Energy of Generalized Complements of a Graph

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Abstract—Let \( G \) be a finite simple graph on \( n \) vertices. Let \( P = \{V_1, V_2, V_3, \ldots, V_k\} \) be a partition of vertex set \( V(G) \) of order \( k \geq 2 \). For all \( V_i \) and \( V_j \) in \( P, i \neq j \), remove the edges between \( V_i \) and \( V_j \) in graph \( G \) and add the edges between \( V_i \) and \( V_j \) which are not in \( G \). The graph \( G^P_k \) thus obtained is called the \( k \)-complement of \( G \) with respect to the partition \( P \). Let \( P = \{V_1, V_2, V_3, \ldots, V_k\} \) be a partition of vertex set \( V(G) \) of order \( k \geq 2 \). For each set \( V_i \) in \( P \), remove the edges of graph \( G \) inside \( V_i \) and add the edges of \( G \) (the complement of \( G \)) joining the vertices of \( V_i \). The graph \( \overline{G}^P_{k(i)} \) thus obtained is called the \( k(i) \)-complement of \( G \) with respect to the partition \( P \).

The energy of a graph \( G \) is defined by Ivan Gutman [4] as the sum of absolute eigenvalues of \( G \). It represents a proper generalization of a formula valid for the total \( \pi \)-electron energy of a conjugated hydrocarbon as calculated by the Huckel molecular orbital (HMO) method in quantum chemistry. For recent mathematical work on the energy of a graph see [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16].

The characteristic polynomial of \( G \) is its characteristic polynomial of adjacency matrix \( A(G) \) denoted as \( \phi(G, \lambda) \). The set of eigenvalues of \( A(G) \) given by \( \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) is called spectrum of graph \( G \). Two graphs \( G \) and \( H \) are said to be equienergetic if they have same energy. Two or more graphs are called co-spectral graphs if they have same spectra. In this paper we study the energy of generalized complements of some classes of graph. For a graph, there exists many \( k \) and \( k(i) \) complements. Hence, it is interesting to study the variation of energy for these complements.

I. INTRODUCTION

Let \( G \) be a graph on \( n \) vertices and \( m \) edges. The complement of a graph \( G \), denoted by \( \overline{G} \), has the same vertex set as that of \( G \), but two vertices are adjacent in \( \overline{G} \) if and only if they are not adjacent in \( G \). If \( G \) is isomorphic to \( \overline{G} \) then \( G \) is said to be self-complementary graph.

Definition 1: [3] Let \( P = \{V_1, V_2, V_3, \ldots, V_k\} \) be a partition of vertex set \( V(G) \) of order \( k \geq 2 \). For all \( V_i \) and \( V_j \) in \( P, i \neq j \), remove the edges between \( V_i \) and \( V_j \) in graph \( G \) and add the edges between \( V_i \) and \( V_j \) which are not in \( G \). The graph \( G^P_k \) thus obtained is called \( k \)-complement of \( G \) with respect to the partition \( P \).

Definition 2: [3] Let \( P = \{V_1, V_2, V_3, \ldots, V_k\} \) be a partition of vertex set \( V(G) \) of order \( k \geq 2 \). For each set \( V_i \) in \( P \), remove the edges of graph \( G \) inside \( V_i \) and add the edges of \( G \) joining vertices of \( V_i \). The graph \( \overline{G}^P_{k(i)} \) thus obtained is called the \( k(i) \)-complement of \( G \) with respect to the partition \( P \).

Example 3:

The energy of a graph is given by Ivan Gutman [4] as the sum of absolute eigenvalues of \( G \). It represents a proper generalization of a formula valid for the total \( \pi \)-electron energy of a conjugated hydrocarbon as calculated by the Hückel molecular orbital (HMO) method in quantum chemistry. For recent mathematical work on the energy of a graph see [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16].

The characteristic polynomial of \( G \) is its characteristic polynomial of adjacency matrix \( A(G) \) denoted as \( \phi(G, \lambda) \). The set of eigenvalues of \( A(G) \) given by \( \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) is called spectrum of graph \( G \). Two graphs \( G \) and \( H \) are said to be equienergetic if they have same energy. Two or more graphs are called co-spectral graphs if they have same spectra. In this paper we study the energy of generalized complements of some classes of graph. For a graph, there exists many \( k \) and \( k(i) \) complements. Hence, it is interesting to study the variation of energy for these complements.

