On the Persistent and Extinction Property of a Discrete Mutualism Model with Time Delays

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***Abstract*—Sufficient conditions are obtained for the permanence and extinction of the following discrete model of mutualism**

$$\begin{aligned} x_1(k+1) &= x_1(k)\exp\Big\{r_1(k)\Big[\frac{K_1(k)+\alpha_1(k)x_2^{\beta_1}(k-\tau_2(k))}{1+x_2^{\beta_1}(k-\tau_2(k))} \\ &\quad -x_1^{\delta_1}(k-\sigma_1(k))\Big]\Big\}, \\ x_2(k+1) &= x_2(k)\exp\Big\{r_2(k)\Big[\frac{K_2(k)+\alpha_2(k)x_1^{\beta_2}(k-\tau_1(k))}{1+x_1^{\beta_2}(k-\tau_1(k))} \\ &\quad -x_2^{\delta_2}(k-\sigma_2(k))\Big]\Big\}, \end{aligned}$$

where we assume that one of the following conditions holds.

(A) r_i, K_i, α_i**,** τ_i **and** $\sigma_i, i=1,2$ **are nonnegative sequences bounded above and below by positive constants, and** $\alpha_i > K_i, i=1,2.$ $\beta_i, \delta_i, i=1,2$ **are all positive constants;**
(B) $r_i, \alpha_i, \tau_i\sigma_i, i=1,2$ **and** K_1 **are nonnegative sequences bounded above and below by positive constants,** K_2 **is a negative sequences bounded above and below by negative constants, and** $\alpha_1 > K_1$, $\beta_i, \delta_i, i=1,2$ **are all positive constants.**
The results obtained here generalize the main result of Fengde Chen.

***Index Terms*—Nonautonomous; Mutualism model; Discrete model; Delays; Permanence.**

I. Introduction

THROUGHOUT this paper, for any bounded sequence $\{h(n)\}$, set $h^u=\sup_{n\in N}\{h(n)\}$ and $h^l=\inf_{n\in N}\{h(n)\}$.

The aim of this paper is to investigate the persistent property of the following discrete model of mutualism

$$\begin{aligned} x_1(k+1) &= x_1(k)\exp\Big\{r_1(k)\Big[f_1(x_2(k-\tau_2(k))) \\ &\quad -x_1^{\delta_1}(k-\sigma_1(k))\Big]\Big\}, \\ x_2(k+1) &= x_2(k)\exp\Big\{r_2(k)\Big[f_2(x_1((k-\tau_1(k))) \\ &\quad -x_2^{\delta_2}(k-\sigma_2(k))\Big]\Big\}, \end{aligned} \tag{1.1}$$

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where

$$\begin{aligned} f_1(x_2(k-\tau_2(k))) &= \frac{K_1(k)+\alpha_1(k)x_2^{\beta_1}(k-\tau_2(k))}{1+x_2^{\beta_1}(k-\tau_2(k))} \\ f_2(x_1((k-\tau_1(k))) &= \frac{K_2(k)+\alpha_2(k)x_1^{\beta_2}(k-\tau_1(k))}{1+x_1^{\beta_2}(k-\tau_1(k))}. \end{aligned}$$

We assume that the coefficients of system (1.1) satisfies:

(A) $r_i, K_i, \alpha_i, \tau_i$and $\sigma_i, i=1,2$ are nonnegative sequences bounded above and below by positive constants, and $\alpha_i > K_i, i=1,2.$ $\beta_i, \delta_i, i=1,2$ are all positive constants.

(B) $r_i, \alpha_i, \tau_i\sigma_i, i=1,2$ and K_1 are nonnegative sequences bounded above and below by positive constants, K_2 is a negative sequences bounded above and below by negative constants, and $\alpha_1 > K_1$, $\beta_i, \delta_i, i=1,2$ are all positive constants.

Let $\tau=\sup_k\{\tau_i(k),\sigma_i(k), i=1,2\}$, we consider (1.1) together with the following initial conditions

$$\begin{aligned} x_i(\theta) &= \varphi_i(\theta)\geq 0, \theta\in N[-\tau,0]=\{-\tau,-\tau+1,...,0\}, \\ \varphi_i(0) &> 0. \end{aligned} \tag{1.2}$$

It is not difficult to see that solutions of (1.1)-(1.2) are well defined for all $k\geq 0$ and satisfy

$$x_i(k)>0, \quad \text{for } k\in Z, i=1,2.$$

During the last decade, many scholars investigated the dynamic behaviors of the mutualism model, see [1]-[26] and the references cited therein. Some excellent results about the existence of positive periodic solution (almost periodic solution), the persistent property of the system etc are obtained.

