A New Marshall Olkin Weibull Distribution

Wei Cui, Zaizai Yan, and Xiuyun Peng

Abstract—An alternative lifetime model called the new Marshall Olkin Weibull (NMOW) distribution is proposed and studied. The importance of the new distribution comes from its ability to model some life distributions with increasing, decreasing, bathtub, and unimodal shapes of hazard rate functions, which are quite common in reliability study. We obtain the origin moments, generating function, conditional moment, conditional moment generating function, mean residual life function, mean deviations, order statistics and Rényi entropies. The maximum likelihood method is used to estimate parameters of the model. Two real data sets are used to illustrate the applicability of this new model.

Index Terms—Marshall-Olkin distribution; Weibull distribution; Maximum likelihood estimation.

I. INTRODUCTION

Weibull distribution is very popular model that is commonly used for analyzing biological, medical and hydrological data sets. However, it does not provide an acceptable fit for some applications, especially, when the hazard rates are bathtub, upside down bathtub, or bimodal shapes. To overcome these drawbacks, several authors have generalized and extended the Weibull distribution to model various types of data. Mudholkar et al. [1] studied an exponentiated Weibull (EW) distribution by adding a shape parameter to the Weibull distribution to allowing bathtub-shaped hazard rate function. Aryal and Tsokos [2] added one parameter to the Weibull distribution using the quadratic rank transmutation map. Carrasco et al. [3] defined a four parameters generalization of the Weibull distribution which has bathtub-shaped hazard rate function. Cordeiro et al. [4] proposed a five-parameter lifetime model called by McDonald Weibull distribution, it contains the Weibull, exponentiated Weibull, beta Weibull and Kumaraswamy Weibull distribution as the special cases. Gauss et al. [5] proposed a new distribution which called by beta-Weibull geometric distribution, whose failure rate function can be decreasing, increasing or upside down bathtub shape. Peng and Yan [6] introduced a new extended Weibull distribution with one scale parameter and two shape parameters which has increasing and upside-down bathtub shaped hazard rate functions. Carrasco et al. [7] studied a generalized modified Weibull distribution (MW).

Moreover, Pogány and Saboor [8] introduced a new four-parameter model called gamma-exponentiated-Weibull distribution. The exponential Weibull lifetime distribution [9], Kumaraswamy Weibull distribution [10] and a five-parameter extension of the Weibull distribution (APW) [11] had been studied by Cordeiro et al. Nassar et al. [12] introduced a new family of generalized distribution based on Alpha power transformation and extended Weibull distribution. Marshall and Olkin [13] introduced a new distribution family by adding a parameter to a initial distribution. Let the survival function of the initial distribution is given by \( \bar{U}(x) = 1 - U(x) \), where \( U(x) \) is the cumulative distribution function (CDF) of the initial distribution, and \( \bar{U}(x) \) is the corresponding probability density function (PDF). The survival function of Marshall-Olkin extend distribution is defined by

\[
\bar{V}(x) = \frac{\theta \bar{U}(x)}{1 - (1 - \theta)\bar{U}(x)}
\]

The corresponding CDF and PDF are given respectively

\[
V(x) = \frac{U(x)}{1 - (1 - \theta)\bar{U}(x)}
\]

\[
v(x) = \frac{\theta u(x)}{\left[1 - (1 - \theta)(1 - \bar{U}(x))\right]^2}
\]


We introduce a new model by inverting the equation (2), it is named as New Marshall-Olkin Weibull distribution (NMOW) in later sections. The NMOW has several desirable properties and more flexible hazard and density functions. The rest of this paper is organized as follows. In section II, we introduce the NMOW model. In section III, we study the properties including quantile function, moments, moment generating function, conditional moment and conditional moment generating function, mean residual life function, mean deviations, order statistics, and Rényi entropies. In section IV, we discuss the maximum likelihood estimates (MLEs) of model parameters. In section V, two real data sets are analyzed to illustrate the potentiality of the new model. The paper is concluded in section VI.