II. PRELIMINARIES

Now we present few results on \( k \) and \( k(i) \) self complementary graphs, characteristic polynomial of cluster graph \( K_{b_p}(k) \) and complete multipartite graph, which are extensively used to prove our main results.

Proposition 4: [3] The \( k \)-complement and \( k(i) \) complement of \( G \) are related as follows:

(i) \( G^P_k \cong G^P_{k(i)} \)
(ii) \( G^P_{k(i)} \cong G^P_k \)

Definition 5: [17] Let \( f_i, i = 1, 2, \ldots, k, 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \) be independent edges of complete graph \( K_p, p \geq 3 \). The graph \( K_{b_p}(k) \) is obtained by deleting \( f_i, i = 1, 2, \ldots, k \) from \( K_p \).

In addition \( K_{b_p}(0) \equiv K_p \).

Proposition 6: [17] For \( p \geq 3 \) and \( 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \),
\[ \phi(K_{b_p}(k), \lambda) = \lambda^k(\lambda + 1)^{p-2k-1}(\lambda + 2)^k-1[\lambda^2 - (p - 3)\lambda - 2(p - k - 1)]. \]

Proposition 7: [10] The characteristic polynomial of complete multipartite graph \( K_{n_1,n_2,\ldots,n_p} \) is
\[ \phi(K_{n_1,n_2,\ldots,n_p}, \lambda) = \lambda^{n-p}(1 - \sum_{i=1}^{p} \frac{n_i}{\lambda + n_i}) \prod_{j=1}^{p} (\lambda + n_j). \]

Lemma 8: [10] Let \( A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix} \) be a \( 2 \times 2 \) block symmetric matrix. Then eigenvalues of \( A \) are the eigenvalues of matrices \( A_0 + A_1 \) and \( A_0 - A_1 \).

III. ENERGY OF GENERALIZED COMPLEMENTS OF CLASSES OF GRAPHS

In this section, we find energy of generalized complements of some standard graphs like complete, complete bipartite, star, path, friendship, double star and cocktail party graph.
We also compute energy of generalized complements of graphs derived from star graphs like $K_{1,n-1}+e$, $K_{1,n-1}+2e$ and $G_2 = G(K_{1,n-1},K_{1,n-1},...,K_{1,n-1})$.

**Theorem 9:** Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of the complete graph $G_n$.

(i) If $V_i \geq K_i$, for $i = 1, 2, \ldots, m$, then $E(K_n)^{P}_{k}) = 2(n-k)$ and $E(K_n)^{P}_{k(i)} = E(K_{n_i},n_2,n_3,\ldots,n_k)$, where $|V_i| = n_i$, $i = 1, 2, \ldots, k$.

(ii) If $|V_i| = 2$ and any one partite $V_i$ has order 1 for odd $n$, then $E(K_n)^{P}_{k}) = \left\{ \begin{array}{ll}
  n & \text{if } n \text{ is even} \\
  n-1 & \text{if } n \text{ is odd}
\end{array} \right.$

$$
\phi((K_n)^{P}_{k(i)}; \lambda) = \lambda^{\frac{n}{2}} (\lambda + 1)^{n-2} \left( \frac{n}{2} \right) - 1
$$

By substituting the value of $k$ in statement (i) of Theorem 9, we obtain $E(K_n)^{P}_{k}) = \left\{ \begin{array}{ll}
  n & \text{if } n \text{ is even} \\
  n-1 & \text{if } n \text{ is odd}
\end{array} \right.$

We observe that $(K_n)^{P}_{k(i)} = Kb_n(\left\{ \frac{n}{2} \right\})$, a graph obtained after deleting the independent edges of $K_n$. Hence, from Proposition 6, we get

$$
\phi((K_n)^{P}_{k(i)}; \lambda) = \lambda^{\frac{n}{2}} (\lambda + 1)^{n-2} \left( \frac{n}{2} \right) - 1
$$

**Remark 10:** (i) From Theorem 9, it follows that $E(K_n)^{P}_{k})$ is independent of the order of each partite set. Also, for fixed integers $n$ and $k$, we obtain non-co-spectral equienergetic graphs. $E(K_n)^{P}_{k})$ decreases as order of the partition set $k$ increases.