Li[1] proposed the following two species discrete model of mutualism

$$\begin{aligned} x_1(k+1) &= x_1(k)\exp\Big\{r_1(k)\Big[g_1(x_2(k-\tau_2(k))) \\ &\quad -x_1(k-\sigma_1(k))\Big]\Big\}, \\ x_2(k+1) &= x_2(k)\exp\Big\{r_2(k)\Big[g_2(x_1(k-\tau_1(k))) \\ &\quad -x_2(k-\sigma_2(k))\Big]\Big\}, \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} g_1(x_2(k-\tau_2(k))) &= \frac{K_1(k)+\alpha_1(k)x_2(k-\tau_2(k))}{1+x_2(k-\tau_2(k))}, \\ g_2(x_1(k-\tau_1(k))) &= \frac{K_2(k)+\alpha_2(k)x_1(k-\tau_1(k))}{1+x_1(k-\tau_1(k))}. \end{aligned}$$

Under the assumption $r_i, K_i, \alpha_i, \tau_i$ and $\sigma_i, i = 1, 2$ are periodic positive sequences with common period ω, and $\alpha_i > K_i$. By applying coincidence degree theory, he showed that system (1.1) admits at least one positive ω-periodic solution. Chen[2] argued that a general nonautonomous nonperiodic system is more appropriate, and thus, he assume that $r_i, K_i, \alpha_i,\ \tau_i$ and $\sigma_i, i = 1, 2$ are nonnegative sequences bounded above and below by positive constants, and $\alpha_i > K_i, i = 1, 2$. Under those assumption, he showed that system (1.3) is permanent.

It bring to our attention that the model (1.3) is based on the following single species discrete model:

$$x(k+1) = x(k)\exp\Big\{r_1(k)\Big[K_1(k) - x_1(k)\Big]\Big\}. \quad (1.4)$$

Already, during the past decades, in his series works, based on the traditional single species Ayala model, Fengde Chen and his coauthors ([16]-[20]) proposed several kind of nonlinear population models, and investigated the extinction, persistent, and stability property of the system. Specially, in [19], they proposed the following discrete n-species Gilpin-Ayala competition model

$$x_i(k+1) = x_i(k)\exp\Big[b_i(k) - \sum_{j=1}^{n} a_{ij}(k)(x_j(k))^{\theta_{ij}}\Big], \quad (1.5)$$

where $i = 1, 2, ..., n$; $x_i(k)$ is the density of competition species i at k-th generation. They investigated the permanence and stability of the system (1.5). Their success motivated us to propose the nonlinear mutualism model (1.1).

The aim of this paper is, by further developing the analysis technique of [2], [19], [20], to obtain a set of sufficient conditions to ensure the permanence of the system (1.1). More precisely, we will prove the following result.

Theorem 1.1. *Under the assumption (A), system (1.1) is permanent, that is, there exist positive constants $m_i, M_i, i = 1, 2$ which are independent of the solutions of system (1.1), such that for any positive solution $(x_1(k), x_2(k))^T$ of system (1.1) with initial condition (1.2), one has:*

$$m_i \le \liminf_{k\to+\infty} x_i(k) \le \limsup_{k\to+\infty} x_i(k) \le M_i, i = 1, 2.$$

Remark 1.1. When $\beta_i = \delta_i = 1$, Theorem 1.1 degenerate to Theorem 1.1 in Chen[2], thus, we generalize the main result of Chen[2] to the nonlinear case.

