Modeling Nonlinear Stochastic Filter by Volterra Transfer Functions

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***Abstract*—The Volterra series are widely employed to represent the input-output relationship of a nonlinear system. This representation is based on the Volterra transfer functions. The Volterra transfer functions are evaluated by a so-called harmonic probing method. In this paper the simulation of a stochastic resonance effect using a nonlinear device is discussed. Three cases are under consideration: harmonic input, Gaussian noise input, and sine wave plus Gaussian noise input. The output signal is determined by the Runge-Kutta method and in terms of the Volterra series. Expressions for the output harmonics were derived. If the Volterra transfer functions are known, the items of interest regarding the output signal can be obtained by substituting them into the general formulas derived from the Volterra series representation. These items include expressions for the output power spectrum. Frequency dependences of the output power spectral density of a non-linear stochastic filter, as well as amplitude characteristics are calculated and analyzed for the case of different parameter values of the filter. Numerical calculations of the output signal by the Runge-Kutta method were carried out to estimate the accuracy and reliability of results obtained. The comparative analysis has shown that the dependences of output signal power spectrum densities, obtained using the numerical calculation and Volterra series, are of the same character.**

***Index Terms*—stochastic resonance, power spectral density, Volterra series, Volterra transfer functions, white Gaussian noise.**

I. Introduction

In the nonlinear systems analysis it is somewhat difficult to obtain explicit expressions for output signal characteristics. Wide prospects are offered by the analysis method using transfer functions, which is developed in more detail for linear systems. At the same time, there is an opportunity either to calculate the output signal spectrum, using a spectral analysis method, or to derive the expression for an output signal instant value by means of the convolution integral. Here it is supposed that the transfer function and the pulse response, being an imposing integral kernel, are related by the Fourier transform [1], [2].

Using a graphic representation of a nonlinear system, it is possible to separate the linear and nonlinear parts and their corresponding characteristics. As is shown in [3], any functional nonlinear system without a feedback, consisting of inertial linear subsystems and inertial nonlinearity, can be described by the Volterra series. Such systems are refer red to as "system with memory". [4]. Thus, the nonlinear system response can be calculated by means of the transfer functions basing on the Volterra series.

The Volterra series is a type of functional series which relates the system input, $x(t)$ [3] – [5] to the system output, $y(t)$, as:

$$y(t)=\sum_{n=1}^{\infty}\frac{1}{n!}\int_{-\infty}^{\infty}du_1\ldots\int_{-\infty}^{\infty}du_n g_n(u_1,\ldots,u_n)\prod_{r=1}^{n}x(t-u_r), \quad (1)$$

where $y(t)$ is the output, $x(t)$ is the input, and the kernels $g_n(u_1,...,u_n)$ describe the nonlinear system. The first-order kernel $g_1(u_1)$ is simply a familiar pulse response of the linear circuit. The higher-order kernels are the higher-order pulse responses and serve to characterize various orders of nonlinearity. The coefficient $1/n!$ has been introduced by A. Bedrosian and D. Rice [4] to simplify many of equations.

Analysis of a nonlinear component of the system is based on the multi-dimensional Fourier transform. As is shown in [3], [4], the n-fold Fourier transform is described by the expression

$$G_n(f_1,\ldots,f_n)=\int_{-\infty}^{\infty}du_1\ldots\int_{-\infty}^{\infty}du_n g_n(u_1,\ldots,u_n)\times$$
$$\times\exp[-j(f_1u_1+\ldots+f_nu_n)]. \quad (2)$$

Here G_0 is identically zero as our Volterra series starts with $n=1$ and $G_1(f_1)$ is the transfer function of the system's linear part. For linear systems possible output frequencies are the same as the input frequencies. However, for the nonlinear systems, the relationship between the input and output frequencies is more complicated [1], [2].

Thus, the transform of the nth-order Volterra kernel is seen to be analogous to the nth -order Volterra transfer function. In many cases G_n can be obtained without first calculation of kernels $g_n(u_1,...,u_n)$.

The complete formulas are infinite series. Fortunately, in the study of communication systems it is often possible to neglect terms of the Volterra series of the order higher then the second or third one. They are usually used because of the fast increase in the computation complexity [3], [4]. The n-fold Fourier transform significantly simplifies the solution of a large number of problems.