(ii) If $P = \{V_1, V_2\}$ is the partition of $K_n$ with $|V_i| = n_i$, $i = 1, 2$, then energy of $k(i)$-complement of $K_n$ is $2\sqrt{mn}$.

**Theorem 11:** Let $K_{m,n}$ be complete bipartite graph with vertex set $V = \{U_m, V_n\}$ and partition $P = \{V_1, V_2\}$.

(i) If $V_1 \geq K_{s_1,s_2}$ and $V_2 \geq K_{m-s_1,n-s_2}$, where $s_1, s_2$ denote number of vertices of $V_1$ such that $s_1$ vertices belong to $U_m$ and $s_2$ vertices belong to $V_n$, then $E(K_{m,n})^P = 2\sqrt{(n-s_1+s_2)(m-s_2+s_1)}$ and $E(K_{m,n})^{P}_{2(i)} = 2(m+n-2)$.

(ii) If $|V_i| = m-1$ such that all the vertices of $V_i$ are from first partite set of $K_{m,n}$ and $|V_2| = n + 1$, then $E(K_{m,n})^P = 2\sqrt{m+n-1}$ and $E(K_{m,n})^{P}_{2(i)} = 2(m+n-2)$.

**Proof:**

(i) If $V_1 \geq K_{s_1,s_2}$ and $V_2 \geq K_{m-s_1,n-s_2}$ then $(K_{m,n})^P = K_{s_1+s_2,m-s_2+s_1}$.

Hence, $E(K_{m,n})^P = 2\sqrt{(n-s_1+s_2)(m-s_2+s_1)}$. Also $(K_{m,n})^{P}_{2(i)} \geq K_m \cup K_n$. Thus, $E(K_{m,n})^{P}_{2(i)} = 2(m+n-2)$.

(ii) If $|V_i| = m-1$ such that all the vertices of $V_i$ are from first partite set of $K_{m,n}$ and $|V_2| = n + 1$, then $(K_{m,n})^P \geq K_{m+n-1}$. Hence, $E(K_{m,n})^P = 2m+n-1$. Also $(K_{m,n})^{P}_{2(i)} \geq K_1 \cup K_{m+n-1}$. Hence, $E(K_{m,n})^{P}_{2(i)} = 2(m+n-2)$.

**Corollary 12:** Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of $K_{m,n}$ such that $|V_1| > K_2$ for $i = 1, 2, \ldots, k$ and $|V_k| > K_2$ for odd path. The characteristic polynomial of $(K_{m,n})^P$ is $\lambda^{n-1} (\lambda^2 + 2m^2 - 2mn + mn)$ and $E(K_{m,n})^{P}_{2(i)} = 2(m+n-1)$.

**Theorem 13:** Let $V_1, V_2, \ldots, V_k$ be a partition of path $P_n$. Then, $E(P_n)^P_{2(i)} = 2(k-1)$ and $E(K_n)^P = E(Kb_n(k-1))$ in the following cases.

(i) Any one of the pendant(non-pendant) vertex is in $V_1$ or $V_k$ and, remaining $V_i$ are such that $|V_i| = K_2$, for odd path.

(ii) $|V_i| > K_2$, for even path, $i = 1, 2, \ldots, k$.

**Proof:** We note that $(P_n)^P_{2(i)}$ is the union of $k-1$ number of $K_2$s and in addition two isolated vertices for odd path. Hence, $E(P_n)^P_{2(i)} = 2(k-1)$. Whereas $(P_n)^P_{2(i)}$ is the graph obtained from $K_n$ by deleting all the independent edges. Thus, $E(P_n)^P_{2(i)} = E(Kb_n(k-1))$, which is estimated using Proposition 6.

**Theorem 14:** Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of cycle $C_n$.

1) If $|V_i| = K_2$, then $E(C_n)^P_{2(i)} = 2k$ and $E(C_n)^P = E(Kb_n(k))$, for even $n$.

2) If $|V_i| > K_2$ and one of $|V_i| = 1$, then $E(C_n)^P_{2(i)} = 2(n-k + \sqrt{2})$, for odd $n$.

