Multi-criteria Group Decision Making Based on the Multiplicative Consistency of Intuitionistic Fuzzy Preference Relation

Tao Li, Liyuan Zhang and Ziyu Yang

Abstract—**During the process of decision making with intuitionistic fuzzy preference relation (IFPR), the underlying normalized intuitionistic fuzzy priority weight vector can be obtained by a mathematical programming model. In the multi-criteria group decision making (GDM) problem, it is reasonable to assume that different decision makers have different criteria weights, this is because that each decision maker has his/her own opinions and preferences, and the importance of criteria should not be the same for different decision makers. The aim of this paper is to develop two techniques for multi-criteria GDM with IFPRs based on the multiplicative consistency. In the first case where the decision maker acts as separate individual, the individual priority weight vector can be derived from the IFPR established by each decision maker with respect to each criterion, after that an overall priority weight vector can be obtained by synthesizing these priority weight vectors together. When the decision makers are taken as a group, the normalized overall intuitionistic fuzzy priority weight vector can be generated directly by building a fractional programming model without using the aggregation operator. An example is given to illustrate the validity and applicability of the proposed methods.**

Index Terms—**multi-criteria, group decision making, multiplicative consistency, intuitionistic fuzzy preference relation.**

I. Introduction

Group decision making (GDM), where some decision makers are involved to select the best alternative, takes place widely in various fields of operations research and management science. For a GDM problem, decision makers may express their preferences by comparing each pair of alternatives and establish some preference relations. It is known that pairwise comparison methods are more accurate than non-pairwise comparison methods [1]. Many preference relations have been studied by scholars, such as multiplicative preference relation [2], fuzzy preference relation (FPR) [3], interval-valued fuzzy preference relation [4] and triangular fuzzy preference relation [5]. Although these preference relations have some advantages, each element of them only utilizes a membership to describe the degree of one alternative preferred to the other, which means the experts' hesitations or indeterminacies are neglected. To circumvent this issue, Szmidt and Kacprzyk [6] introduced the intuitionistic fuzzy preference relation (IFPR) whose elements are intuitionistic fuzzy values, which can express the membership degree, non-membership degree and hesitant degree jointly. As a result, it is more natural to use the IFPR to describe the uncertainties of pairwise comparisons between the alternatives. Inspired by the interval-valued fuzzy preference relation, the interval-valued IFPRs [7], [8] were also considered by some researchers.

In recent years, GDM with IFPRs has become a hot research topic. Since the lack of consistency of preference relations in a decision making problem may lead to unreasonable conclusions, the consistency of IFPRs has attracted many scholars' attentions. From the existing research achievements, there are two main types of consistency: additive consistency and multiplicative consistency. Wang [9] developed a method for GDM with IFPRs based on the additive consistency, and some mathematical programming models were constructed to obtain the priority weights of alternatives. Liao and Xu [10] pointed out that the additive consistency has some disadvantages, because it is conflict with the [0,1] scale used for providing the preference values, but the multiplicative consistency does not have this limitation. Thus, the multiplicative consistency is more appropriate than the additive consistency in expressing the decision maker's preferences. Inspired by the definition of additive consistency [9], Liao and Xu [10] introduced the concept of multiplicative consistent IFPR. Then, based on this multiplicative consistency, Liao and Xu [11] provided three algorithms for intuitionistic fuzzy GDM from two aspects: aggregating individual intuitionistic fuzzy priorities and aggregating individual intuitionistic fuzzy judgments. The framework of intuitionistic fuzzy GDM was proposed by Liao et al. [12], and this complex GDM can be divided into three subproblems, which are the consistency checking and inconsistency repairing process, the consensus checking and reaching process and the selection process. More studies about IFPRs with multiplicative consistency can also be found in [13]-[16].

In our daily life, some decision making problems are involving various criteria, and multi-criteria decision making (MCDM) is pertaining to structure and solve these problems [17], [18]. A lot of intuitionistic fuzzy MCDM methods have been developed, readers can refer to [19]-[21]. In these literature, the weights of the criteria are assumed to be the same for all decision makers. Nevertheless, since each expert has his/her own opinions and preferences, it is reasonable to suppose that different decision makers should have different criteria weights in a real decision making problem. Thus, in this paper, based on the multiplicative consistency of IFPR given by Liao and Xu [10], we consider a multi-

Manuscript received July 3th, 2019; revised January 17th, 2020. This work was supported in part by the Humanities and Social Sciences Project of the Ministry of Education of China (No.18YJCZH239).