To calculate the transfer functions, we use the harmonic input method [4]. This method is based on the fact that a harmonic input must result in a harmonic output when (1) takes place. The system under consideration in [3], [4] is described by the nonlinear differential equation

$$F(d/dt)y+\sum_{l=2}^{\infty}a_l y^l = x(t) \quad (3)$$

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under condition that the system satisfies the causality principle ($y(t)$ vanishes if $x(t)$ is zero). We assumed that one and only one such solution exists (this is proved in [3]) and the system is stable. $F(d/dt)$ is a polynomial in d/dt, and the coefficients in $F(d/dt)$ and the coefficients a_l are independent on *t*, *x* and *y*.

As is shown in [3], equation (3) has a unique solution for the given initial conditions.

The Volterra transfer functions for (3) can be written as [4]

$$G_n(f_1,...,f_n) = -\frac{\sum_{l=2}^{n} a_l G_n^{(l)}(f_1,...,f_n)}{F(j\omega_1+...+j\omega_n)}.$$

The last equation is the recurrence relation as $G_n^{(l)}$ is given by

$$G_n^{(l)}(f_1,...,f_n) = l!\sum_{(v;l,n)}\sum_{l=2}^{n} a_l G_{v1}(f_1,...,f_{v1})\times$$
$$\times G_{v2}(f_{v1+1},...,f_{v1+v2})...G_{vl}(f_\mu,...,f_n)$$

for the *n*-fold Fourier transform of the *n*-th kernel in the Volterra series for $[y(t)]^l$, l being a positive integer, and $1 \le l \le n$. $G_n^{(l)}(f_1,...,f_n)$ is zero for $l > n$ and $G_n^{(n)}(f_1,...,f_n)$ equals to $n!G_1(f_1)G_1(f_2)...G_1(f_n)$.

It is usual to regard the noise in the system as a negative factor and so the noise abatement is a problem of current importance for radio systems. Low-noise devices and methods for noise immunity increase, noise-proof codes and digital communication signals with necessary correlation properties are developed. However, the research conducted recently in the field of theoretical and experimental physics has shown that in some cases a weak input signal can be amplified and optimized with the assistance of noise [6] – [8]. The integral characteristics of the process at the system output, such as the spectral power amplification, the signal-to-noise ratio (SNR) contains a peak at a certain optimal noise level.

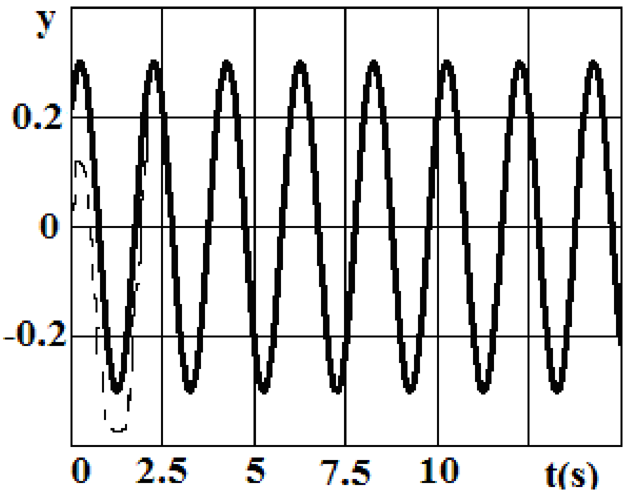

Fig. 1. The output signal in the case of a sinusoidal input signal obtained by different methods: Volterra series (solid line); Runge-Kutta method (dotted line)

The notion of a stochastic resonance (SR) defines a group of phenomena where the nonlinear system response to a weak input signal can be significantly increased by appropriate tuning of the noise intensity. SR refers to a generic physical phenomenon typical for nonlinear systems.

A weak input signal significantly increases with noise intensity increasing and reaches its maximum at a certain noise level in the nonlinear systems where SR occurs.