**Proof:**

1) For even $n$, we observe that $(C_n)^P_{2(i)}$ is the union of $k$ number of $K_2$s. Hence, $E(C_n)^P_{2(i)} = 2k$. Whereas $(C_n)^P_{2(i)}$ is the graph obtained from $K_n$ by deleting all the independent edges. Thus, $E(C_n)^P_{2(i)} = E(Kb_n(k))$ which can be computed by Proposition 6.

2) In case of odd $n$, $(C_n)^P_{2(i)}$ is the union of $K_{1,2}$ and $k-2$ number of $K_2$’s. Thus, $E(C_n)^P_{2(i)} = 2(n-k + \sqrt{2})$.

**Theorem 15:** Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of star graph $S_n$. Then the following statements are true.

(i) For $k = 2$, if $|V_1| > S_r$, $r \geq 2$, then $E(S_n)^P_{2(i)} = \sqrt{(r-1)(n-r+1)}$ and $E(S_n)^P_{2(i)} = 2(n-3)$.

(ii) If $|V_1| = 1$ such that the central vertex belongs to $V_i$ and $|V_i| > n_i K_1$, $1 \leq r \leq n-i, i > 2$ then $E(S_n)^P_{2(i)} = E(K_1 \cup K_{n_i,n_2,n_3,\ldots,n_k})$, the energy of complete multipartite graph of order $n-1$. 

Volume 28, Issue 1: March 2020
Proof:

(i) For a partition $P$ in (i), $(S_n)^e_2$ is the complete bipartite graph $K_{r+1,n-r+1}$. Hence $E(S_n)^e_2 = \sqrt{(r-1)(n-r+1)}$ and $E(S_n)^e_2(\tau_2) = 2(n-3)$, since $(S_n)^e_2(\tau_2)$ is the union of two complete subgraphs of order $r+1$ and $n-r$.

(ii) Suppose $|V_1| = 1$ such that central vertex belongs to $V_1$ and $< V_1 >= n_r K_1, 1 \leq r \leq n-1, i \geq 2$. Then $(S_n)^e_2 = K_1 \cup K_{n_2,n_3,\ldots,n_r}$. Hence, $E(S_n)^e_2 = E(K_1 \cup K_{n_2,n_3,\ldots,n_r})$ which is given by Proposition 7.

Theorem 16: If $\tau = K_{1,n+1} + e$ is a unicyclic graph of order $n$ obtained by adding an edge between two pendant vertices of star graph $K_{n-1}$, then

1) For partition $P = \{V_1, V_2\}$ such that $< V_1 >= C_3$ and $< V_2 >= (n-2)K_1, E(\tau_2^P) = E(K_{1,1,n-2})$ and $E(\tau_2^P) = 2(n-3)$.

2) For partition $P = \{V_1, V_2\}$ such that $< V_1 >= K_2$ and $< V_2 >= (n-2)K_1, E(\tau_2^P) = 2\sqrt{n-2}$.

3) For partition $P = \{V_1, V_2, V_3\}$ such that only central vertex is in $V_1, < V_2 >= K_2$ and $< V_3 >= (n-3)K_1, E(\tau_3^P) = E(K_{1,1,1,n-3})$.

![Fig. 1. $\tau = K_{1,5} + e$](image)

Proof:

1) For the given partition $P$, $\tau_2^P$ results into complete multipartite graph $K_{1,1,1,n-2}$. Hence $E(\tau_2^P) = E(K_{1,1,1,n-2})$, which can be computed using Proposition 7. Note that $\tau_2^P$ is an union of $K_{n-2}$ and two isolated vertices. So, $E(\tau_2^P) = 2(n-3)$.

2) For $\tau = \{V_1, V_2\}$ such that $< V_1 >= K_2$ and $< V_2 >= (n-2)K_1$, the resultant graph $\tau_2^P$ is an union of isolated vertex and a star graph $K_{1,n-2}$. Therefore, $E(\tau_2^P) = 2\sqrt{n-2}$.

