A New Marshall Olkin Weibull Distribution

Wei Cui, Zaizai Yan, and Xiuyun Peng

***Abstract*—An alternative lifetime model called the new Marshall Olkin Weibull (NMOW) distribution is proposed and studied. The importance of the new distribution comes from its ability to model some life distributions with increasing, decreasing, bathtub, and unimodal shapes of hazard rate functions, which are quite common in reliability study. We obtain the origin moments, generating function, conditional moment, conditional moment generating function, mean residual life function, mean deviations, order statistics and Rényi entropies. The maximum likelihood method is used to estimate parameters of the model. Two real data sets are used to illustrate the applicability of this new model.**

***Index Terms*—Marshall-Olkin distribution; Weibull distribution; Maximum likelihood estimation.**

I. Introduction

WEIBULL distribution is very popular model that is commonly used for analyzing biological, medical and hydrological data sets. However, it does not provide an acceptable fit for some applications, especially, when the hazard rates are bathtub, upside down bathtub, or bimodal shapes. To overcome these drawbacks, several authors have generalized and extended the Weibull distribution to model various types of data. Mudholkare et al. [1] studied an exponentiated Weibull (EW) distribution by adding an shape parameter to the Weibull distribution to allowing bathtub-shaped hazard rate function. Aryal and Tsokos [2] added one parameter to the Weibull distribution using the quadratic rank transmutation map. Carrasco et al. [3] defined a four parameters generalization of the Weibull distribution which has bathtub-shaped hazard rate function. Cordeiro et al. [4] proposed a five-parameter lifetime model called by McDonald Weibull distribution, it contains the Weibull, exponentiated Weibull, beta Weibull and Kumaraswamy Weibull distribution as the special cases. Gauss et al. [5] proposed a new distribution which called by beta-Weibull geometric distribution, whose failure rate function can be decreasing, increasing or an upside down bathtub shape. Peng and Yan [6] introduced a new extended Weibull distribution with one scale parameter and two shape parameters which has increasing and upside-down bathtub shaped hazard rate functions. Carrasco et al. [7] studied a generalized modified Weibull distribution (MW). Pogány and Saboor [8] introduced a new four-parameter model called gamma-exponentiated-Weibull distribution. The exponential Weibull lifetime distribution [9], Kumaraswamy Weibull distribution [10] and a five-parameter extension of the Weibull distribution (APW) [11] had been studied by Cordeiro et al. Nassar et al. [12] introduced a new family of generalized distribution based on Alpha power transformation and extended Weibull distribution. Marshall and Olkin [13] introduced a new distribution family by adding a parameter to a initial distribution. Let the survival function of the initial distribution is given by $\bar{U}(x) = 1 - U(x)$, where $U(\cdot)$ is the cumulative distribution function (CDF) of the initial distribution, and $u(\cdot)$ is the corresponding probability density function (PDF). The survival function of Mashall-Olkin extend distribution is defined by

$$\bar{V}(x) = \frac{\theta \bar{U}(x)}{1-(1-\theta)\bar{U}(x)} \tag{1}$$

The corresponding CDF and PDF are given respectively

$$V(x) = \frac{U(x)}{1-(1-\theta)\bar{U}(x)} \tag{2}$$

$$v(x) = \frac{\theta u(x)}{[1-(1-\theta)(1-U(x))]^2} \tag{3}$$

Marshall-Olkin method has been used to obtain new distributions by researchers. For example, Marshall and Olkin [13] introduced the Marshall-Olkin Weibull distribution. Benkhelifa [14] studied a new three-parameter model called the Marshall-Olkin extended generalized Lindley distribution. Mirmostafaee et al. [15] introduced the Marshall-Olkin extended generalized Rayleigh distribution. In 2016, Sanoor and Pogany [16] introduced a Marshall-Olkin variant of the provost type gamma-Weibull probability distribution .

We introduce a new model by inverting the equation (2), it is named as New Marshall-Olkin Weibull distribution (NMOW) in later sections. The NMOW has several desirable properties and more flexible hazard and density functions. The rest of this paper is organized as follows. In section II, we introduce the NMOW model. In section III, we study the properties including quantile function, moments, moment generating function, conditional moment and conditional moment generating function, mean residual life function, mean deviations, order statistics, and Rényi entropies. In section IV, we discuss the maximum likelihood estimates (MLEs) of model parameters. In section V, two real data sets are analyzed to illustrate the potentiality of the new model. The paper is concluded in section VI.