II. NMOW DISTRIBUTION

Let \( g(x) \) and \( G(x) \), respectively, are the PDF and CDF of a continuous random variable \( X \). Then the CDF \( F(x) \) of the new proposed distribution is obtained by inverting the equation (2) as follows:

\[
F(x) = \frac{\theta G(x)}{1 + (\theta - 1)G(x)}
\]
functions of the NMOW model for some parameters $\beta$, $\theta$, $\lambda > 0$. Results from Figures 1 and 2 show that the PDF can be left-skewed or right-skewed, and the hazard function has increasing, decreasing, bathtub, and upside down bathtub shapes. Therefore, the PDF and hazard rate functions of the NMOW model are very flexible.

### III. Properties of the NMOW Distribution

#### A. Quantile

**Theorem 2.1.** If a random variable $X$ follows the NMOW distribution with parameters $(\theta, \lambda, \beta)$, then the $p$th quantile is given by

$$x_p = \left[ \frac{1}{\beta} \log \left( \frac{\theta (p-1)}{p(\theta - 1) - \theta} \right) \right]^{1/\lambda}$$

**proof.** It is obtained easily from the equation $F_{NMOW}(x_p) = p$. 

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**Fig. 1.** Density functions of NMOW model for some values of $\beta$ and $\theta$. 

$$f(x) = \frac{\theta g(x)}{1 + (\theta - 1)G(x)^2}$$

(5)

Note that when $\theta \rightarrow 1$, $f(x)$ reduced to $g(x)$. From (4) and (5), the reliability function and hazard rate function are given by

$$S(x) = \frac{1 - G(x)}{1 + (\theta - 1)G(x)}$$

(6)

$$h(x) = \frac{\theta g(x)}{[1 + (\theta - 1)G(x)][1 - G(x)]}$$

(7)

In this paper, let $\phi = (\theta, \lambda, \beta)^T$, $X$ follows the Weibull distribution with CDF $G(x) = 1 - \exp \{-\lambda x^\beta\}, x > 0$. Based on (4) and (5), the CDF and PDF of the NMOW distribution are defined as

$$F_{NMOW}(x; \phi) = \frac{\theta (1 - \exp \{-\lambda x^\beta\})}{\theta + (1 - \theta) \exp \{-\lambda x^\beta\}}$$

(8)

$$f_{NMOW}(x; \phi) = \frac{\theta \lambda \beta x^{\beta - 1} \exp \{-\lambda x^\beta\}}{\left[\theta + (1 - \theta) \exp \{-\lambda x^\beta\}\right]^2}$$

(9)

where $\beta, \theta > 0$ are the shape parameters and $\lambda > 0$ is the scale parameter. The survival function and the hazard rate function for $x > 0$, are respectively given by

$$S_{NMOW}(x; \phi) = \frac{\exp \{-\lambda x^\beta\}}{\theta + (1 - \theta) \exp \{-\lambda x^\beta\}}$$

(10)

$$h_{NMOW}(x; \phi) = \frac{\theta \lambda \beta x^{\beta - 1}}{\left[\theta + (1 - \theta) \exp \{-\lambda x^\beta\}\right]}$$

(11)

Figures 1 and 2 are the curves of density and hazard rate functions of the NMOW model for some parameters $\beta$ and $\theta$, and $\lambda = 1$. Results from Figures 1 and 2 show that the PDF can be left-skewed or right-skewed, and the hazard rate function has increasing, decreasing, bathtub, and upside down bathtub shapes. Therefore, the PDF and hazard rate functions of the NMOW model are very flexible.
B. Moment and moment generating function

In this subsection, we present thth moment and the moment generating function of the NMOW distribution. By using the following series representation,

\[(1 - z)\rho = \sum_{j=0}^{\infty} \frac{\Gamma(\rho + j)}{\Gamma(\rho)}z^j, \quad |z| < 1, \rho > 0\]

the denominator in (9) can be expressed as

\[\left(\theta - (\theta - 1)\exp\left\{-\lambda x^\beta\right\}\right)^{-2}\]

\[= \frac{1}{\theta^2} \sum_{j=0}^{\infty} (j + 1)(1 - 1/\theta)^j \exp \{-\lambda j x^\beta\},\]

and substitutes above expression into (9), where \(\Gamma(\alpha) = \int_{0}^{\infty} x^{\alpha-1} e^{-x} dx\). The equation (9) can be rewritten as follows

\[f_{NMOW}(x; \phi) = \frac{\lambda x^{\beta-1} \sum_{j=0}^{\infty} (1 - 1/\theta)^j (j + 1)}{\theta^2} \times \exp \{-\lambda x^\beta (1 + j)\}.\]