3) For $P = \{V_1, V_2, V_3\}$ such that only central vertex is in $V_1$, $< V_2 >= K_2$ and $< V_3 >= (n-3)K_1$, the graph $\tau_3^P$ is a disconnected graph with $K_1$ and $K_{1,1,1,n-3}$ as components. Thus $E(\tau_3^P) = E(K_{1,1,1,n-3})$ which can be evaluated by Proposition 7.

Theorem 17: Let $S(m,n)$ be double star graph with partition $P = \{V_1, V_2\}$, such that the vertices of $V_1$ and $V_2$ are of distance two. Then characteristic polynomial of $(S(m,n))^e_2$ is $(\lambda + 1)^m n + 3[\lambda^3 - (n + m - 3)\lambda^2 + (n - 3)(n - 2)\lambda - (n - 3)]$.

Proof: Let $P = \{V_1, V_2\}$ be a partition of vertices of $(S(m,n))^e_2$ such that the vertices of $V_1$ and $V_2$ are of distance two i.e., $V_1=\{v_1, v_2, v_3, \ldots, v_m-1, u_1\}$ and $V_2=\{u_m, u_2, u_3, \ldots, u_{n-1}, u_n\}$. We have $A(S(m,n))^e_2 =
\[-1\begin{array}{cccc}
(J-I)_{m-1} & J_{m-1} & 0_{m-1 \times n-1} & J_{m-1} \\
J_{m-1} & (J-I)_{2} & J_{m-1} & 0_{m-1 \times n-1} \\
0_{n-1 \times m-1} & J_{n-1 \times 2} & (J-I)_{n-1}
\end{array}\]

Consider det$(\lambda I - A((S(m,n))^e_2))$

Step 1: Replacing $R_i$ by $R_i - R_{i+1}$, for $i = v_1, v_2, v_3, \ldots, v_m-2, v_m$ and $R_i$ by $R_i - R_{i-1}$, for $i = u_n, u_{n-1}, \ldots, u_3, u_2$, the determinant reduces to

\[det(D) = \begin{vmatrix}
\lambda - (n-2) & -1 & -1 & 0 \\
0 & 1 & -1 & 0 \\
1 & -m & -1 & \lambda & 1 - n \\
0 & -1 & -1 & \lambda - n + 2
\end{vmatrix}\]

Hence, $det(D) = \lambda^3 - (n + m - 3)\lambda^2 + (n - 3)(n - 2)\lambda - (n - 3)$. Thus, the characteristic polynomial of $(S(m,n))^e_2$ is

\[(\lambda + 1)^m n + 3[\lambda^3 - (n + m - 3)\lambda^2 + (n - 3)(n - 2)\lambda - (n - 3)]\]

We have $S(m,n)_P^e \cong K_{m-1,n-1} \cup 2K_1$.

Hence, $E(S(m,n)_P^e) = E(K_{m-1,n-1} + 2E(K_1) = 2\sqrt{(m-1)(n-1)}$.

Theorem 18: Let $P = \{V_1, V_2, V_3\}$ be a partition of double star graph $S(m,n)$ such that $< V_1 >= (m - 1)K_1, < V_2 >= K_2$ and $< V_3 >= (n - 1)K_1$. Then $E(S(m,n)_P^e) = 2\sqrt{mn}$ and $E(S(m,n)_P^e) = 2(m+n-2)$.

Proof: For the partition $P = \{V_1, V_2, V_3\}$ such that $< V_1 >= (m-1)K_1, < V_2 >= K_2$ and $< V_3 >= (n-1)K_1$, we observe that $S(m,n)_P^e$ is a complete bipartite graph $K_{m,n}$. Hence, $E(S(m,n)_P^e) = 2\sqrt{mn}$. Also, $S(m,n)_P^e$ is an union of two subgraphs $K_m$ and $K_n$. Thus, $E(S(m,n)_P^e) = E(K_m) + E(K_n) = 2(m+n-2)$.

Definition 19: [18] A graph $G_s$ obtained by completely connecting the central vertices of $r$ star graphs $K_{1,n_1-1}, K_{1,n_2-1}, \ldots, K_{1,n_r-1}$ is called generalized star graph. It is denoted by $G_s = (K_{1,n_1-1}, K_{1,n_2-1}, \ldots, K_{1,n_r-1})$.