On the other hand, by considering the relationship of customer fish and cleaner fish, Jiang, Xie and Ye[27] established the following two species obligate mutualism model:

$$\begin{aligned} \dot{x}_1 &= r_1x_1\Big(1 - \tfrac{x_1}{N_1} + \sigma_1\tfrac{x_2}{N_2}\Big), \\ \dot{x}_2 &= r_2x_2\Big(-1 + \sigma_2\tfrac{x_1}{N_1} - \tfrac{x_2}{N_2}\Big), \end{aligned} \quad (1.6)$$

where x_1 and x_2 are the densities of first and second species at time t, respectively. r_1 is the intrinsic growth rate of the first species, r_2 is the death rate of the second species, $N_i, i = 1, 2$ are the carrying capacity of the i-th species, respectively. $\sigma_i, i = 1, 2$ reflects the efficiency of cooperation. Here, obligate means that the first species benefiting from the presence of each other, however it may also survive in the absence of each other, while the second species may survive only by association. The authors investigated the local stability property of the equilibria of system (1.6). By constructing a suitable Lyapunov function, Chen, Yang, Han et al[28] obtained sufficient conditions which ensure the global asymptotical stability of the positive equilibrium and boundary equilibrium of above system. They showed that the conditions which ensure the local stability of the nonnegative equilibria is enough to ensure their global asymptotical stability. Chen[29] and Wu et al[30] also proposed the commensalism model with one party could not survival independently. All of their study shows that with the help of other species, the system may become permanent. However, to this day, still no scholars investigated the discrete type cooperation system with one species could not survival independently. This leads us to study the dynamic behaviors of system (1.1) under the assumption (B).
In system (1.1), without the help of the first species, then the second species satisfies the equation

$$x_2(k+1) = x_2(k)\exp\Big\{r_2(k)\Big[K_2(k) - x_2^{\delta_2}(k - \sigma_2(k))\Big]\Big\},$$

Since $K_2(k)$ is negative sequence, one could easily see that $x_2(k) \to 0$ as $k \to +\infty$. Now, for system (1.1), under the assumption (B), is it possible for us to investigate the persistent or extinct property of the system? We will give the affirm answer to this problem, indeed, we have the following results:

Theorem 1.2. *Assume that*

$$\frac{r_2^l K_2^l}{1 + M_1^{\beta_2}} + r_2^u \alpha_2^u < 0 \quad (H)$$

and (B) hold, then the species x_2 will be driven to extinction, and the species x_1 is permanent, that for any positive solution $(x_1(k), x_2(k))^T$ of system (1.1) with initial condition (1.2), one has:

$$\lim_{k\to+\infty} x_2(k) = 0,$$
$$m_1 \le \liminf_{k\to+\infty} x_1(k) \le \limsup_{k\to+\infty} x_1(k) \le M_1$$

Theorem 1.3. *Assume that*

$$\frac{K_2^l + \alpha_2^l m_1^{\beta_2}}{1 + m_1^{\beta_2}} > 0 \quad (F)$$

and (B) hold, then system (1.1) is permanent.

We will prove Theorem 1.1 - 1.3 in the next section, and end the paper by a briefly discussion.

II. Proof of the main results

Now we state several lemmas which will be useful in proving of our main results.

Lemma 2.1.[2] *Assume that $\{x(k)\}$ satisfies $x(k) > 0$ and*

$$x(k+1) \le x(k)\exp\Big\{a(k) - b(k)x(k)\Big\}$$

for $k \in N$, where $a(k)$ and $b(k)$ are nonnegative sequences bounded above and below by positive constants. Then

$$\limsup_{k\to+\infty} x(k) \le \frac{1}{b^l}\exp(a^u - 1).$$

Lemma 2.2.[2] *Assume that* $\{x(k)\}$ *satisfies*

$$x(k+1) \geq x(k)\exp\Big\{a(k) - b(k)x(k)\Big\}, \quad k \geq N_0,$$

$\limsup_{k\to+\infty} x(k) \leq x^*$ *and* $x(N_0) > 0$*, where* $a(k)$ *and* $b(k)$ *are nonnegative sequences bounded above and below by positive constants and* $N_0 \in N$*. Then*

$$\liminf_{k\to+\infty} x(k) \geq \min\left\{\frac{a^l}{b^u}\exp\{a^l - b^u x^*\}, \frac{a^l}{b^u}\right\}.$$

Now we are in the position to prove the main results of this paper.