**Theorem 2.2.** If \(X\) follows the NMOW distribution with parameters \((\theta, \lambda, \beta)\), then the \(r\)th moments are given by

\[E(X^r) = \frac{\lambda \beta}{\theta^r} \sum_{j=0}^{\infty} (1 - 1/\theta)^j (j + 1) \int_{0}^{\infty} x^{\beta + r - 1} \times \Gamma(\frac{r}{\beta} + 1), \quad r = 1, 2, \ldots\] (13)

**Proof.** From the definition of moments, we get

\[E(X^r) = \int_{0}^{\infty} x^r f_{NMOW}(x; \phi) dx\]

\[= \frac{\lambda \beta}{\theta^r} \sum_{j=0}^{\infty} (1 - 1/\theta)^j (j + 1) \int_{0}^{\infty} x^{\beta + r - 1} \times \exp \{-\lambda x^\beta (1 + j)\} dx,\]

let \(\lambda (1 + j) x^\beta = y\) above equation, we can get (13).

**Theorem 2.3.** If \(X\) follows the NMOW distribution with parameters \((\theta, \lambda, \beta)\), then the moment generating function is given by

\[M_X(t) = \frac{\lambda \beta}{\theta} \sum_{j=0}^{\infty} (1 - 1/\theta)^j (j + 1) \times \sum_{k=0}^{\infty} \frac{t^k}{k!} \left[\frac{1}{\lambda(1 + j)}\right]^{1 + \frac{k}{\beta}} \Gamma\left(\frac{k}{\beta} + 1\right)\]

\[= \frac{\lambda t^k}{\theta^\beta} \sum_{j=0}^{\infty} (1 - 1/\theta)^j (j + 1) \sum_{k=0}^{\infty} \frac{t^k}{k!} \times \int_{0}^{\infty} x^{\beta + k - 1} \exp \{-\lambda x^\beta (1 + j)\} dx,\]

let \(\lambda (1 + j) x^\beta = y\), we can get (14).

Theorems 2.1 and 2.2 are used to obtain the median, mean, variance, skewness and kurtosis of the NMOW distribution.

<table>
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<th>(\beta)</th>
<th>(\theta)</th>
<th>median</th>
<th>mean</th>
<th>variance</th>
<th>skewness</th>
<th>kurtosis</th>
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<tr>
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<tr>
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<td>0.7935</td>
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<td>2.1282</td>
<td>41.1612</td>
<td></td>
</tr>
</tbody>
</table>

The median is obtained by setting \(q = 0.5\) in Theorem 2.1. Then mean \(\mu\) is obtained by setting \(r = 1\) in Theorem 2.2. The variance \(\sigma^2\), skewness \(\gamma_1\) and kurtosis \(\gamma_2\) are obtained using the formulas \(\sigma^2 = E[X - \mu]^2\), \(\gamma_1 = E[\frac{X - \mu}{\sigma}]^3\), \(\gamma_2 = E[\frac{X - \mu}{\sigma}]^4\). These values are reported in Table I for various \(\theta\) and \(\beta\), where the scale parameter \(\lambda = 1\). From Table I, it is noted that for fixed \(\theta\) and \(\lambda\), the mean, moment of the NMOW are decreasing function of \(\beta\), and the skewness is increasing function of \(\beta\). Also, for fixed \(\beta\) and \(\lambda\), the median, mean are decreasing function of \(\theta\), and the skewness is increasing function of \(\theta\).