Theorem 20: Let $G_s = (K_{1,n_1-1}, K_{1,n_2-1}, \ldots, K_{1,n_r-1})$ be a generalized star graph with partition $P = \{V_1, V_2, \ldots, V_{r+1}\}$ such that $< V_j >= (n_j - 1)K_1, j = 1, 2, \ldots, r$ and $< V_{r+1} >= K_r$. Then $E(G_s)_P^e = (E(K_{1,n_1-1}) + \ldots + E(K_{1,n_r-1}) = 2(n_1 + n_2 + \ldots + n_r - r)$.

Proof: For partition $P, (G_s)_P^e$ results into complete multipartite graph and thus its energy is $E((G_s)_P^e) = E(K_{1,n_1-1} + \ldots + E(K_{1,n_r-1})$ which can be obtained using Proposition 7.

Theorem 21: If $\mathcal{H} = K_{1,n-1} + 2e$ is a bicyclic graph of order $n$ obtained by adding an edge between a pendant vertex and a vertex of degree two of the graph $\tau = K_{1,n-1} + e$ of star graph $K_{n-1}$, then for partition $P = \{V_1, V_2\}$ such that $< V_1 >= (n-4)K_2$ and $< V_2 >= K_4 - e$, $E(H_{2})^e = E(K_{1,n-4}) + E(H_{2}^e) = 2(n-3)$.
Fig. 2. \( \mathcal{H} = K_{1,6} + 2e \)

**Proof:** We observe that for the given partition \( P \), graph \( \mathcal{H}_P^P \) reduces to complete tripartite graph \( K_{1,3,n-4} \). Hence, \( \mathcal{E}(\mathcal{H}_P^P) = \{ K_{1,3,n-4} \} \) which is obtained using Proposition 7. On the other hand, \( \mathcal{H}_P^P \) will be an union of disconnected complete subgraphs of order 1,2 and \( n-3 \) respectively. Thus, \( \mathcal{E}(\mathcal{H}_P^P) = \{ K_1 \cup U \} \), and hence, \( \mathcal{E}(\mathcal{H}_P^P) = nE(K_2) + E(K_1) = 2n \).

**Theorem 23:** Let \( F_n \) be friendship graph with partition \( P = \{ V_1, V_2 \} \), such that \( < V_1 >= K_3 \) and \( < V_2 >= n-2K_2 \). Then \( \mathcal{E}(\mathcal{H}_P^P) = (2n - \sqrt{n^2 - 2n} + 2) \).

**Proof:** For partition \( P = \{ V_1, V_2 \} \), such that \( V_1 = K_3 \) and \( \leq V_2 = n-2K_2 \) of graph \( F_n \), the adjacency matrix of \( k(i) \) complement of \( F_n \) is given by:

\[
\begin{pmatrix}
0 & 0 & 1 & \ldots & 1 \\
0 & 1 & 0 & \ldots & 0 \\
1 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \ldots & 0
\end{pmatrix}
\]

Consider \( \det (\lambda - A(F_n^P)_{2(i)}) \)

**Step 1:** Replace \( R_i \) by \( R_i - R_i \), where \( i = v_{2i+1}, v_{2i+2}, \ldots, v_{2n} \) and replace \( R_{v_4} \) by \( R_{v_4}, R_{v_4} \). Then \( \det (\lambda - A(F_n^P)_{2(i)}) \) reduces to the form \( \lambda^{n-1} \).

**Step 2:** In \( \det (D) \), replace \( R_i \) by \( R_i - R_i \), where \( i = v_{2i+1}, v_{2i+2}, \ldots, v_{2n} \). We get a new determinant, let it be \( \det (E) \).

**Step 3:** In \( \det (E) \), we replace \( C_i \) by \( C_i + C_{i+1} \), where \( i = v_{2i+1}, v_{2i+2}, \ldots, v_{2n} \). It simplifies to

\[
\det (E) = (\lambda + 2)^{n-2} \begin{vmatrix}
\lambda & 0 & 0 & 2 - n \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\lambda - 1 & 0 & 0 & \lambda + 2
\end{vmatrix}
\]

Thus, \( \det (E) = (\lambda + 2)^{n-2} \).

Hence \( \det (D) = (\lambda - (n - 2 + \sqrt{n^2 - 2n} + 2))((\lambda - (n - 2 - \sqrt{n^2 - 2n} + 2))) \).