T. Li is with School of Mathematics and Statistics, Shandong University of Technology, Zibo 255049, China.

L. Y. Zhang (e-mail: zhangliyuancsu@163.com) and Z. Y. Yang are with Business School, Shandong University of Technology, Zibo 255049, China. Liyuan Zhang is the corresponding author.

criteria GDM problem, where the weights of criteria for different decision makers can be different. By comparing each pair of alternatives, a decision maker can construct an IFPR with respect to each criterion. Inspired by the transformation formula in [10], we first give a similar formula to convert the normalized intuitionistic fuzzy weights into a multiplicative consistent IFPR. Then, by minimizing the absolute deviation between the original judgment and the converted multiplicative consistent IFPR, the priority weight vector can be derived by a fractional programming model. When the decision maker acts as separate individual, an aggregation operator is used to synthesize all individual priority weight vectors together, then an overall priority vector is obtained. When the experts act as one individual, the overall intuitionistic fuzzy priority weight vector can be generated directly by building a fractional programming model without using the aggregation method.

By considering the hierarchal structure of a decision making problem, Xu and Liao [22] proposed the intuitionistic fuzzy analytic hierarchy process, where a decision maker also constructs an IFPR by comparing each pair of alternatives with respect to each criterion. Compared with Xu and Liao's approach [22], our method still have some advantages.

(1) The multiplicative consistency condition used by Xu and Liao [22] is based on the concept given by Xu et al. [23]. However, Liao and Xu [10] have pointed out that the multiplicative consistency condition in [22], [23] may be too strict for an IFPR, and it loses the original foundation of multiplicative consistency. The multiplicative consistency used in this paper is based on the membership and nonmembership degrees of the intuitionistic fuzzy judgments directly [10], it can reflect the original information of the decision maker. Moreover, we also prove that the definition we used is robust to the permutations of decision maker's judgments and independent of alternative labels. Thus, based on this multiplicative consistency, our priority weights will be more reasonable than that obtained by Xu and Liao's approach [22].

(2) Xu and Liao [22] generated the interval priority weights by transforming an IFPR into an interval-valued preference relation, sometimes the original information losing may take place if the number of alternatives is too large. Although the derived interval priority weight $[w_i, v_i]$ in [22] can be equivalently considered as an intuitionistic fuzzy number (IFN) $(w_i, 1-v_i)$, we can not guarantee that the converted intuitionistic fuzzy priority weight vector is still normalized. In our method, based on the original IFPR, we use a fractional programming model to obtain the normalized intuitionistic fuzzy priority weights.

(3) Compared with Xu and Liao's approach [22], where only one decision maker is involved to construct IFPRs, our proposed method is for a GDM problem, which will be more useful and powerful. And two cases are studied, the decision makers act as separate individuals or one individual. Moreover, since each decision maker has his/her own opinions and preferences, we also assume that the weights of criteria can be different for different decision makers, which is more appropriate for a real decision making problem.

The rest of the paper is organized as follows. Section II presents some basic knowledge about IFPRs and IFNs. Based on the multiplicative consistency condition, Section III gives a transformation formula to convert the normalized intuitionistic fuzzy weights into a multiplicative consistent IFPR. Using the fractional programming model, Section IV proposes a method for multi-criteria intuitionistic fuzzy GDM which takes the decision makers as separate individuals, the method for the second case where the group acts as one individual is provided in Section V. Finally, Section VI makes a comparative analysis and Section VII gives some conclusions.

II. Preliminaries

For a decision making problem, let $X = \{x_1, x_2, \cdots, x_n\}$ be the set of alternatives, and a decision maker is asked to provide his/her preferences by comparing each pair of alternatives. A FPR on the set X is represented by a complementary matrix $R = \left(r_{ij}\right)_{n\times n}$ with $r_{ij} \geq 0$, $r_{ij}+r_{ji} = 1$, $r_{ii} = 0.5$, where r_{ij} indicates the degree to which the alternative x_i is preferred to x_j.