C. Conditional moment and Conditional moment generating function

**Theorem 2.4.** If \(X\) follows the NMOW distribution with parameters \((\theta, \lambda, \beta)\), then the \(r\)th conditional moments and conditional moment generating function of a random variable \(X\), are given by

\[E(x^r|T > t) = \frac{\lambda \beta}{\theta^r} \sum_{j=0}^{\infty} (1 - 1/\theta)^j (j + 1) \times \Gamma\left(\frac{r}{\beta} + 1, \lambda t^\beta (1 + j)\right), r = 1, 2, \ldots\] (15)

\[E(e^{tx}|T > t) = \frac{\lambda \beta}{\theta^\beta} \sum_{j=0}^{\infty} (1 - 1/\theta)^j (j + 1) \sum_{k=0}^{\infty} \frac{t^k}{k!} \times \int_{0}^{\infty} x^{\beta + k - 1} \exp \{-\lambda x^\beta (1 + j)\} dx,\]

where \(\Gamma(\alpha, x) = \int_{x}^{\infty} t^{\alpha-1} e^{-t} dt\).

**Proof.** According to the definition of conditional moment and conditional moment generating functions

\[E(x^r|T > t) = \frac{1}{F(t)} \int_{t}^{\infty} x^r f(x) dx,\]

\[E(e^{tx}|T > t) = \frac{1}{F(t)} \int_{t}^{\infty} e^{tx} f(x) dx,\]

we can use the same method as above as to get the results easily.
D. Mean residual life

The mean residual life (MRL) function describes the aging process that is very important in reliability and survival analysis. The MRL function of a lifetime random variable \( X \) is given by

\[
\mu(x) = \frac{1}{F(x)} \int_x^\infty t f(t)dt - x
\]  

(17)

**Theorem 2.5.** If \( X \) follows the NMOW distribution with parameters \((\theta, \lambda, \beta)\), the MRL function is given by

\[
\mu(x) = \frac{1}{F(x)} \sum_{j=0}^{\infty} \left( 1 - \frac{1}{\theta} \right)^j (j + 1) \left[ \frac{1}{\lambda(1+j)} \right]^{1+\frac{1}{\beta}} 
\times \Gamma\left( \frac{1}{\beta} + 1, \lambda(1+j) \right)
\]  

(18)

E. Mean deviations

The mean deviation of \( X \) about the mean and about the median, measure of spread in a population, are given by

\[
\delta_1 = E(|X - \mu|) = 2\mu F(\mu) - 2 \int_0^\mu x f(x)dx
\]  

(19)

and

\[
\delta_2 = E(|X - M|) = \mu - M + 2MF(M) - 2 \int_0^\mu x f(x)dx
\]  

(20)

respectively, where the mean \( \mu = E(X) \) and \( M \) denotes the median of \( X \).

**Theorem 2.6.** If \( X \) follows the NMOW distribution with parameters \((\theta, \lambda, \beta)\), the Mean deviations are given by

\[
\delta_1 = 2\mu F(\mu) - 2 \frac{\lambda}{\theta} \sum_{j=0}^{\infty} \left( 1 - \frac{1}{\theta} \right)^j (j + 1) \left[ \frac{1}{\lambda(1+j)} \right]^{1+\frac{1}{\beta}} 
\times \Gamma\left( \frac{1}{\beta} + 1, \lambda(1+j) \right)
\]  

(21)

\[
\delta_2 = \mu - M + 2MF(M) - 2 \frac{1}{\theta} \sum_{j=0}^{\infty} (1 - \frac{1}{\theta})^j \left[ \frac{1}{\lambda(1+j)} \right]^{1+\frac{1}{\beta}} 
\times \Gamma\left( \frac{1}{\beta} + 1, \lambda(1+j) \right)
\]  

(22)

here \( \gamma(a,x) = \int_0^x t^{a-1} e^{-t}dt \).