**Step 4:** For partition \( P = \{ V_1, V_2 \} \), such that \( V_1 = K_3 \) and \( V_2 = n-2K_2 \). The characteristic polynomial of \( (F_n^P)_{2(i)} \) is \((\lambda + 1)^n(\lambda - 1)^{n-2}[\lambda^2 - 4\lambda - (4n - 3)]\).

**Proof:** Let \( P = \{ V_1, V_2 \} \) be a partition of vertices of \( F_n \), where \( V_1 = K_3 \) and \( V_2 = \).

**Step 1:** Replacing \( R_i \) by \( R_i - R_i \), where \( i = v_{2i+1}, v_{2i+2}, \ldots, v_{2n} \), we obtain a new determinant say, \( \det (D) \).

**Step 2:** In \( \det (D) \), replacing \( C_{v_3} \) by \( C_{v_3} - C_{v_4} \), we get a new determinant, let it be \( \det (E) \).
Step 3: In det(E), expanding over last row, we see that,
\[ \text{det}(E) = (-1)^{2n-3}(-1) \text{det}(P) + (-1)^{n+1}(-1) \text{det}(Q) + (-1)^n \lambda \text{det}(R). \]

(a) In det(P), replacing \( C_i = C_i + C_{i+1}, \) where \( i = v_{2n-1}, v_{2n-2}, v_{2n-3}, \ldots, v_3, \) we get a new determinant, let it be det(S).

In det(S), replacing \( R_i \) by \( R_i + R_{i-1}, \) where \( i = v_5, v_7, v_9, \ldots, v_{2n-1} \) and by simplifying, we get det(S) = \((-1)^n(\lambda^2 - 2\lambda + 2)\).

(b) In det(Q), replacing \( R_i \) by \( R_i + R_{i-1}, \) where \( i = v_5, v_7, v_9, \ldots, v_{2n-1} \) and by simplifying we obtain det(Q) = \((-1)(\lambda - 2)^2\).

(c) In det(R), replacing \( R_i \) by \( R_i + R_{i-1}, \) where \( i = v_5, v_7, v_9, \ldots, v_{2n-1} \) and further reducing det(R), we see that det(R) = \((-1)^n(\lambda^2 - 2\lambda + 2)\).

Then from step 3, we obtain det(E) = \((-1)^n(\lambda^2 - 2\lambda + 2)\).

Theorem 25: Let \( K_{n} \) be any root \( 2n \) partitioned

\[ A \begin{bmatrix} X \\ Y \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} (\lambda + 1)I - J & I - J \\ I - J & (\lambda + 1)I - J \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} - \begin{bmatrix} X \\ Y \end{bmatrix} = 0 
\]

\[ \text{det}(E) = (\lambda - 2)^2. \]

Proof: Let \( K_{n} \) be any root \( 2n \) partitioned

\[ A \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} = 0 \]

\[ \text{det}(E) = (\lambda - 2)^2. \]

Case 1: Let \( X = I_n \) and \( Y = I_n \). Suppose \( \lambda \) is any root of the equation

\[ ([\lambda + 1] - n)I_n - I_n = 0 \]

\[ (\lambda - n)I_n = 0 \]

\[ \lambda = n \]

We conclude that \( n \) is an eigenvalue of \( A(K_{n} \times 2) \), with multiplicity at least one.

Case 2: Let \( X = I_n \) and \( Y = -I_n \).

Now

\[ ([\lambda + 1] - n)I_n + I_n = (\lambda - n + 2)I_n = 0 \]

From Equation (27), we note that \( \lambda - n + 2 = 0 \). Thus \( \lambda = n - 2 \) is an eigenvalue, with multiplicity at least one.

Case 3: Let \( X = X_i \) be an eigenvector with first element 1 and \( i^{th} \) element -1, for \( i = 2, 3, \ldots, n \) and remaining elements equal to zero. Now \( Y = (\lambda + 1)X_i \), where \( \lambda \) is any root of \( \lambda^2 + 2\lambda = 0 \).