Let us consider a nonlinear system which can be described by the following stochastic differential equation [6], [7]

$$\frac{dy}{dt} = ay(t) - by(t)^3 + x(t), \quad (4)$$

where *a* and *b* are positive, usually given in terms of system parameters, $x(t) = s(t) + n(t)$, $s(t) = A\sin(2\pi f_0 t + \varphi)$ is the driving signal, $n(t)$ is the input noise. The input signal consists of the driving signal $s(t)$ and the additive white Gaussian noise $n(t)$ [6], [8]. This equation describes a stochastic resonance effect (SR) [6] – [8]. The Volterra transfer functions for $y(t)$ are given in Table I for the general case (equation 3) and for the SR equation (equation 4).

The solution of the nonlinear equation can be obtained in terms of the Volterra transfer functions without calculation, firstly, of kernels $g_n(u_1,...,u_n)$.

II. Sinusoidal Input

If there is no noise, the input signal can be written as $x(t) = s(t) = A\sin(2\pi f_0 t + \varphi)$. We solve the SR equation for this case. The leading terms of the output signal are given in Table II.

We calculate the output signal of the non-linear system by the Runge-Kutta method and by the Volterra series. The calculation results are given in Fig. 1 for $A = 1$, $f_0 = 0.5$ Hz.

Results from Fig.1 show that the numerical calculation by the Runge-Kutta method has a transient having duration of about two signal periods. Further the output signals calculation results obtained by the both methods are coinciding.

Let us determine the power spectral density (PSD) of the output signal. The powers of the first and third harmonics are given in Table III.

Results in Table III show that the PSD of the output signal decreases drastically with its frequency increasing. The third harmonic also increases sharply with sine signal amplitude increasing, that is a negative factor. Therefore, to obtain the powerful output signal, the input sine wave amplitude, as well as its frequency, should be low.

III. Gaussian Noise Input

In the following the input signal $x(t)$ is a zero-mean stationary Gaussian process with a two-sided power spectrum, $W_I(f)$. The output signal $y(t)$ is a stationary process, and the ensemble average $\langle [y(t)]^i \rangle$ and associated cumulants k_i do not change with t.

The leading terms for the 2d moment of $y(t)$ are given in Table IV, where the first column contains the common expressions obtained in [4] and the second column contains the SR calculation results.

The result can be written as

$$\langle y^2(t)\rangle \approx \begin{cases} W_I/2a - 3bW_I^2/4a^3, & \text{if } a>0, \\ -W_I/2a + 3bW_I^2/4a^3, & \text{if } a<0. \end{cases}$$

Using the Volterra transfer functions, we determine the two-sided power spectrum of the output signal $W_y(f)$. The leading terms in the series for the two-sided power spectrum $W_y(f)$ of $y(t)$ are shown in Table V.

The components of the double-sided power spectrum of the output signal for the initial values n are

$$W_y(f) \approx \begin{cases} W_I \dfrac{(3bW_I - 2a^2)^2 + 4a^2\omega^2}{4a^2(a^2+\omega^2)^2} + \dfrac{9b^2}{2(a^2+\omega^2)a^2(\omega^2+9a^2)} W_I^3, & \text{if } a>0, \\ W_I \dfrac{(3bW_I + 2a^2)^2 + 4a^2\omega^2}{4a^2(a^2+\omega^2)^2} + \dfrac{9b^2}{2(a^2+\omega^2)a^2(\omega^2+9a^2)} W_I^3 & \text{if } a<0. \end{cases}$$

Using this formula, we construct a graph of the power spectral density (PSD) output signal and compare it with the numerical calculation by the Runge-Kutta method (Fig. 2).

The graphs are similar: the output signal PSD of a stochastic nonlinear filter decreases with its frequency increasing, that is characteristic for the nonlinear systems [1], [2].

IV. Sine Wave Plus Noise Input

In the following $x(t) = A\cos\omega t + n(t)$, where $n(t)$ is a zero-mean stationary Gaussian process with the two-sided power spectrum $W_I(f)$.

The ensemble average of $x(t)$ at time t is $\langle x(t)\rangle = A\cos\omega t$. Similarly, the ensemble average of $y(t)$ consists of a sum of sinusoidal harmonics of $\cos\omega t$. The leading terms of the expression for $\langle y(t)\rangle$ are shown in Table VI.