F. Order statistics

**Theorem 2.7.** If a random variable \( X \) follows the NMOW distribution with parameters \((\theta, \lambda, \beta)\), let \( X_1, X_2, \ldots, X_n \) be its a random sample with size \( n \), then, the PDF and CDF of the \( i \)th order statistics \( X_{i:n} \) are given by

\[
f_{i:n}(x) = \frac{\lambda x^{\beta-1} \exp(-\lambda x^\beta) n! i^{n-i} (n-i)! (1-i)!}{(i-1)!} \left( \frac{1}{\theta + (1 - \theta) \exp(-\lambda x^\beta)} \right)^{i+1} 
\times \theta^{i+1} \left[ 1 - \exp(-\lambda x^\beta) \right]^{i+1} 
\times \Gamma\left( \frac{1}{\beta} + 1, \lambda x^\beta \right)
\]  

(23)

\[
F_{i:n}(x) = \sum_{j=0}^{n-i} \binom{n-j}{i} (1-i)^j (1-\theta)^{i+1} \theta^{i+1} \left[ 1 - \exp(-\lambda x^\beta) \right]^{i+1}
\times \frac{\exp(-\lambda x^\beta (j+1))}{\theta + (1 - \theta) \exp(-\lambda x^\beta)}
\times \Gamma\left( \frac{1}{\beta} + 1, \lambda x^\beta \right)
\]  

(24)

G. Rényi entropy

The entropy of a random variable \( X \) measures the variation of the uncertainty. The Rényi entropy, say \( R_{\alpha}(v) \), is defined as

\[
R_{\alpha}(v) = \frac{1}{1-\alpha} \log \left( \int_0^\infty f(x)^\alpha dx \right)
\]  

(25)

**Theorem 2.8.** If \( X \) follows the NMOW distribution with parameters \((\theta, \lambda, \beta)\), the Rényi entropy is given by

\[
R_{\alpha}(v) = \frac{1}{1-\alpha} \log \left( \int_0^\infty f(x)^\alpha dx \right)
\]  

(26)

IV. Maximum likelihood

Let \( x = (x_1, \ldots, x_n) \) be a random sample from the NMOW distribution with parameters \( \phi = (\theta, \lambda, \beta) \), then the log-likelihood function is given by

\[
l(x|\phi) = n \left[ \log(\theta) + \log(\beta) + \log(\lambda) \right] + (\beta - 1) \sum_{i=1}^{n} \log(x_i) 
\]  

\[ 
- \lambda \sum_{i=1}^{n} x_i^\beta - 2 \sum_{i=1}^{n} \log(\theta + (1 - \theta) \exp \{- \lambda x_i^\beta \})
\]  

(27)

The MLEs \((\hat{\theta}, \hat{\lambda}, \hat{\beta})\) are obtained by using the following likelihood equations:

\[
\frac{\partial l}{\partial \theta} = n + 2 \sum_{i=1}^{n} \left[ \left( 1 - \theta \right) \frac{\lambda x_i^\beta}{\theta + (1 - \theta) \exp \{- \lambda x_i^\beta \}} \right] = 0
\]  

(28)

\[
\frac{\partial l}{\partial \lambda} = n - \sum_{i=1}^{n} x_i^\beta + 2 \sum_{i=1}^{n} \left[ \left( 1 - \theta \right) \exp \{- \lambda x_i^\beta \} x_i^\beta \right] = 0
\]  

(29)

\[
\frac{\partial l}{\partial \beta} = n - \lambda \sum_{i=1}^{n} x_i^\beta - \lambda \sum_{i=1}^{n} \log(x_i) - \lambda \sum_{i=1}^{n} x_i^\beta - 2 \sum_{i=1}^{n} \log(\theta + (1 - \theta) \exp \{- \lambda x_i^\beta \}) = 0
\]  

(30)

However, the above equations are very complex and no analytical solutions. We use Newton-Raphson method to compute the MLEs \((\hat{\theta}, \hat{\lambda}, \hat{\beta})\). We can also get the estimation of the unknown parameters using Bayesian approach which can be seen \([17], [18], [19]\).

V. FITTING RELIABILITY DATA

In this section we analysis real data to illustrate that the NMOW is a better lifetime model by comparing with many known distributions, such as Weibull (W), inverse Weibull (IW), exponential Weibull (EW) distributions[4], and their PDFs are expressed as follows.