Proof of the Theorem 1.1. Let $(x_1(k), x_2(k))$ be any positive solution of system (1.1) with initial condition (1.2). From the first equation of system (1.1) it follows that

$$\begin{aligned} & x_1(k+1) \\ \leq\ & x_1(k)\exp\Big\{r_1(k)\Big[\frac{K_1(k) + \alpha_1(k)x_2^{\beta_1}(k-\tau_2(k))}{1 + x_2^{\beta_1}(k-\tau_2(k))}\Big]\Big\} \\ \leq\ & x_1(k)\exp\Big\{r_1(k)\Big[\frac{\alpha_1(k) + \alpha_1(k)x_2^{\beta_1}(k-\tau_2(k))}{1 + x_2^{\beta_1}(k-\tau_2(k))}\Big]\Big\} \\ =\ & x_1(k)\exp\Big\{r_1(k)\alpha_1(k)\Big\} \\ \leq\ & x_1(k)\exp\Big\{r_1^u\alpha_1^u\Big\}. \end{aligned} \tag{2.1}$$

By using (2.1), one could easily obtain that

$$x_1(k - \sigma_1(k)) \geq x_1(k)\exp\{-r_1^u\alpha_1^u\tau\}. \tag{2.2}$$

Substituting (2.2) into the first equation of system (1.1), it follows that

$$\begin{aligned} x_1(k+1) \ \leq\ & x_1(k)\exp\Big\{r_1^u\alpha_1^u \\ & -r_1^l\exp\{-\delta_1 r_1^u\alpha_1^u\tau\}x_1^{\delta_1}(k)\Big\}. \end{aligned} \tag{2.3}$$

That is

$$\begin{aligned} x_1^{\delta_1}(k+1) \ \leq\ & x_1^{\delta_1}(k)\exp\Big\{\delta_1 r_1^u\alpha_1^u \\ & -r_1^l\delta_1\exp\{-\delta_1 r_1^u\alpha_1^u\tau\}x_1^{\delta_1}(k)\Big\}. \end{aligned} \tag{2.4}$$

Set

$$u_1(k) = x_1^{\delta_1}(k), \tag{2.5}$$

then,

$$\begin{aligned} u_1(k+1) \ \leq\ & u_1(k)\exp\Big\{\delta_1 r_1^u\alpha_1^u \\ & -r_1^l\delta_1\exp\{-\delta_1 r_1^u\alpha_1^u\tau\}u_1(k)\Big\}. \end{aligned} \tag{2.6}$$

As a direct corollary of Lemma 2.1, according to (2.6), one has

$$\limsup_{k\to+\infty} u_1(k) \leq \frac{1}{r_1^l\delta_1}\exp\Big\{\delta_1 r_1^u\alpha_1^u(\tau+1) - 1\Big\} \stackrel{\text{def}}{=} u_1^*. \tag{2.7}$$

Consequently,

$$\limsup_{k\to+\infty} x_1(k) \leq \left(\frac{1}{r_1^l\delta_1}\right)^{\frac{1}{\delta_1}}\exp\Big\{r_1^u\alpha_1^u(\tau+1) - \frac{1}{\delta_1}\Big\} \stackrel{\text{def}}{=} M_1. \tag{2.8}$$

By using the second equation of system (1.1), similar to the analysis of (2.1)-(2.8), we can obtain

$$\limsup_{k\to+\infty} x_2(k) \leq \left(\frac{1}{r_2^l\delta_2}\right)^{\frac{1}{\delta_2}}\exp\Big\{r_2^u\alpha_2^u(\tau+1) - \frac{1}{\delta_2}\Big\} \stackrel{\text{def}}{=} M_2. \tag{2.9}$$

For any small positive constant $\varepsilon > 0$, from (2.8)-(2.9) it follows that there exists a $N_1 > 0$ such that for all $k > N_1$ and $i = 1, 2$,

$$x_i(k) < M_i + \varepsilon. \tag{2.10}$$

For $k \geq N_1 + \tau$, from (2.10) and the first equation of system (1.1), we have

$$\begin{aligned} & x_1(k+1) \\ =\ & x_1(k)\exp\Big\{r_1(k)\Big[\frac{K_1(k) + \alpha_1(k)x_2^{\beta_1}(k-\tau_2(k))}{1 + x_2^{\beta_1}(k-\tau_2(k))} \\ & -x_1^{\delta_1}(k-\sigma_1(k))\Big]\Big\} \\ \geq\ & x_1(k)\exp\Big\{r_1(k)\Big[\frac{K_1(k) + K_1(k)x_2^{\beta_1}(k-\tau_2(k))}{1 + x_2^{\beta_1}(k-\tau_2(k))} \\ & -x_1^{\delta_1}(k-\sigma_1(k))\Big]\Big\} \\ \geq\ & x_1(k)\exp\Big\{r_1^lK_1^l - r_1^u(M_1+\varepsilon)^{\delta_1}\Big\}. \end{aligned} \tag{2.11}$$