If $n(t)$ is identically zero, $W_I(f)$ is zero, and Table VI reduces to Table IV if $x(t) = A\cos\omega t$. If $A = 0$, Table VI reduces to Table V if $x(t)$ is a Gaussian noise.

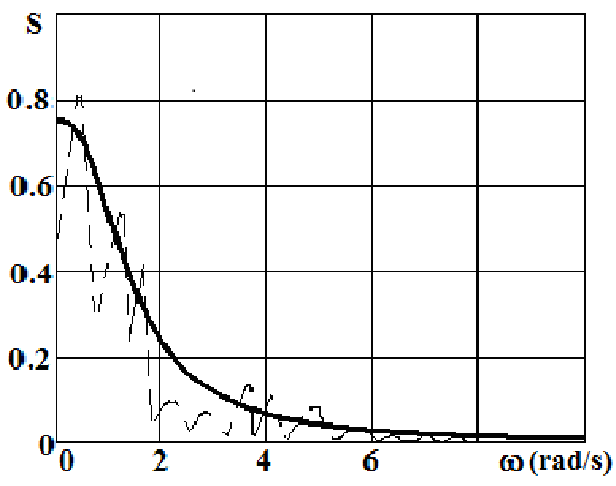

Fig. 2. The output signal PSD in the case of a sinusoidal input obtained by different methods: Volterra series (black line); Runge-Kutta method (blue line), $a = b = 1$

It is possible to study the influence of the PSD noise, sine wave amplitude and system parameters on the output signal.

Having summed up components PSD of the output signal, we will get a resultant power at the stochastic filter output (Fig. 3).

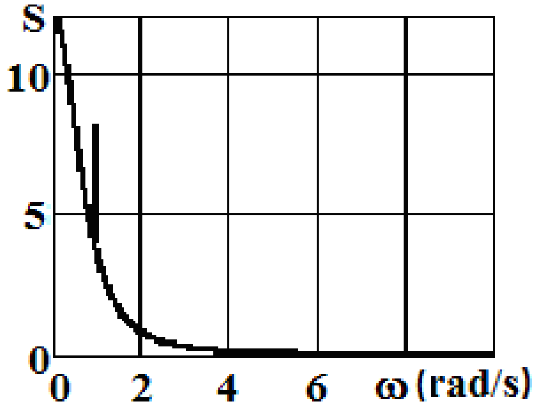

Fig. 3. The power spectral density of an output signal for different values of input noise power ($A=1$; $W_I=1$; $\omega_0=1$)

The spectral power density curve, which is characteristic for nonlinear systems [9], drastically falls with frequency increasing. Since ω = 4 Gz the value of spectral power density is almost equal to 0.

CONCLUSIONS

The Volterra series is a powerful tool that can be used to describe a wide class of nonlinear systems. In this paper to develop algorithms we have applied the Volterra series for nonlinear stochastic filter analysis.

The Volterra transfer functions for a nonlinear stochastic filter were determined. The solution of the nonlinear equation can be obtained in terms of the Volterra transfer functions without calculation of kernels.

Three cases were considered: sinusoidal input, Gaussian noise input, and sine wave plus Gaussian noise input.

The sinusoidal input investigation results show the following:

- in the numerical calculation by the Runge-Kutta method a transient takes place during about two signal periods. Further the output signal calculation results, obtained using the both means (numerical calculation by the Runge-Kutta method and Volterra series) are coinciding.
- the output signal PSD decreases drastically with its frequency increasing. The third harmonic also increases sharply with sine amplitude increasing that is a negative factor.

The Gaussian noise input investigation results show the following:

- the graphs of the PSD output signal (numerical calculation by the Runge-Kutta method and Volterra series) are of the same character, the output signal PSD of the stochastic nonlinear filter decreases with its frequency increasing, that is a feature of nonlinear systems.

The expressions for the PSD output signal of the sine wave plus Gaussian noise input are derived. We expected that they can be applied for studying the influence of the PSD noise, sine wave amplitude and system parameters.