By noting \( JX_i = X_i \) and from Equation (26),

\[ -(J + \lambda - 1)X_i + I(\lambda - 1)X_i = -((\lambda + 1)X_i + (\lambda - 1)X_i) = 0 \]

and

\[ -I_nX_i = [\lambda + 1]I_n - J(\lambda + 1)X_i = (\lambda^2 + 2\lambda)X_i. \]

From Equation (28), we obtain \( \lambda^2 + 2\lambda = 0 \). Thus \( \lambda = 0 \) and \( \lambda = -2 \) are eigenvalues, each with multiplicity at least \((n-1)\), as there are \((n-1)\) independent vectors of the form \( X_i \).

Since order of the graph is \( 2n \), spectrum of \( (K_{n} \times 2)^P \) is

\[ \text{det}(E) = (\lambda - 2)^2. \]

Theorem 29: Let \( K_{n} \) be root \( 2n \) partitioned

\[ \begin{bmatrix} 0 & n & n-2 \\ n-1 & 1 & n-1 \\ 1 & n & 1 \end{bmatrix} \]

Hence, \( E(K_{n} \times 2)^P = E(S_n^1) \).

Theorem 30: For the given partition, \( (K_{n} \times 2)^P \) is the totally disconnected graph. Hence, \( E(K_{n} \times 2)^P = 0 \). We have \( (K_{n} \times 2)^P \) disconnected.

Proof: Let \( K_{n} \times 2 \) be root \( 2n \) partitioned

\[ A \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} = 0 \]

\[ \text{det}(E) = (\lambda - 2)^2. \]

It is of the form \( A^1 \), \( A_2 \). Hence, we can apply Lemma 8. First we shall compute spectrum of \( A_0 + A_1 \).

Step 1: Consider det(A_0 + A_1)

(i) Replacing \( R_i \) by \( R_i - R_{i-1}, \) for \( i = v_1, v_2, v_3, \ldots, v_{2n}, \) the det(A_0 + A_1) will reduce to \((\lambda + 1)^n\) det(D).

(ii) In det(D), replacing \( C_i \) by \( C_i + C_{i-1}, \) for \( i = v_2, v_3, v_4, \ldots, v_{2n}, \) we get a new determinant det(E).

(iii) In det(E), replacing \( R_i \) by \( R_i - R_{i-1}, \) where \( i = v_2, v_4, v_6, \ldots, v_{2n-2}, \) we get \((\lambda + 3)^{n-1}\) det(F).

(iv) Expanding det(F) over first row we obtain det(F) = \[ \begin{bmatrix} \lambda^2 - (4n - 3)\lambda - 8n \end{bmatrix} \]

\[ \frac{1}{2}\begin{bmatrix} 4n - 3 + \sqrt{4n - 3 + 4n - 8n + 8} \\ 1 \end{bmatrix} \]

To compute spectrum of \( A_0 - A_1 \)

Step 2: Consider det(A_0 - A_1)
(i) Replacing $R_i$ by $R_i - R_{i+1}$ for $i = v_1, v_2, v_3, \ldots, v_{2n-1}$, we see that $\det(A_0 - A_1)$ is of the form $(\lambda - 1)^n \det(D)$.

(ii) Replacing $C_i$ by $C_i - C_{i+1}$ for $i = v_2, v_3, v_4, \ldots, v_{2n}$ in $\det(D)$, we obtain determinant of lower triangular matrix. Hence, $\det(D) = \lambda(\lambda + 1)^n$. Thus, we have $\det(A_0 - A_1) = \lambda(\lambda - 1)^n \det(D)$.

Spectrum of $A_0 - A_1$ is $\{0, -1, n\}$. Then, we have $K_{2n+1}$.

Combining spectra of $A$ and $A_0$, for the energy of generalized complements of some classes of graphs by taking partition of order 2, 3 and $k \geq 3$ in some cases.

The energy of a graph is one of the emerging subject of the Serbian Chemical Society, vol. 47, no. 1, pp. 123-129, 2017.

### IV. CONCLUSION

The energy of a graph is one of the emerging subject within graph theory. It serves as a frontier between Chemistry and mathematics. The energy of several graphs is found in literature. In our study, we have derived explicit expression for the energy of generalized complements of some classes of graphs by taking partition of order 2, 3 and $k \geq 3$ in some cases.

### REFERENCES


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