II. NMOW distribution

Let $g(x)$ and $G(x)$, respectively, are the PDF and CDF of a continuous random variable X. Then the CDF $F(x)$ of the new proposed distribution is obtained by inverting the equation (2) as follows:

$$F(x) = \frac{\theta G(x)}{1+(\theta-1)G(x)} \tag{4}$$

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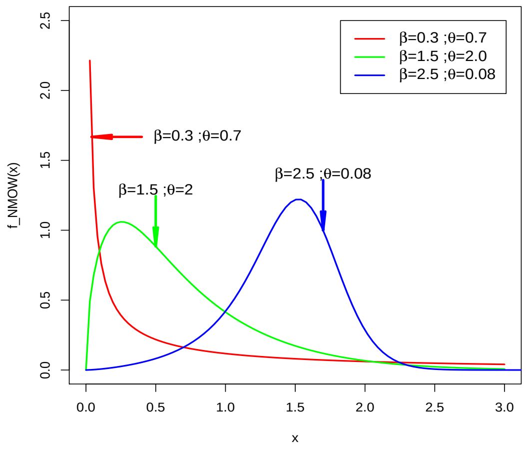

Fig. 1. Density functions of NMOW model for some values of β and θ.

$$f(x) = \frac{\theta g(x)}{[1 + (\theta - 1)G(x)]^2} \tag{5}$$

Note that when $\theta \longrightarrow 1$, $f(x)$ reduced to $g(x)$. From (4) and (5), the reliability function and hazard rate function are given by

$$S(x) = \frac{1 - G(x)}{1 + (\theta - 1)G(x)} \tag{6}$$

$$h(x) = \frac{\theta g(x)}{[1 + (\theta - 1)G(x)][1 - G(x)]} \tag{7}$$

In this paper, let $\phi = (\theta, \lambda, \beta)^T$, X follows the Weibull distribution with CDF $G(x) = 1 - \exp\{-\lambda x^{\beta}\}, x > 0$. Based on (4) and (5), the CDF and PDF of the NMOW distribution are defined as

$$F_{NMOW}(x;\phi) = \frac{\theta\Big(1 - \exp\{-\lambda x^{\beta}\}\Big)}{\theta + (1-\theta)\exp\{-\lambda x^{\beta}\}} \tag{8}$$

$$f_{NMOW}(x;\phi) = \frac{\theta\lambda\beta x^{\beta-1}\exp\{-\lambda x^{\beta}\}}{\Big(\theta + (1-\theta)\exp\{-\lambda x^{\beta}\}\Big)^2} \tag{9}$$

where $\beta, \theta > 0$ are the shape parameters and $\lambda > 0$ is the scale parameter. The survival funciton and the hazard rate function for $x > 0$, are respectively given by

$$S_{NMOW}(x;\phi) = \frac{\exp\{-\lambda x^{\beta}\}}{\theta + (1-\theta)\exp\{-\lambda x^{\beta}\}} \tag{10}$$

$$h_{NMOW}(x;\phi) = \frac{\theta\lambda\beta x^{\beta-1}}{\Big(\theta + (1-\theta)\exp\{-\lambda x^{\beta}\}\Big)} \tag{11}$$

Figures 1 and 2 are the curves of density and hazard rate functions of the NMOW model for some parameters β and θ, and $\lambda = 1$. Results from Figures 1 and 2 show that the PDF can be left-skewed or right-skewed, and the hazard rate function has increasing, decreasing, bathtub, and upside down bathtub shapes. Therefore, the PDF and hazard rate functions of the NMOW model are very flexible.