Thus, by using (2.11) we obtain

$$\begin{aligned} x_1(k - \sigma_1(k)) \ \leq\ & x_1(k)\exp\Big\{-\Big[r_1^lK_1^l \\ & -r_1^u(M_1+\varepsilon)^{\delta_1}\Big]\tau\Big\}. \end{aligned} \tag{2.12}$$

Substituting (2.12) into the first equation of system (1.1), for $t \geq N_1 + \tau$, it follows that

$$x_1(k+1) \ \geq\ x_1(k)\exp\Bigg[r_1^lK_1^l - H_1x_1^{\delta_1}(k)\Bigg], \tag{2.13}$$

where

$$H_1 = r_1^u\exp\Big\{-\delta_1\Big[r_1^lK_1^l - r_1^u(M_1+\varepsilon)^{\delta_1}\Big]\tau\Big\},$$

and so

$$x_1^{\delta_1}(k+1) \geq x_1^{\delta_1}(k)\exp\Bigg[\delta_1r_1^lK_1^l - \delta_1H_1x_1^{\delta_1}(k)\Bigg]. \tag{2.14}$$

Set

$$u_1(k) = x_1^{\delta_1}(k), \tag{2.15}$$

then,

$$u_1(k+1) \geq u_1(k)\exp\Bigg[\delta_1r_1^lK_1^l - \delta_1H_1u_1(k)\Bigg], \tag{2.16}$$

Thus, as a direct corollary of Lemma 2.2, according to (2.7) and (2.16), one has

$$\liminf_{k\to+\infty} u_1(k) \geq \min\Big\{A_{1\varepsilon}, A_{2\varepsilon}\Big\}, \tag{2.17}$$

where

$$A_{1\varepsilon} = \frac{r_1^l K_1^l}{r_1^u} \exp\Big\{\delta_1\Big[r_1^l K_1^l - r_1^u (M_1+\varepsilon)^{\delta_1}\Big]\tau\Big\}, \tag{2.18}$$

$$A_{2\varepsilon} = A_{1\varepsilon} \exp\Bigg[\delta_1 r_1^l K_1^l - \delta_1 H_1 u_1^*\Bigg]. \tag{2.19}$$

Noticing that

$$\begin{aligned} & r_1^l K_1^l - r_1^u (M_1+\varepsilon)^{\delta_1} \\ \leq\ & K_1^l - M_1^{\delta_1} \\ =\ & K_1^l - \frac{1}{r_1^l \delta_1}\exp\Big\{r_1^u \alpha_1^u \delta_1(\tau+1) - 1\Big\} \\ \leq\ & K_1^l - \frac{1}{r_1^l \delta_1^l} r_1^u \alpha_1^u \delta_1 (\tau+1) \\ <\ & K_1^l - \alpha_1^u \\ \leq\ & 0, \end{aligned} \tag{2.20}$$

and so, by using (2.20), one has

$$\begin{aligned} & \delta_1 r_1^l K_1^l - \delta_1 H_1 u_1^* \\ =\ & \delta_1 r_1^l K_1^l - \delta_1 H_1 \frac{1}{r_1^l \delta_1}\exp\Big\{\delta_1 r_1^u \alpha_1^u (\tau+1) - 1\Big\} \\ <\ & \delta_1 r_1^l K_1^l - \frac{r_1^u}{r_1^l}\exp\Big\{\delta_1 r_1^u \alpha_1^u(\tau+1) - 1\Big\} \\ \leq\ & \delta_1 r_1^l K_1^l - \delta_1 r_1^u \alpha_1^u (\tau+1) \\ \leq\ & 0. \end{aligned} \tag{2.21}$$

Therefore,

$$\liminf_{k\to+\infty} u_1(k) \geq A_{2\varepsilon}. \tag{2.22}$$

And so,

$$\liminf_{k\to+\infty} x_1(k) \geq \Big(A_{2\varepsilon}\Big)^{\frac{1}{\delta_1}}.$$

