cdot ||(I - rG_1)(I + rG_1)^{-1}||_2^n \cdot ||(I - rG_2)(I + rG_2)^{-1}||_2^{n-1} \cdot ||(I - rG_2)||_2.$$

Hence

$$||G^n||_2 \le ||(I - rG_2)||_2 \le ||(I - rG_2)||_\infty \cdot ||(I - rG_2)||_1 \le 1 + 160|r|.$$

This shows that the alternating parallel difference method (28)-(29) is unconditionally stable.

V. Numerical Experiments

In this section, we perform numerical experiments for (1) using the following model problem

$$f(x) = \sin x, \ \alpha = 1, \ l = \pi.$$

$$g_1(t) = g_2(t) = g_3(t) = g_4(t) = 0.$$

The exact solution of this problem is

$$u(x,t) = e^{-t}\sin x.$$

The discrete initial-boundary value conditions are

$$\begin{aligned} &U_j^0 = \sin(x_j), j = 0, 1, 2, \cdots, J \\ &U_0^n = U_{-1}^n + U_1^n = U_{-2}^n + U_2^n = 0, \\ &U_J^n = U_{J-1}^n + U_{J+1}^n = U_{J-2}^n + U_{J+2}^n = 0, \\ &n = 0, 1, 2, \cdots, N. \end{aligned}$$

We first illustrate the convergence rates in space for the new alternating difference parallel method. Let $v_j^n = u(x_j, t^n)$ be the exact solution of the problem (1) and u_j^n be the approximate solution. We introduce the following $L_\infty-$norm error and L_2-norm error

$$E_{\infty,h} = \max_j |v_j^n - u_j^n|, E_{2,h} = \big(\sum_j |v_j^n - u_j^n|^2 h\big)^{\frac{1}{2}}.$$

Thus, we can calculate the rates of convergence by the following definitions

$$rate = \frac{\log(E_{\infty,h_1}/E_{\infty,h_2})}{\log(h_1/h_2)}, rate = \frac{\log(E_{2,h_1}/E_{2,h_2})}{\log(h_1/h_2)}.$$

where h_1 and h_2 are the space mesh steps. Let 'NASC-N' represents the new alternating parallel difference method described above, 'ASC-N' represents the alternating segment Crank-Nicolson scheme in [13], and 'Exact' represents the values of the exact solution $u(x_j, t^n)$. For the NASC-N method, we give the $L_\infty-$norm errors, L_2-norm errors and the convergence rates in Tables I–II. We can see from these tables that the convergence rate of NASC-N method appears to be O(h^4) in space, which is coincident with our theoretical analysis, while the ASC-N method in [13] appears to be O(h^2) in space. In addition, although the boundary schemes could reduce the accuracy of the NASC-N method, it does not affect the convergence rate O(h^4) in space from Table I and Table II.

Next, we compare the errors for the NASC-N method with the ASC-N method at the same time t in Tables III–IV, respectively, where the absolute error $ae = |u_j^n - u(x_j, t^n)|$, the relative error $re = \dfrac{|u_j^n - u(x_j, t^n)|}{|u(x_j, t^n)|} \times \%$. The results show that the NASC-N method is more accurate than the ASC-N scheme in [13]. In addition, from Fig. 2–5, we can see clearly that the NASC-N method solutions are more accurate than the ASC-N method solutions.

Third, we verify the stability of the NASC-N method. From Tables V and VI, we can easily find that the NASC-N method is unconditionally stable

Finally, let's give a brief discussion on parallelism of the NASC-N mathod. When we compute the interface values by

TABLE I
THE CONVERGENCE RATE OF THE NASC-N SCHEME, $t = 1, \tau = 1 \times 10^{-8}$

h	$\pi/25$	$\pi/49$	$\pi/73$	$\pi/97$
L_∞	2.663E-6	1.871E-7	3.887E-8	1.373E-8
Rate	–	3.946	3.942	3.861
L_2	3.344E-6	2.346E-7	4.873E-8	1.721E-8
Rate	–	3.949	3.942	3.862

TABLE II
THE CONVERGENCE RATE OF THE NASC-N SCHEME $t = 0.01, \tau = 1 \times 10^{-8}$

h	$\pi/25$	$\pi/49$	$\pi/73$	$\pi/97$
L_∞	1.109E-8	9.693E-9	2.139E-9	7.034E-10
Rate	–	3.947	3.952	3.897
L_2	8.999E-8	6.308E-9	1.302E-9	4.655E-10
Rate	–	3.949	3.958	3.918

TABLE III
THE ERRORS OF NUMERICAL SOLUTION $J = 120, \tau = 1 \times 10^{-7}, t = 1$

scheme	error	j=20	j=40	j=60	j=80	j=100
NASC-N	$ae(10^{-7})$	3.640	6.332	7.350	6.426	3.804
	$re(10^{-6})$	1.994	1.997	1.998	1.997	1.994
ASC-N	$ae(10^{-5})$	2.051	3.561	4.133	3.615	2.144
	$re(10^{-4})$	1.124	1.124	1.124	1.1235	1.124
Exact	(10^{-1})	1.826	3.169	3.678	3.214	1.908