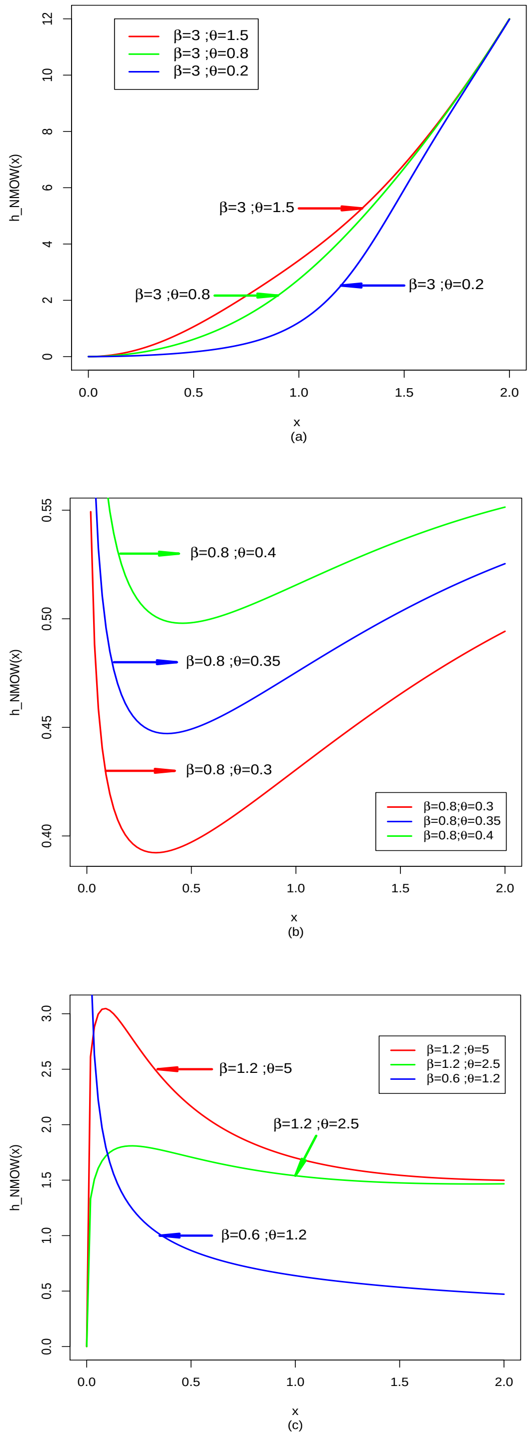

Fig. 2. Hazard rate functions (a), (b) and (c) of NMOW model for some values of β and θ.

III. Properties of the NMOW distribution

A. Quantile

Theorem 2.1. If a random variable X follows the NMOW distribution with parameters (θ, λ, β), then the pth quantile is given by

$$x_p = [-\frac{1}{\lambda}\log(\frac{\theta(p-1)}{p(\theta-1)-\theta})]^{\frac{1}{\beta}} \tag{12}$$

proof. It is obtained easily from the equation $F_{NMOW}(x_p) = p$.

B. Moment and moment generating function

In this subsection, we present rth moment and the moment generating function of the NMOW distribution. By using the following series representation,

$$(1-z)^{-\rho} = \sum_{j=0}^{\infty} \frac{\Gamma(\rho+j)}{\Gamma(\rho)j!} z^j, |z| < 1, \rho > 0$$

the denominator in (9) can be expressed as

$$\begin{aligned} & \Big(\theta - (\theta-1)\exp\big\{-\lambda x^{\beta}\big\}\Big)^{-2} \\ = \ & \frac{1}{\theta^2}\sum_{j=0}^{\infty}(j+1)(1-\frac{1}{\theta})^j \exp\big\{-\lambda j x^{\beta}\big\}, \end{aligned}$$

and substitutes above expression into (9), where $\Gamma(a) = \int_0^{\infty} x^{a-1}\mathrm{e}^{-x}\mathrm{d}x$. The equation (9) can be rewritten as follows

$$\begin{aligned} f_{NMOW}(x;\phi) &= \frac{\lambda\beta}{\theta}x^{\beta-1}\sum_{j=0}^{\infty}(1-\frac{1}{\theta})^j(j+1) \\ &\quad \times \exp\big\{-\lambda x^{\beta}(1+j)\big\}. \end{aligned}$$

Theorem 2.2. If X follows the NMOW distribution with parameters (θ, λ, β), then the rth moments are given by

$$\begin{aligned} E(X^r) &= \frac{1}{\theta}(\frac{1}{\lambda})^{\frac{r}{\beta}}\sum_{j=0}^{\infty}(1-\frac{1}{\theta})^j\Big[\frac{1}{(1+j)}\Big]^{\frac{r}{\beta}}\Gamma(\frac{r}{\beta}+1) \\ &\quad \times\Gamma(\frac{r}{\beta}+1),\ r=1,2,\cdots \end{aligned} \tag{13}$$

Proof. From the definition of moments, we get

$$\begin{aligned} E(X^r) &= \int_0^{\infty} x^r f_{NMOW}(x;\phi)\mathrm{d}x \\ &= \frac{\lambda\beta}{\theta}\sum_{j=0}^{\infty}(1-\frac{1}{\theta})^j(j+1)\int_0^{\infty} x^{\beta+r-1} \\ &\quad \times \exp\big\{-\lambda x^{\beta}(1+j)\big\}\mathrm{d}x, \end{aligned}$$

let $\lambda(1+j)x^{\beta} = y$ in above equation, we can get (13).