Setting $\varepsilon \to 0$, then

$$\liminf_{k\to+\infty} x_1(k) \geq \frac{1}{2}\Big(A_2\Big)^{\frac{1}{\delta_1}} \stackrel{\text{def}}{=} m_1. \tag{2.23}$$

where

$$\begin{aligned} A_2 &= A_1 \exp\Bigg[\delta_1 r_1^l K_1^l - \delta_1 H_1^* u_1^*\Bigg] \\ A_1 &= \frac{r_1^l K_1^l}{r_1^u}\exp\Big\{\delta_1\Big[r_1^l K_1^l - r_1^u (M_1)^{\delta_1}\Big]\tau\Big\}. \end{aligned} \tag{2.24}$$

Similarly to the analysis of (2.11)-(2.24), by applying (2.9), from the second equation of system (1.1), we also have that

$$\liminf_{k\to+\infty} x_2(k) \geq \frac{1}{2}\Big(B_2\Big)^{\frac{1}{\delta_2}} \stackrel{\text{def}}{=} m_2 > 0, \tag{2.25}$$

where

$$\begin{aligned} B_2 &= B_1 \exp\Bigg[\delta_2 r_2^l K_2^l - \delta_2 H_2 u_2^*\Bigg] \\ B_1 &= = \frac{r_2^l K_2^l}{r_2^u}\exp\Big\{\delta_2\Big[r_2^l K_2^l - r_2^u (M_2)^{\delta_2}\Big]\tau\Big\} \\ H_1^* &= r_1^u \exp\Big\{-\delta_1\Big[r_1^l K_1^l - r_1^u M_1^{\delta_1}\Big]\tau\Big\} \\ H_2 &= r_2^u \exp\Big\{-\delta_2\Big[r_2^l K_2^l - r_2^u (M_2)^{\delta_2}\Big]\tau\Big\}, \\ u_2^* &= \frac{1}{r_2^l \delta_2}\exp\Big\{\delta_2 r_2^u \alpha_2^u (\tau+1) - 1\Big\}. \end{aligned} \tag{2.26}$$

(2.8), (2.9) (2.23) and (2.26) show that system (1.1) is permanent. The proof of the Theorem 1.1 is completed.

Proof of the Theorem 1.2. Let $(x_1(k), x_2(k))$ be any positive solution of system (1.1) with initial condition (1.2). Similar to the analysis of (2.1)-(2.8), (2.11)-(2.23), we can obtain

$$m_1 \leq \liminf_{k\to+\infty} x_1(k) \leq \limsup_{k\to+\infty} x_1(k) \leq M_1. \tag{2.27}$$

From (H), there exists a small enough positive constant ε such that

$$\frac{r_2^l K_2^l}{1 + (M_1+\varepsilon)^{\beta_2}} + r_2^u \alpha_1^u < 0. \tag{2.28}$$

From (2.27), for above ε, there exists a $N_2 > 0$ such that for $k > N_2$,

$$x_1(k) < M_1 + \varepsilon. \tag{2.29}$$

Substituting (2.29) into the second equation of system (1.1), for $k \geq N_2 + \tau$, it follows that

$$\begin{aligned} & x_2(k+1) \\ =\ & x_2(k)\exp\Big\{r_2(k)\Big[\frac{K_2(k) + \alpha_2(k) x_1^{\beta_2}(k-\tau_1(k))}{1 + x_1^{\beta_2}(k-\tau_1(k))} \\ & - x_2^{\delta_2}(k - \sigma_2(k))\Big]\Big\} \\ \leq\ & x_2(k)\exp\Big\{r_2(k)\Big[\frac{K_2(k)}{1 + x_1^{\beta_2}(k-\tau_1(k))} + \alpha_2(k)\Big]\Big\} \\ \leq\ & x_2(k)\exp\Big\{\frac{r_2^l K_2^l}{1 + (M_1+\varepsilon)^{\beta_2}} + r_2^u \alpha_2^u\Big\}. \end{aligned} \tag{2.30}$$

For condition (2.28), there exists small enough positive $\gamma > 0$, such that

$$\frac{r_2^l K_2^l}{1 + (M_1+\varepsilon)^{\beta_2}} + r_2^u \alpha_2^u < -\gamma < 0. \tag{2.31}$$

Substituting (2.31)to (2.30), for all $k \geq N_2 + \tau$, it follows

$$x_2(k+1) \leq x_2(k)\exp\{-\gamma\}. \tag{2.32}$$

Therefore,

$$x_2(k+1) < x_2(N_2+\tau)\exp\Big\{-[k - (N_2+\tau)]\gamma\Big\}, \tag{2.33}$$

which yields

$$\lim_{k\to+\infty} x_2(k) = 0. \tag{2.34}$$

The proof of the Theorem 1.2 is completed.