The Weibull distribution with PDF

\[
f(x; \beta, \lambda) = \lambda x^{\beta-1} \exp \{- \lambda x^\beta \}, \quad x \geq 0, \lambda > 0, \beta > 0
\]  

(31)

The IW distribution with PDF

\[
f(x; \beta, \lambda) = \lambda x^{-(\beta+1)} \exp \{- \lambda x^{-\beta} \}, \quad x \geq 0, \lambda > 0, \beta > 0
\]  

(32)
The EW distribution with PDF
\[
    f(x; \beta, \lambda, \theta) = \theta \lambda x^{\beta-1} \exp\left\{ -\lambda x^{\beta} \right\},
\]
\[x \geq 0, \quad \lambda > 0, \quad \beta > 0, \quad \theta > 0\]

(33)

Application 1: The carbon fibers data set.
We shall consider the uncensored data set on the breaking stress of carbon fibers [16]. The observations are shown as follows.

3.70 2.74 2.73 2.50 3.60 3.11 3.27 2.87 1.47 3.11 3.56 4.42 2.41 3.19 3.22 1.69 3.28 3.09 1.87 3.15 4.90 1.57 2.35 2.55 2.03

Application 2: The cancer patients data set.
The second data set represents the remission times (in month) of random sample of 128 bladder cancer patients as reported in Lee and Wang [20]. The observations are shown as follows.

0.08 2.09 3.48 4.87 6.94 8.66 13.11 23.63 0.20 2.23 3.52 4.98 6.97 9.02 13.29 2.64 0.40 2.26 3.57 5.06 7.09 9.22 13.80 25.74 0.50 2.26 3.64 5.09 7.26 9.47 14.24 25.82 0.81 2.54 3.70 5.17 7.28 9.74 14.76 26.31 2.64 3.82 5.32 7.32 10.06 14.77 32.15 2.69 3.88 5.32 7.39 10.34 14.83 34.26 0.90 2.69 4.18 5.34 7.59 10.66 15.96 36.66 2.69 4.23 5.41 7.62 10.75 16.62 43.01 1.19 4.33 4.26 5.41 7.63 11.12 17.12 46.12 1.26 5.49 7.66 11.25 17.14 79.05 1.35 8.27 5.62 5.49 7.66 11.25 17.14 79.05 1.35 8.27 5.62 7.87 11.64 17.36 1.40 3.02 4.34 5.71 7.93 11.79 18.10 1.46 4.40 5.85 8.26 11.98 19.13 1.76 3.25 4.50 6.25 8.37 12.02 2.02 3.31 4.51 6.54 8.53 12.03 20.28 2.02 3.36 6.76 12.07 21.73 2.07 3.36 6.93 8.65 12.63 22.69

For both data sets, we calculate the MLEs of the parameters of all the models and the Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), and Kolmogorov-Smirnov (K-S) distances and corresponding p-values, and they are listed in Tables II and III, respectively. We observe that the NMOW has the lowest AIC, BIC, CAIC and K-S values and largest p-values by comparison with other models. Therefore, the NMOW could be chosen as the better model. Figures 3 and 4 show the histogram for data and the density estimates of all models. From Figures 3 and 4 we can also found the NMOW is better fitted to the data. Figures 5 and 6 are the MLEs changing of reliability and hazard rate along with time of data. From Figures 5 and 6 we can see the hazard rate of carbon fibers is increasing which has inflection point and the hazard rate of cancer patients data is bathtub shaped.

VI. CONCLUSIONS
In this paper, we propose a new three-parameter model, called the New Marshall-Olkin Weibull distribution (NMOW) which has more forms of hazard rate functions. The NMOW distribution is motivated by inverting the Marshall-Olkin distribution family in terms of the Weibull distribution in order to provide more flexibility to analyze lifetime data. We derive explicit expressions for the quantile, moments, moment generating function, conditional moment
and conditional moment generating function, mean residual life mean deviations, order statistics, Rényi entropies. We obtain the MLEs of the parameters for NMOW. Two applications illustrate that the proposed model may attract wider applications in reliability analysis. Finally, we use the new model to analyze the reliability of the real data.

ACKNOWLEDGMENT

The authors would like to thank the editor and the referees for their valuable comments and suggestions which greatly improved the paper.

REFERENCES