Theorem 2.3. If X follows the NMOW distribution with parameters (θ, λ, β), then the moment generating function is given by

$$\begin{aligned} M_X(t) &= \frac{\lambda}{\theta}\sum_{j=0}^{\infty}(1-\frac{1}{\theta})^j(j+1) \\ &\quad \times\sum_{k=0}^{\infty}\frac{t^k}{k!}\Big[\frac{1}{\lambda(1+j)}\Big]^{1+\frac{k}{\beta}}\Gamma(\frac{k}{\beta}+1) \end{aligned} \tag{14}$$

Proof. The moment generating function is defined as

$$M_X(t) = \int_0^{\infty} \mathrm{e}^{tx} f_{NMOW}(x;\phi)\mathrm{d}x,$$

by using $\mathrm{e}^t = \sum_{k=0}^{\infty}\frac{t^k}{k!}$ we obtain

$$\begin{aligned} M_X(t) &= \frac{\lambda\beta}{\theta}\sum_{j=0}^{\infty}(1-\frac{1}{\theta})^j(j+1)\sum_{k=0}^{\infty}\frac{t^k}{k!} \\ &\quad \times\int_0^{\infty} x^{\beta+k-1}\exp\big\{-\lambda x^{\beta}(1+j)\big\}\mathrm{d}x, \end{aligned}$$

let $\lambda(1+j)x^{\beta} = y$, we can get (14).

Theorems 2.1 and 2.2 are used to obtain the median, mean, variance, skewness and kurtosis of the NMOW distribution.

TABLE I
MEDIAN, MEAN, VARIANCE, SKEWNESS, AND KURTOSIS OF THE NMOW DISTRIBUTION FOR SOME VALUES OF β, θ AND $\lambda = 1$

β	θ	median	mean	variance	skewness	kurtosis
0.8	0.5	1.1247	1.6567	3.0194	0.4201	11.8139
	0.8	0.7695	1.2865	2.3304	0.7304	17.5059
	1.2	0.5348	1.0179	1.8204	1.2244	26.1135
	1.8	0.3602	0.7935	1.3921	2.1282	41.1612
1.2	0.5	1.0815	1.2526	0.7955	1.6074	34.9602
	0.8	0.8398	1.0355	0.6768	2.4933	41.1525
	1.2	0.6589	0.8672	0.5737	3.7667	50.3666
	1.8	0.5603	0.7174	0.4753	5.8616	65.2156
2	0.5	1.0481	1.0810	0.2178	6.2607	197.6212
	0.8	0.9005	0.9448	0.2229	7.5531	156.7927
	1.2	0.7785	0.8934	0.2069	7.5531	156.7927
	1.8	0.6647	0.7394	0.1880	11.0312	165.9672

The median is obtained by setting $q = 0.5$ in Theorem 2.1. Then mean μ is obtained by setting $r = 1$ in Theorem 2.2. The variance σ^2, skewness γ_3 and kurtosis γ_4 are obtained using the formulas $\sigma^2 = E[X-\mu]^2$, $\gamma_3 = E[\frac{X-\mu}{\sigma}]^3$, $\gamma_4 = E[\frac{X-\mu}{\sigma}]^4$. These values are reported in Table 1 for various of θ and β, where the scale parameter $\lambda = 1$. From Table I, it is noted that for fixed θ and λ, the mean, moment of the NMOW are decreasing function of β, and the skewness is increasing function of β. Also, for fixed β and λ, the median, mean are decreasing function of θ, and the skewness is increasing function of θ.