Proof of the Theorem 1.3. Let $(x_1(k), x_2(k))$ be any positive solution of system (1.1) with initial condition (1.2). Let $(x_1(k), x_2(k))$ be any positive solution of system (1.1) with initial condition (1.2). Similar to the analysis of (2.1)-(2.24), we can obtain

$$\begin{aligned} m_1 \le \liminf_{k\to+\infty} x_1(k) &\le \limsup_{k\to+\infty} x_1(k) \le M_1 \\ \limsup_{k\to+\infty} x_2(k) &\le M_2. \end{aligned} \tag{2.35}$$

From (F), there exists a small enough positive constant ε, such that

$$\frac{K_2^l + \alpha_2^l (m_1-\varepsilon)^{\beta_2}}{1+(m_1-\varepsilon)^{\beta_2}} > 0. \tag{2.36}$$

From (2.35) for above ε, there exists a $N_3 > 0$, such that for $k > N_3$

$$x_1(k) > m_1 - \varepsilon,\ x_2(k) < M_2 + \varepsilon. \tag{2.37}$$

Substituting (2.37) into the second equation of system (1.1), for $k \ge N_3 + \tau$, it follows that

$$\begin{aligned} & x_2(k+1) \\ = \ & x_2(k)\exp\Big\{r_2(k)\Big[\frac{K_2(k)+\alpha_2(k)x_1^{\beta_2}(k-\tau_1(k))}{1+x_1^{\beta_2}(k-\tau_1(k))} \\ & -x_2^{\delta_2}(k-\sigma_2(k))\Big]\Big\} \\ \ge \ & x_2(k)\exp\Big\{\frac{r_2^l[K_2^l+\alpha_2^l(m_1-\varepsilon)^{\beta_2}]}{1+(m_1-\varepsilon)^{\beta_2}} - r_2^u(M_2+\varepsilon)^{\delta_1}\Big\}. \end{aligned} \tag{2.38}$$

By using (2.38) we obtain

$$\begin{aligned} x_2(k-\sigma_2(k)) \le \ & x_2(k)\exp\Big\{-\Big[r_2^l\frac{K_2^l+\alpha_2^l(m_1-\varepsilon)^{\beta_2}}{1+(m_1-\varepsilon)^{\beta_2}} \\ & -r_2^u(M_2+\varepsilon)^{\delta_2}\Big]\tau\Big\}. \end{aligned} \tag{2.39}$$

Substituting (2.39) into the second equation of system (1.1), for $k \ge N_3 + \tau$, it follows that

$$\begin{aligned} x_2(k+1) \ge \ & x_2(k)\exp\Big\{\frac{r_2^l[K_2^l+\alpha_2^l(m_1-\varepsilon)^{\beta_2}]}{1+(m_1-\varepsilon)^{\beta_2}} \\ & - H_3 x_2^{\delta_2}(k)\Big\}, \end{aligned} \tag{2.40}$$

where

$$H_3 = r_2^u \exp\Big\{-\delta_2\Big[r_2^l\frac{K_2^l+\alpha_2^l(m_1-\varepsilon)^{\beta_2}}{1+(m_1-\varepsilon)^{\beta_2}} - r_2^u(M_2+\varepsilon)^{\delta_2}\Big]\tau\Big\},$$

and so

$$\begin{aligned} x_2^{\delta_2}(k+1) \ge \ & x_2^{\delta_2}(k)\exp\Big\{\frac{\delta_2 r_2^l[K_2^l+\alpha_2^l(m_1-\varepsilon)^{\beta_2}]}{1+(m_1-\varepsilon)^{\beta_2}} \\ & - \delta_2 H_3 x_2^{\delta_2}(k)\Big\}, \end{aligned} \tag{2.41}$$