APPENDIX

TABLE I
VOLTERRA TRANSFER FUNCTIONS

	Volterra transfer functions for (3) [2]	Volterra transfer functions for (5)
$G_1(f_1)$	$1/F(j\omega_1)$	$\dfrac{1}{-a+j\omega_1}$
$G_2(f_1,f_2)$	$-2a_2G_1(f_1)G_1(f_2)/F(j\omega_1+j\omega_2)$	0
$G_3(f_1,f_2,f_3)$	$-\dfrac{2a_2\sum_3' G_1(f_1)G_2(f_2,f_3)+6a_3G_1(f_1)G_1(f_2)G_1(f_3)}{F(j\omega_1+j\omega_2+j\omega_3)}$	$\dfrac{-6b}{(-a+j\omega_1)(-a+j\omega_2)(-a+j\omega_3)(-a+j\omega_1+j\omega_2+j\omega_3)}$

TABLE II
THE LEADING TERMS OF THE OUTPUT SIGNAL

The output signal for .(3) [2]	The output signal for (4)
$\left[\dfrac{A^2}{4}G_2(f_0,-f_0)+...\right]$	0
$e^{j\omega_0 t}\left[\dfrac{A}{2}G_1(f_0)+\dfrac{A^3}{16}G_3(f_0,f_0,-f_0)+...\right]+$ $+e^{-j\omega_0 t}\left[\dfrac{A}{2}G_1(-f_0)+\dfrac{A^3}{16}G_3(-f_0,-f_0,f_0)+...\right]$	$\dfrac{A}{(a^2+\omega_0^2)}\left[-a+\dfrac{3A^2b(-a^2+\omega_0^2)}{4(a^2+\omega_0^2)^2}\right]\cos\omega_0 t+$ $+\dfrac{A}{(a^2+\omega_0^2)}\left[\omega_0+\dfrac{3A^2b\omega_0}{2(a^2+\omega_0^2)^2}\right]\sin\omega_0 t$
$e^{j2\omega_0 t}\left[\dfrac{A^2}{8}G_2(f_0,f_0)+...\right]+$ $+e^{-j\omega_0 t}\left[\dfrac{A^2}{8}G_2(-f_0,-f_0)+...\right]$	0
$e^{j3\omega\omega_0 t}\left[\dfrac{A^3}{48}G_3(f_0,f_0,f_0)+...\right]+$ $+e^{-j3\omega\omega_0 t}\left[\dfrac{A^3}{48}G_3(-f_0,-f_0,-f_0)+...\right]+...$	$\dfrac{A^3b}{4(a^2+\omega_0^2)^3(a^2+9\omega_0^2)}\left[\begin{matrix}-(a^4-12a^2\omega_0^2+3\omega_0^4)\cos 3\omega_0 t+\\+2a\omega_0(5\omega_0^2-3a^2)\sin 3\omega_0 t\end{matrix}\right]$

TABLE III
THE PSD OF THE OUTPUT SIGNAL

	The PSD of the output signal for . (3) [2]	The PSD of the output signal for .(4)
S_{ω_0}	$\left\lvert\dfrac{A}{2}G_1(f_0)+\dfrac{A^3}{16}G_3(f_0,f_0,-f_0)\right\rvert^2$	$\dfrac{a^2A^2}{4(a^2+\omega_0^2)^2}-\dfrac{3A^4ba}{8(a^2+\omega_0^2)^3}+\dfrac{9A^6b^2}{64(a^2+\omega_0^2)^4}+\dfrac{A^2\omega_0^2}{4(a^2+\omega_0^2)^2}$
$S_{3\omega_0}$	$\left\lvert\dfrac{P^3}{48}G_3(f_0,f_0,f_0)\right\rvert^2$	$\dfrac{A^6b^2}{64(a^2+\omega_0^2)^3(a^2+9\omega_0^2)}$

TABLE IV
THE LEADING TERMS FOR THE 2TH MOMENT OF $y(t)$

The leading terms of $\left\langle y(t)^2\right\rangle$ for .(3) [2]	The leading terms of $\left\langle y(t)^2\right\rangle$ for (4)
$\left\langle y(t)\right\rangle^2$	0
$\int_{-\infty}^{\infty} df_1 W_I(f_1)G_1(f_1)G_1(-f_1)$	$W_I/2a$, if $a>0$, $-W_I/2a$, if $a<0$.
$\int_{-\infty}^{\infty} df_1\int_{-\infty}^{\infty} df_2 W_I(f_1)W_I(f_2)[G_1(f_1)G_3(-f_1,f_2,-f_2)$	$-3bW_I^2/4a^3$, if $a>0$, $3bW_I^2/4a^3$, if $a<0$

TABLE V
THE LEADING TERM OF THE TWO-SIDED POWER SPECTRUM OF THE OUTPUT SIGNAL.