C. Conditional moment and Conditional moment generating function

Theorem 2.4. If X follows the NMOW distribution with parameters (θ, λ, β), then the rth conditional moments and conditional moment generating function of a random variable X, are given by

$$\begin{aligned} E(x^r|T>t) &= \frac{\lambda}{\theta}\sum_{j=0}^{\infty}(1-\frac{1}{\theta})^j(j+1)\Big[\frac{1}{\lambda(1+j)}\Big]^{1+\frac{r}{\beta}} \times \\ &\quad \Gamma(\frac{r}{\beta}+1, \lambda t^{\beta}(1+j)), r=1,2,\cdots \end{aligned} \tag{15}$$

$$\begin{aligned} E(e^{tx}|T>t) &= \frac{\lambda}{\theta}\sum_{j=0}^{\infty}(1-\frac{1}{\theta})^j(j+1)\sum_{k=0}^{\infty}\Gamma(\frac{k}{\beta}+1, \lambda t^{\beta}) \\ &\quad \times\frac{t^k}{k!}(1+j)\Big[\frac{1}{\lambda(1+j)}\Big]^{1+\frac{k}{\beta}} \end{aligned} \tag{16}$$

where $\Gamma(a,x) = \int_x^{\infty} t^{a-1}\mathrm{e}^{-t}\mathrm{d}t$.

Proof. According to the definition of conditional moment and conditional moment generating functions

$$E(x^r|T>t) = \frac{1}{\bar{F}(t)}\int_t^{\infty} x^k f(x)\mathrm{d}x,$$

$$E(\mathrm{e}^{tx}|T>t) = \frac{1}{\bar{F}(t)}\int_t^{\infty} \mathrm{e}^{tx} f(x)\mathrm{d}x,$$

we can use the same method as above as to get the results easily.

D. Mean residual life

The mean residual life (MRL) function describes the aging process that is very important in reliability and survival analysis. The MRL function of a lifetime random variable X is given by

$$\mu(x) = \frac{1}{\bar{F}(x)} \int_x^\infty t f(t) \mathrm{d}t - x \tag{17}$$

Theorem 2.5. If X follows the NMOW distribution with parameters (θ, λ, β), the MRL function is given by

$$\mu(x) = \frac{1}{\bar{F}(x)} \frac{\lambda}{\theta} \sum_{j=0}^{\infty} (1 - \frac{1}{\theta})^j (j+1) \Big[\frac{1}{\lambda(1+j)}\Big]^{1+\frac{1}{\beta}} \times \Gamma(\frac{1}{\beta} + 1, \lambda t^\beta (1+j)) \tag{18}$$

E. Mean deviations

The mean deviation of X about the mean and about the median, measure of spread in a population, are given by

$$\delta_1 = E(|X - \mu|) = 2\mu F(\mu) - 2\int_0^\mu x f(x) \mathrm{d}x \tag{19}$$

and

$$\delta_2 = E(|X - M|) = \mu - M + 2MF(M) - 2\int_0^\mu x f(x) \mathrm{d}x \tag{20}$$

respectively, where the mean $\mu = E(X)$ and M denotes the median of X.

Theorem 2.6. If X follows the NMOW distribution with parameters (θ, λ, β), the Mean deviations are given by

$$\delta_1 = 2\mu F(\mu) - 2\frac{\lambda}{\theta} \sum_{j=0}^{\infty} (1 - \frac{1}{\theta})^j (j+1) \Big[\frac{1}{\lambda(1+j)}\Big]^{1+\frac{1}{\beta}} \times \gamma(\frac{1}{\beta} + 1, \lambda(1+j)\mu^\beta) \tag{21}$$

$$\delta_2 = \mu - M + 2MF(M) - 2\frac{1}{\theta}(\frac{1}{\lambda})^{\frac{1}{\beta}} \sum_{j=0}^{\infty} (1 - \frac{1}{\theta})^j \Big[\frac{1}{(1+j)}\Big]^{\frac{1}{\beta}} \times \gamma(\frac{1}{\beta} + 1, \lambda(1+j)\mu^\beta) \tag{22}$$

here $\gamma(a, x) = \int_0^x t^{a-1} e^{-t} \mathrm{d}t$.