Definition 1: ([24]) A FPR $R = (r_{ij})_{n\times n}$ is called multiplicative consistent if the following multiplicative transitivity is satisfied:

$$r_{ij} \cdot r_{jk} \cdot r_{ki} = r_{ik} \cdot r_{kj} \cdot r_{ji}, \quad i, j, k = 1, 2, \cdots, n. \quad (1)$$

Let ω_i $(i = 1, 2, \cdots, n)$ be the underlying priority weights of the alternatives and satisfies $0 \leq \omega_i \leq 1$, $\sum_{i=1}^{n} \omega_i$, then a multiplicative consistent FPR $R = (r_{ij})_{n\times n}$ can be shown as [25]:

$$r_{ij} = \frac{\omega_i}{\omega_i + \omega_j}, \quad i, j = 1, 2, \cdots, n. \quad (2)$$

Since the elements in FPRs only describe the intensities of preferences but cannot depict the degrees of non-preferences, it is effective and suitable to describe the decision maker's judgments in the intuitionistic fuzzy set, then an IFPR on the set of alternatives X can be constructed. The concept of IFPR is given as follows:

Definition 2: ([26]) An IFPR on the set $X = \{x_1, x_2, \cdots, x_n\}$ is defined as a matrix $R = \left(r_{ij}\right)_{n\times n}$ with $r_{ij} = (\mu_{ij}, \nu_{ij})$, where $\mu_{ij}, \nu_{ij} \in [0, 1]$, $\mu_{ij} + \nu_{ij} \leq 1$, $\mu_{ij} = \nu_{ji}$, $\nu_{ij} = \mu_{ji}$ and $\mu_{ii} = \nu_{ii} = 0.5$. μ_{ij} indicates the degree to which the object x_i is preferred to the object x_j, ν_{ij} means the degree to which the object x_i is not preferred to the object x_j, and $\pi_{ij} = 1 - \mu_{ij} - \nu_{ij}$ is explained as an indeterminacy degree or a hesitancy degree.

For any two IFNs $r_{ij} = (\mu_{ij}, \nu_{ij})$ and $r_{kl} = (\mu_{kl}, \nu_{kl})$, Xu [26] introduced the following operations:

$$r_{ij} \oplus r_{kl} = (\mu_{ij} + \mu_{kl} - \mu_{ij}\mu_{kl}, \nu_{ij}\nu_{kl}); \quad (3)$$

$$r_{ij} \otimes r_{kl} = (\mu_{ij}\mu_{kl}, \nu_{ij} + \nu_{kl} - \nu_{ij}\nu_{kl}); \quad (4)$$

$$\lambda r_{ij} = \left(1 - (1-\mu_{ij})^{\lambda}, \nu_{ij}^{\lambda}\right), \quad \lambda > 0; \quad (5)$$

$$r_{ij}^{\lambda} = \left(\mu_{ij}^{\lambda}, 1 - (1-\nu_{ij})^{\lambda}\right), \quad \lambda > 0. \quad (6)$$

In order to rank IFNs, some methods have been proposed. For an IFN $\alpha_i = (\mu_i, \nu_i)$, Xu [26] defined an accuracy function $H(\alpha_i) = \mu_i + \nu_i$, Zhang and Xu [27] introduced a similarity function $L(\alpha_i) = \frac{1-\nu_i}{1+\pi_i}$. Then, a total order method for ranking any two IFNs $\alpha_i = (\mu_i, \nu_i)$ and $\alpha_j = (\mu_j, \nu_j)$ can be given as follows [27]:

If $L(\alpha_i) > L(\alpha_j)$, then $\alpha_i > \alpha_j$.
If $L(\alpha_i) = L(\alpha_j)$, and
if $H(\alpha_i) > H(\alpha_j)$, then $\alpha_i > \alpha_j$;

if $H(\alpha_i) = H(\alpha_j)$, then $\alpha_i = \alpha_j$.

Liao and Xu [10] have pointed out that the ranking method in [27] is reasonable, since it not only obtain a consistent result when little changes of the parameters of IFNs take place, but also generate a total order of IFNs. Thus, in this paper, we will use Zhang and Xu's ranking method [27] to compare IFNs.

Let $\alpha_i = (\mu_i, \nu_i)$ $(i = 1, 2, \cdots, n)$ be a collection of IFNs, $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ be a permutation of $(1, 2, \cdots, n)$, such that $\alpha_{\sigma(i-1)} \geq \alpha_{\sigma(i)}$ for any i, $w = (w_1, w_2, \cdots, w_n)^T$ be the aggregation-associated vector with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Based on the above operational laws (3)-(6), some important aggregation operators have been introduced [28], [29].