The leading terms of the two-sided power spectrum of the output signal for (3) [2]	The leading terms of the two-sided power spectrum of the output signal for (4)
$\langle y(t)\rangle^2\,\delta(f)$	0
$W_I(f)\left\lvert G_1(f)+\frac{1}{2}\int_{-\infty}^{\infty} df_1 W_I(f_1)G_3(f,f_1,-f_1)\right\rvert^2$	$W_I\,\frac{(3bW_I-2a^2)^2+4a^2\omega^2}{4a^2(a^2+\omega^2)^2}$, if $a>0$, $W_I\,\frac{(3bW_I+2a^2)^2+4a^2\omega^2}{4a^2(a^2+\omega^2)^2}$ if $a<0$.
$\frac{1}{2!}\int_{-\infty}^{\infty} df_1 W_I(f_1)W_I(f-f_1)\lvert G_2(f_1,f-f_1)\rvert^2$	0
$\frac{1}{3!}\int_{-\infty}^{\infty} df_1\int_{-\infty}^{\infty} df_2 W_I(f_1)W_I(f_2)W_I(f-f_1-f_{2\partial})\lvert G_3(f_1,f_2,f-f_1-f_2)\rvert^2$	$W_I^3\,\frac{9b^2}{2a^2(a^2+\omega^2)(a^2+9\omega^2)}$

where $\langle y(t)\rangle$ is the dc component of $y(t)$ and $\delta(f)$ is the unit impulse function

TABLE VI
LEADING TERMS OF THE OUTPUT POWER SPECTRUM OF THE STOCHASTIC NONLINEAR FILTER

Leading terms of the output power spectrum $W_Y(f)$ [2]	Leading terms of the output power spectrum for SR
Spikes due to sine waves	
1	2
$\delta(f-f_0)\left\lvert\frac{A}{2}G_1(f_0)+\frac{A^3}{16}G_3(f_0,f_0,-f_0)+\right.$ $\left.+\frac{A}{4}\int_{-\infty}^{\infty}df_1W_1(f_1)G_3(f_1,-f_1,f_0)\right\rvert^2$	if $a>0$ $\delta(f-f_0)A^2\,\frac{[2(a^2+\omega_0^2)\{2a^2+3bW_I\}+3A^2ba]^2+16a^2\omega_0^2(a^2+\omega_0^2)^2}{64a^2(a^2+\omega_0^2)^4}$ if $a<0$ $\delta(f-f_0)A^2\,\frac{[2(a^2+\omega_0^2)\{2a^2-3bW_I\}+3A^2ba]^2+16a^2\omega_0^2(a^2+\omega_0^2)^2}{64a^2(a^2+\omega_0^2)^4}$
$\delta(f-3f_0)\left\lvert\frac{A^3}{48}G_3(f_0,f_0,f_0)\right\rvert^2$	$\delta(f-3f_0)\frac{A^6b^2}{64(a^2+\omega_0^2)^3(a^2+9\omega_0^2)}$
{terms with $-\omega_0,\ -f_0$ for $\omega_0,\ f_0$ in $e^{jk\omega_0 t}[...],\ k=1,2,...$ }, where $f_0=\omega_0/2\pi$	
$\delta(f+f_0)\left\lvert\frac{A}{2}G_1(f_0)+\frac{A^3}{16}G_3(f_0,f_0,-f_0)+\right.$ $\left.+\frac{A}{4}\int_{-\infty}^{\infty}df_1W_1(f_1)G_3(f_1,-f_1,f_0)\right\rvert^2$	if $a>0$ $\delta(f-f_0)A^2\,\frac{[2(a^2+\omega_0^2)\{2a^2+3bW_I\}+3A^2ba]^2+16a^2\omega_0^2(a^2+\omega_0^2)^2}{64a^2(a^2+\omega_0^2)^4}$ $\delta(f+f_0)A^2\,\frac{[2(a^2+\omega_0^2)\{2a^2+3bW_I\}+3A^2ba]^2+16a^2\omega_0^2(a^2+\omega_0^2)^2}{16a^2(a^2+\omega_0^2)^4}$ if $a<0$ $\delta(f+f_0)A^2\,\frac{[2(a^2+\omega_0^2)\{2a^2-3bW_I\}+3A^2ba]^2+16a^2\omega_0^2(a^2+\omega_0^2)^2}{16a^2(a^2+\omega_0^2)^4}$
$\delta(f+3f_0)\left\lvert\frac{A^3}{48}G_3(-f_0,-f_0,-f_0)\right\rvert^2$	$\delta(f+3f_0)\frac{A^6b^2}{64(a^2+\omega_0^2)^3(a^2+9\omega_0^2)}$
$W_1(f)\left\lvert G(f)+\frac{A^2}{4}G_3(f_0,-f_0,f)+\right.$ $\left.+\frac{1}{2}\int_{-\infty}^{\infty}df_1W_1(f_1)G_3(f_1,-f_1,f)\right\rvert^2$	if $a>0$ $W_I\,\frac{[(2a^2+3bW_I)(a^2+\omega_0^2)+3A^2ab]^2+4a^2\omega^2(a^2+\omega_0^2)^2}{4a^2(a^2+\omega_0^2)^2(a^2+\omega^2)^2}$ if $a<0$ $W_I\,\frac{[(-2a^2+3bW_I)(a^2+\omega_0^2)-3A^2ab]^2+4a^2\omega^2(a^2+\omega_0^2)^2}{4a^2(a^2+\omega_0^2)^2(a^2+\omega^2)^2}$