. Set

$$u_2(k) = x_2^{\delta_2}(k), \tag{2.42}$$

then,

$$\begin{aligned} u_2(k+1) \ge \ & u_1(k)\exp\Big\{\frac{\delta_2 r_2^l[K_2^l+\alpha_2^l(m_1-\varepsilon)^{\beta_2}]}{1+(m_1-\varepsilon)^{\beta_2}} \\ & - \delta_2 H_3 u_2(k)\Big\}. \end{aligned} \tag{2.43}$$

According to (2.9), we have

$$\limsup_{k\to+\infty} u_2(k) \le \frac{1}{r_2^l\delta_2}\exp\Big\{\delta_2 r_2^u \alpha_2^u(\tau+1)-1\Big\} = u_2^*. \tag{2.44}$$

Thus, as a direct corollary of Lemma 2.2, according to (2.43) and (2.44), one has

$$\liminf_{k\to+\infty} u_2(k) \ge \min\Big\{C_{1\varepsilon}, C_{2\varepsilon}\Big\}, \tag{2.45}$$

where

$$\begin{aligned} C_{1\varepsilon} = \ & \frac{r_2^l[K_2^l+\alpha_2^l(m_1-\varepsilon)^{\beta_2}]}{r_2^u[1+m_1-\varepsilon]^{\beta_2}}\exp\Big\{\delta_2 \\ & \Big[\ r_2^l\frac{K_2^l+\alpha_2^l(m_1-\varepsilon)^{\beta_2}}{1+(m_1-\varepsilon)^{\beta_2}} \\ & - r_2^u(M_2+\varepsilon)^{\delta_2}\Big]\tau\Big\}, \end{aligned} \tag{2.46}$$

$$\begin{aligned} C_{2\varepsilon} = \ & C_{1\varepsilon}\exp\Big\{\frac{\delta_2 r_2^l[K_2^l+\alpha_2^l(m_1-\varepsilon)^{\beta_2}]}{1+(m_1-\varepsilon)^{\beta_2}} \\ & - \delta_2 H_3 u_2^*\Big\}. \end{aligned} \tag{2.47}$$

And so

$$\liminf_{k\to+\infty} x_2(k) \ge \min\Big\{\Big(C_{1\varepsilon}\Big)^{\frac{1}{\delta_1}}, C_{2\varepsilon}\Big)^{\frac{1}{\delta_2}}\Big\}.$$

Setting $\varepsilon \to 0$, then

$$\liminf_{k\to+\infty} x_2(k) \ge \frac{1}{2}\min\Big\{\Big(C_1\Big)^{\frac{1}{\delta_1}}, \Big(C_2\Big)^{\frac{1}{\delta_1}}\Big\} \overset{\text{def}}{=} m_3. \tag{2.48}$$

where

$$\begin{aligned} C_2 = \ & C_1\exp\Big\{\frac{\delta_2 r_2^l[K_2^l+\alpha_2^l m_1^{\beta_2}]}{1+m_1^{\beta_2}} - \delta_2 H_3^* u_2^*\Big\} \\ C_1 = \ & \frac{r_2^l(K_2^l+\alpha_2^l m_1^{\beta_2})}{r_2^u(1+m_1^{\beta_2})}\exp\Big\{\delta_2\Big[\frac{r_2^l(K_2^l+\alpha_2^l m_1^{\beta_2})}{1+m_1^{\beta_2}} \\ & - r_2^u M_2^{\delta_2}\Big]\tau\Big\}, \end{aligned} \tag{2.49}$$

where

$$H_3^* = r_2^u \exp\Big\{-\delta_2\Big[r_2^l\frac{K_2^l+\alpha_2^l m_1^{\beta_2}}{1+m_1^{\beta_2}} - r_2^u M_2^{\delta_2}\Big]\tau\Big\}. \quad (2.50)$$

(2.35), (2.49) and (2.50) show that system (1.1) is permanent. The proof of the Theorem 1.3 is completed.

III. Discussion

In this paper, we proposed the nonlinear discrete cooperative system (1.1), which can be seen as the generalization of the model (1.2). Under the assumption (A), we show that system (1.1) is permanent, the result generalize the main result of Chen [2]. Also, it bring to our attention that in the nature, many species could not be survival without the help of the other species, this motivated us to study the dynamic behaviors of the system (1.1) under the assumption (B). By developing some new analysis technique, we finally obtain sufficient conditions which ensure the partial survival or permanence of the system.

IV. Declarations

Competing interests

The author declare that there is no conflict of interests.

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Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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