F. Order statistics

Theorem 2.7. If a the random variable X follows the NMOW distribution with parameters (θ, λ, β), let $X_1, X_2, \cdots, X_n$ be its a random sample with size n, then, the PDF and CDF of the ith order statistics $X_{i:n}$ are give by

$$f_{i:n}(x) = \frac{\lambda\beta x^{\beta-1} exp(-\lambda x^\beta) n!}{(i-1)!(n-i)!} \sum_{l=0}^{n-i} \binom{n-i}{l} (-1)^l \times \frac{\theta^{i+l}[1 - exp(-\lambda x^\beta)]^{i+l-1}}{[\theta + (1-\theta) exp(-\lambda x^\beta)]^{i+l+1}} \tag{23}$$

$$F_{i:n}(x) = \sum_{j=i}^{n} \sum_{l=0}^{n-j} \binom{n}{j} \binom{n-j}{l} (-1)^l \theta^{j+l} (1-\theta)^{j+l} \times \frac{exp(-\lambda x^\beta (j+l))}{[\theta + (1-\theta) exp(-\lambda x^\beta)]^{i+l}} \tag{24}$$

G. Réyi entropy

The entropy of a random variable X measures the variation of the uncertainty. The Rényi entropy, say $RE_X(v)$, is defined as

$$RE_X(v) = \frac{1}{1-v} \log\Big(\int_0^\infty f(x)^v \mathrm{d}x\Big) \tag{25}$$

Theorem 2.8. If X follows the NMOW distribution with parameters (θ, λ, β), the Rényi entropy is given by

$$RE_X(v) = \frac{1}{1-v} \log\Big(a^{v-1} (\frac{b}{\theta})^v \sum_{j=0}^{\infty} \frac{\Gamma(-2v+j)}{\Gamma(-2v) j!} (1 - \frac{1}{\theta})^j \times (\frac{1}{b(v+j)})^{v - \frac{v}{a} + \frac{1}{a}} \Gamma(v - \frac{v}{a} + \frac{1}{a}) \Big) \tag{26}$$

IV. Maximum likelihood

Let $x = (x_1, \cdots, x_n)$ be a random sample form the NMOW distribution with parameters $\phi = (\theta, \lambda, \beta)$, then the log-likelihood function is given by

$$l(x|\phi) = n\big[\log(\theta) + \log(\beta) + \log(\lambda)\big] + (\beta - 1)\sum_{i=1}^{n} \log(x_i) - \lambda \sum_{i=1}^{n} x_i^\beta - 2\sum_{i=1}^{n} \log\Big[\theta + (1-\theta)\exp\big\{-\lambda x_i^\beta\big\}\Big] \tag{27}$$

The MLEs $(\hat{\theta}, \hat{\lambda}, \hat{\beta})$ are obtained by using the following likelihood equations:

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + 2\sum_{i=1}^{n} \frac{(1-\theta)\lambda\beta x_i^{\beta-1} \exp\big\{-\lambda x_i^\beta\big\}}{\Big[\theta + (1-\theta)\exp\big\{-\lambda x_i^\beta\big\}\Big]} = 0 \tag{28}$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i^\beta + 2\sum_{i=1}^{n} \frac{(1-\theta)\exp\big\{-\lambda x^\beta\big\} x_i^\beta}{\Big[\theta + (1-\theta)\exp\big\{-\lambda x_i^\beta\big\}\Big]} = 0 \tag{29}$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \log(x_i) - \lambda\beta \sum_{i=1}^{n} x_i^{\beta-1} + 2\sum_{i=1}^{n} \frac{(1-\theta)\exp\big\{-\lambda x^\beta\big\} \lambda\beta x_i^{\beta-1}}{\Big[\theta + (1-\theta)\exp\big\{-\lambda x_i^\beta\big\}\Big]} = 0 \tag{30}$$

However, the above equations are very complex and no analytical solutions. We use Newton-Raphson method to compute the MLEs $(\hat{\theta}, \hat{\lambda}, \hat{\beta})$. We can also get the estimation of the unknown parameters using Bayesian approach which can be seen ([17], [18], [19]).

V. Fitting reliability data

In this section we analysis real data to illustrate that the NMOW is a better lifetime model by comparing with many known distributions, such as Weibull (W), inverse Weibull (IW), exponential Weibull (EW) distributions[4], and their PDFs are expressed as follows.

The Weibull distribution with PDF

$$f(x; \beta, \lambda) = \lambda\beta x^{\beta-1} \exp\big\{-\lambda x^\beta\big\}, \quad x \geq 0, \ \lambda > 0, \ \beta > 0 \tag{31}$$

The IW distribution with PDF

$$f(x; \beta, \lambda) = \lambda\beta x^{-(\beta+1)} \exp\big\{-\lambda x^{-\beta}\big\}, \ x \geq 0, \lambda > 0, \beta > 0 \tag{32}$$

The EW distribution with PDF

$$f(x;\beta,\lambda,\theta) = \theta\lambda\beta x^{\beta-1}\exp\{-\lambda x^{\beta}\}, \quad x\geq 0,\ \ \lambda>0,\ \ \beta>0,\ \ \theta>0 \tag{33}$$

Application 1: The carbon fibers data set.