Continue TABLE VI

1	2
$W_I(f-3f_0)\left\|\frac{A^2}{8}G_3(f_0,f_0,f-2f_0)\right\|^2$	$\frac{9A^4b^2W_I}{16(a^2+\omega_0^2)^2(a^2+[\omega-2\omega_0]^2)(a^2+\omega^2)}$
term with $-f_0$ for f_0 in $W_I(f-kf_0)\|...\|^2, k=1,2,..$ $W_I(f+3f_0)\left\|\frac{A^2}{8}G_3(-f_0,-f_0,f+2f_0)\right\|^2$	$\frac{9A^4b^2W_I}{16(a^2+\omega_0^2)^2(a^2+[\omega+2\omega_0]^2)(a^2+\omega^2)}$
$\frac{1}{2!}\int_{-\infty}^{\infty}df_1W_I(f_1)W_I(f-f_1-f_0)\left\|\frac{A}{2}G_3(f_1,f_0,f-f_1-f_0)\right\|^2$	if $a>0$ $\frac{9A^2b^2W_I}{2a(a^2+\omega_0^2)(a^2+\omega^2)(4a^2+(\omega-\omega_0)^2)}$ if $a<0$ $-\frac{9A^2b^2W_I}{2a(a^2+\omega_0^2)(a^2+\omega^2)(4a^2+(\omega-\omega_0)^2)}$
$\frac{1}{2!}\int_{-\infty}^{\infty}df_1W_I(f_1)W_I(f-f_1+f_0)\left\|\frac{A}{2}G_3(f_1,-f_0,f-f_1+f_0)\right\|^2$	if $a>0$ if $a<0$ $-\frac{9A^2b^2W_I}{2a(a^2+\omega_0^2)(a^2+\omega^2)(4a^2+(\omega+\omega_0)^2)}$
$\frac{1}{3!}\int_{-\infty}^{\infty}df_1\int_{-\infty}^{\infty}df_2W_I(f_1)W_I(f_2)W_I(f-f_1-f_2)\times$ $\times\left\|G_3(f_1,f_2,f-f_1-f_2)\right\|^2$	$\frac{9b^2W_I^3}{2a^2(a^2+\omega^2)(\omega^2+9a^2)}$

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