TABLE IV
THE ERRORS OF NUMERICAL SOLUTION $J = 240, \tau = 1 \times 10^{-8}, t = 0.01$

scheme	error	j=40	j=80	j=120	j=160	j=200
NASC-N	$ae(10^{-9})$	5.823	9.783	9.983	9.651	5.957
	$re(10^{-8})$	1.181	1.226	1.237	1.226	1.181
ASC-N	$ae(10^{-7})$	1.396	2.420	2.802	2.439	1.427
	$re(10^{-7})$	2.829	2.829	2.829	2.829	2.830
Exact	(10^{-1})	4.932	8.553	9.900	8.617	5.043

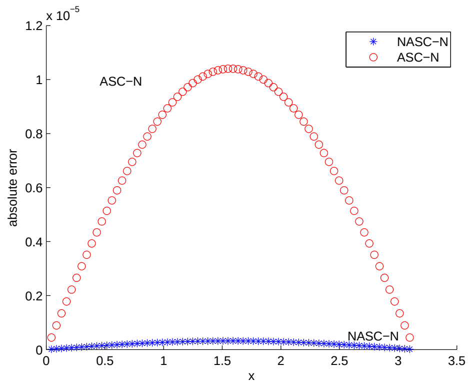

Fig. 2. Comparison of the Absolute Error $h = \pi/73, \tau = 10^{-6}, t = 5$

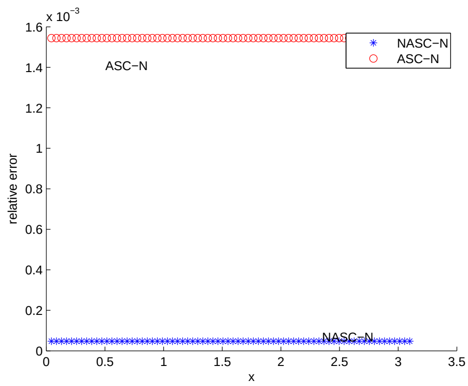

Fig. 3. Comparison of the Relative Error $h = \pi/73, \tau = 10^{-6}, t = 5$

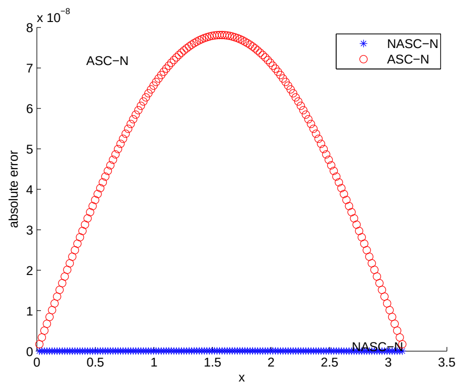

Fig. 4. Comparison of the Absolute Error $h = \pi/145, \tau = 10^{-8}, t = 0.001$

TABLE V
THE ERRORS OF NUMERICAL SOLUTION $J = 48, \tau = 10^{-7}, t = 1$

r	error	j=9	j=18	j=27	j=36	j=45
$r_1 = r$	$ae(10^{-7})$	1.050	1.668	1.757	1.318	1.001
	$re(10^{-7})$	4.839	4.838	4.838	4,838	4.837
$r_2 = 10r$	$ae(10^{-7})$	1.008	1.278	1.374	1.024	1.001
	$re(10^{-7})$	3.717	3.798	3.783	3.762	3.857
$r_3 = 100r$	$ae(10^{-5})$	1.725	2.918	3.145	2.353	1.015
	$re(10^{-5})$	8.593	8.675	8.659	8.638	8.734

the asymmetric difference schemes $(3) - (14)$, the global domain of definition is divided into some small independent segments, and can be computed in parallel, the parallelism is clarity.

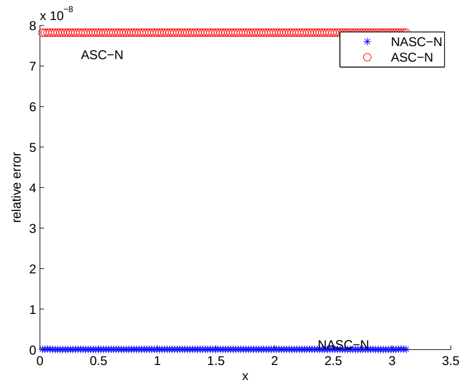

Fig. 5. Comparison of the Relative Error $h = \pi/145, \tau = 10^{-8}, t = 0.001$

TABLE VI
THE ERRORS OF NUMERICAL SOLUTION $J = 144, \tau = 10^{-9}, t = 0.001$

r	error	j=30	j=60	j=90	j=120	j=140
$r_1 = r$	$ae(10^{-11})$	3.400	5.302	5.100	2.781	3.509
	$re(10^{-11})$	5.624	5.508	5.496	5.399	3.248
$r_2 = 10r$	$ae(10^{-11})$	3.428	5.203	5.057	2.653	5.073
	$re(10^{-11})$	5.670	5.406	5.450	5.152	4.697
$r_3 = 100r$	$ae(10^{-9})$	3.791	5.779	5.616	2.963	5.752
	$re(10^{-9})$	6.269	6.003	6.052	5.753	5.325

VI. CONCLUSION

In this paper, we first constructed a group of new asymmetric schemes and Crank-Nicolson scheme, basing on the idea of the alternating schemes, we designed the highly accurate alternating parallel difference method for the fourth-order heat equation. The theoretics analysis and the numerical simulations show that this new alternating parallel difference method constructed in the paper has obvious parallelism, unconditional stability and fourth-order accuracy, which is more accurate than the existing methods in [13]. We hope the result of this paper makes an essential contribution in this direction.

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