We shall consider the uncensored data set on the breaking stress of carbon fibers [16].The observations are shown as follows.

3.70	2.74	2.73	2.50	3.60	3.11	3.27	2.87	1.47	3.11	3.56
4.42	2.41	3.19	3.22	1.69	3.28	3.09	1.87	3.15	4.90	1.57
2.67	2.93	3.22	3.39	2.81	4.20	3.33	2.55	3.31	3.31	2.85
1.25	4.38	1.84	0.39	3.68	2.48	0.85	1.61	2.79	4.70	2.03
1.89	2.88	2.82	2.05	3.65	3.75	2.43	2.95	2.97	3.39	2.96
2.35	2.55	2.59	2.03	1.61	2.12	3.15	1.08	2.56	1.80	2.53
4.20										

Application 2: The cancer patients data set.

The second data set represents the remission times (in month) of random sample of 128 bladder cancer patients as reported in Lee and Wang [20]. The observations are shown as follows.

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63
0.20	2.23	3.52	4.98	6.97	9.02	13.29	0.40
2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50
2.46	3.64	5.09	7.26	9.47	14.24	25.82	0.51
2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81
2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64
3.88	5.32	7.39	10.34	14.83	34.26	0.90	2.69
4.18	5.34	7.59	10.66	15.96	36.66	1.05	2.69
4.23	5.41	7.62	10.75	16.62	43.01	1.19	2.75
4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33
5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62
7.87	11.64	17.36	1.40	3.02	4.34	5.71	7.93
11.79	18.10	1.46	4.40	5.85	8.26	11.98	19.13
1.76	3.25	4.50	6.25	8.37	12.02	2.02	3.31
4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76
12.07	21.73	2.07	3.36	6.93	8.65	12.63	22.69

For both data sets, we calculate the MLEs of the parameters of all the models and the Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), and Kolmogorov-Smirnov (K-S) distances and corresponding p-values, and they are listed in Tables II and III, respectively. We observe that the NMOW has the lowest AIC, BIC, CAIC and K-S values and largest p-values by comparison with other models. Therefore, the NMOW could be chosen as the better model. Figures 3 and 4 show the histogram for data and the density estimates of all models. From Figures 3 and 4 we can also found the NMOW is better fitted to the data. Figures 5 and 6 are the MLEs changing of reliability and hazard rate along with time of data. From Figures 5 and 6 we can see the hazard rate of carbon fibers is increasing which has inflection point and the hazard rate of cancer patients data is bathtub shaped.

VI. Conclusions

In this paper, we propose a new three-parameter model, called the New Marshall-Olkin Weibull distribution (NMOW) which has more forms of hazard rate functions. The NMOW distribution is motivated by inverting the Marshall-Olkin distribution family in terms of the Weibull distribution in order to provide more flexibility to analyze lifetime data. We derive explicit expressions for the quantile, moments, moment generating function, conditional moment

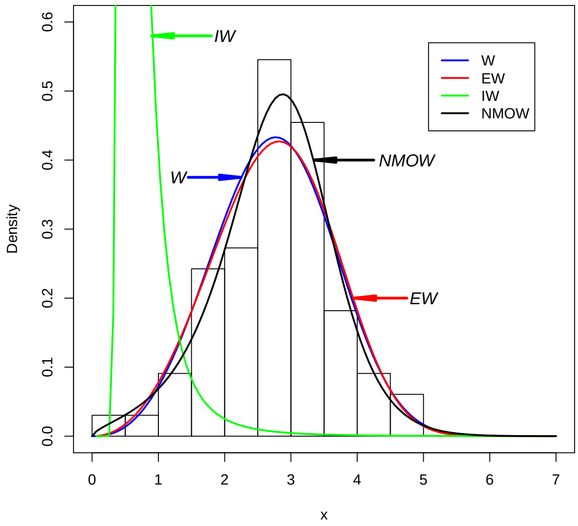

Fig. 3. The fitted PDFs of the W, EW, IW and NMOW for the carbon fibers data.

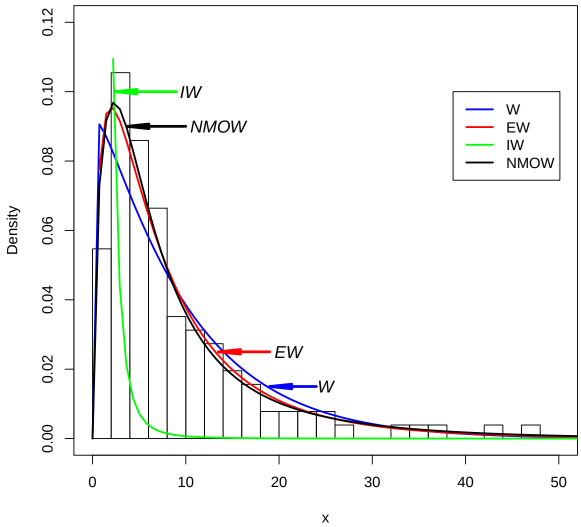

Fig. 4. The fitted PDFs of the W, EW, IW and NMOW for the cancer patients data.

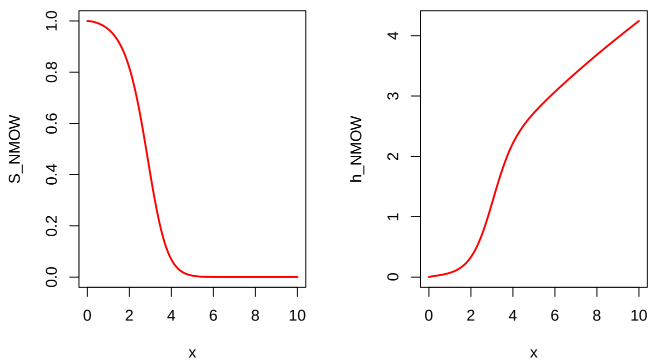

Fig. 5. MLEs of reliability and hazard rate functions based on data 1.

TABLE II
MLEs, AIC, BIC, CAIC, K-S, P-VALUE BASED ON THE FIRST DATA

Distribution	MLEs			AIC	BIC	CAIC	K-S	p-value
	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$					
W	3.4392	0.0213		176.1352	180.5145	176.3257	0.0826	0.7584
IW	3.2262	1.6480		246.3898	250.7691	246.5803	0.2303	0.0018
EW	3.8045	0.0120	0.8369	177.8999	184.4689	178.2870	0.0813	0.7746
NMOW	1.6339	0.6036	0.0406	175.4308	181.9998	175.8179	0.6122	0.9656

TABLE III
MLEs, AIC, BIC, CAIC, K-S, P-VALUE BASED ON THE SECOND DATA

Distribution	MLEs			AIC	BIC	CAIC	K-S	p-value
	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$					
W	1.0478	0.0939		832.1738	837.8778	832.2698	0.0700	0.5572
IW	2.4314	0.7521		892.0015	897.7056	892.0975	0.1408	0.0125
EW	0.6544	0.4538	2.7966	827.3602	835.9163	827.5538	0.0450	0.9577
NMOW	1.5081	0.001	8.3819	826.5759	835.1320	826.7695	0.0380	0.9926

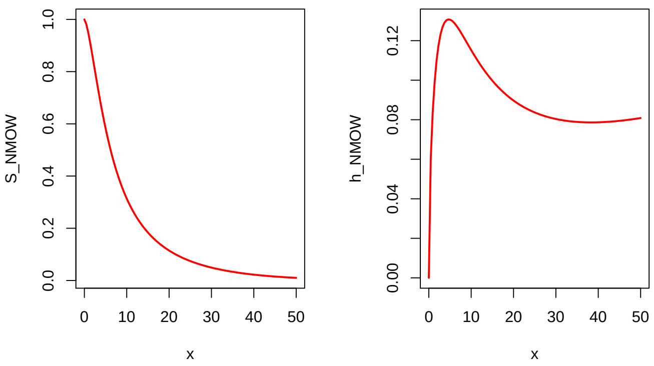

Fig. 6. MLEs of reliability and hazard rate functions based on data 2.

and conditional moment generating function, mean residual life mean deviations, order statistics, Rényi entropies. We obtain the MLEs of the parameters for NMOW. Two applications illustrate that the proposed model may attract wider applications in reliability analysis. Finally, we use the new model to analyze the reliability of the real data.

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