An intuitionistic fuzzy weighted averaging (IFWA) operator is a mapping such that

$$IFWA_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \bigoplus_{i=1}^{n}(w_i\alpha_i)$$
$$= \Big(1 - \prod_{i=1}^{n}(1-\mu_i)^{w_i}, \prod_{i=1}^{n}\nu_i^{w_i}\Big).$$

An intuitionistic fuzzy weighted geometric (IFWG) operator is a mapping such that

$$IFWG_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \bigotimes_{i=1}^{n}(w_i\alpha_i)$$
$$= \Big(\prod_{i=1}^{n}\mu_i^{w_i}, 1 - \prod_{i=1}^{n}(1-\nu_i)^{w_i}\Big).$$

An intuitionistic fuzzy ordered weighted averaging (I-FOWA) operator is a mapping such that

$$IFOWA_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \bigoplus_{i=1}^{n}(w_i\alpha_{\sigma(i)})$$
$$= \Big(1 - \prod_{i=1}^{n}(1-\mu_{\sigma(i)})^{w_i}, \prod_{i=1}^{n}\nu_{\sigma(i)}^{w_i}\Big).$$

An intuitionistic fuzzy ordered weighted geometric (I-FOWG) operator is a mapping such that

$$IFOWG_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \bigotimes_{i=1}^{n}(w_i\alpha_{\sigma(i)})$$
$$= \Big(\prod_{i=1}^{n}\mu_{\sigma(i)}^{w_i}, 1 - \prod_{i=1}^{n}(1-\nu_{\sigma(i)})^{w_i}\Big).$$

Clearly, when $w = (1/n, 1/n, \cdots, 1/n)^T$, the IFWA operator and IFOWA operator both reduce to an intuitionistic fuzzy averaging (IFA) operator, the IFWG operator and IFOWG operator both become an intuitionistic fuzzy geometric (IFG) operator.

III. Relationship Between the Multiplicative Consistent IFPR and a Normalized Intuitionistic Fuzzy Weight Vector

In the course of decision making with IFPRs, how to generate the underlying priority weights is a significant issue. Moreover, when the preference relation is an IFPR, it is reasonable to use the intuitionistic fuzzy priority weight vector to reflect the importance of alternatives. Recently, many approaches have been proposed to derive the intuitionistic fuzzy priority weights, such as additive consistency based method and multiplicative consistency based method. Consistency is the basic property for any preference relation and the lack of consistency may lead to unreasonable conclusions. Since additive consistency is in conflict with the [0,1] scale used for providing the preference values [10], more and more researchers have paid attentions to the multiplicative consistency. Liao and Xu [10] gave a general definition of multiplicative consistent IFPR, shown as follows:

Definition 3: ([10]) An IFPR $R = (r_{ij})_{n\times n}$ with $r_{ij} = (\mu_{ij}, \nu_{ij})$ is called multiplicative consistent if the following multiplicative transitivity is satisfied:

$$\mu_{ij}\cdot\mu_{jk}\cdot\mu_{ki} = \nu_{ij}\cdot\nu_{jk}\cdot\nu_{ki}, \quad i,j,k = 1,2,\cdots,n. \quad (7)$$

If IFNs $r_{ij} = (\mu_{ij}, \nu_{ij})$ satisfy $\mu_{ij} + \nu_{ij} = 1$ for all $i, j = 1, 2, \cdots, n$, then the IFPR R is equivalent to a FPR $R = (\mu_{ij})_{n\times n}$ and Eq. (7) is degraded to Eq. (1).

Theorem 4: ([14]) Let $R = (r_{ij})_{n\times n}$ be an IFPR with $r_{ij} = (\mu_{ij}, \nu_{ij})$, the following statements are equivalent:

i) $\mu_{ij}\cdot\mu_{jk}\cdot\mu_{ki} = \nu_{ij}\cdot\nu_{jk}\cdot\nu_{ki}, \quad i,j,k = 1,2,\cdots,n,$

ii) $\mu_{ij}\cdot\mu_{jk}\cdot\mu_{ki} = \nu_{ij}\cdot\nu_{jk}\cdot\nu_{ki}, \quad i<j<k.$

Based on the above Theorem, the following corollary can be obtained.

Corollary 1: Given an IFPR $R = (r_{ij})_{n\times n}$ with $r_{ij} = (\mu_{ij}, \nu_{ij})$, the following statements are equivalent:

1) $R\ is\ multiplicative\ consistent,$
2) $\mu_{ij}\cdot\mu_{jk}\cdot\mu_{ki} = \nu_{ij}\cdot\nu_{jk}\cdot\nu_{ki}, \quad i,j,k = 1,2,\cdots,n,$
3) $\mu_{ij}\cdot\mu_{jk}\cdot\mu_{ki} = \mu_{ik}\cdot\mu_{kj}\cdot\mu_{ji}, \quad i,j,k = 1,2,\cdots,n,$
4) $\nu_{ij}\cdot\nu_{jk}\cdot\nu_{ki} = \nu_{ik}\cdot\nu_{kj}\cdot\nu_{ji}, \quad i,j,k = 1,2,\cdots,n,$
5) $\mu_{ij}\cdot\mu_{jk}\cdot\mu_{ki} = \nu_{ij}\cdot\nu_{jk}\cdot\nu_{ki}, \quad i<j<k,$
6) $\mu_{ij}\cdot\mu_{jk}\cdot\mu_{ki} = \mu_{ik}\cdot\mu_{kj}\cdot\mu_{ji}, \quad i<j<k,$
7) $\nu_{ij}\cdot\nu_{jk}\cdot\nu_{ki} = \nu_{ik}\cdot\nu_{kj}\cdot\nu_{ji}, \quad i<j<k.$

Let $R^\delta = (r^\delta_{ij})_{n\times n}$ be an IFPR with $r^\delta_{ij} = (\mu^\delta_{ij}, \nu^\delta_{ij}) = (\mu_{\delta(i)\delta(j)}, \nu_{\delta(i)\delta(j)})$, where δ is a permutation of $\{1, 2, \cdots, n\}$.

Theorem 5: Given an IFPR $R = (r_{ij})_{n\times n}$ with $r_{ij} = (\mu_{ij}, \nu_{ij})$, then R is multiplicative consistent if and only if R^δ is multiplicative consistent for any permutation δ.

Proof: Necessity. Assume IFPR $R = (r_{ij})_{n\times n}$ is multiplicative consistent, then from the Definition 3, we have $\mu_{ij}\cdot\mu_{jk}\cdot\mu_{ki} = \nu_{ij}\cdot\nu_{jk}\cdot\nu_{ki}$ for all $i, j, k = 1, 2, \cdots, n$. Thus, we can get $\mu^\delta_{ij}\cdot\mu^\delta_{jk}\cdot\mu^\delta_{ki} = \mu_{\delta(i)\delta(j)}\cdot\mu_{\delta(j)\delta(k)}\cdot\mu_{\delta(k)\delta(i)} = \nu_{\delta(i)\delta(j)}\cdot\nu_{\delta(j)\delta(k)}\cdot\nu_{\delta(k)\delta(i)} = \nu^\delta_{ij}\cdot\nu^\delta_{jk}\cdot\nu^\delta_{ki}$ for all $i, j, k = 1, 2, \cdots, n$. By means of the Definition 3, IFPR $R^\delta = (r^\delta_{ij})_{n\times n}$ is multiplicative consistent.

Sufficiency. If R^δ is multiplicative consistent, we have $\mu^\delta_{ij}\cdot\mu^\delta_{jk}\cdot\mu^\delta_{ki} = \nu^\delta_{ij}\cdot\nu^\delta_{jk}\cdot\nu^\delta_{ki}$, i.e., $\mu_{\delta(i)\delta(j)}\cdot\mu_{\delta(j)\delta(k)}\cdot\mu_{\delta(k)\delta(i)} = \nu_{\delta(i)\delta(j)}\cdot\nu_{\delta(j)\delta(k)}\cdot\nu_{\delta(k)\delta(i)}$ for all $i, j, k = 1, 2, \cdots, n$. Since δ is a permutation of $\{1, 2, \cdots, n\}$, if we denote $\delta(i) = i'$, $\delta(j) = j'$ and $\delta(k) = k'$, one can obtain $\mu_{i'j'}\cdot\mu_{j'k'}\cdot\mu_{k'i'} = \nu_{i'j'}\cdot\nu_{j'k'}\cdot\nu_{k'i'}$ for all $i', j', k' = 1, 2, \cdots, n$. According to the Definition 3, IFPR R is multiplicative consistent. ■

For a decision making problem with n decision alternatives, the decision maker's pairwise comparison can yield $n!$ IFPRs by differently labeling the n alternatives, and these IFPRs can be denoted by a set $\{R^\delta | \delta\ is\ a\ permutation\ of\ \{1, 2, \cdots, n\}\}$. Theorem 5 reveals that the multiplicative consistency given in Definition 3 is robust to the permutations of decision maker's judgments and independent of alternative labels. Therefore, the Definition 3 gives a good measurement for the multiplicative consistency of IFPR.

In the process of decision making with IFPRs, an important thing is to obtain the underlying intuitionistic fuzzy priority weights. Let $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T = \big((\omega_1^\mu, \omega_1^\nu), (\omega_2^\mu, \omega_2^\nu), \cdots, (\omega_n^\mu, \omega_n^\nu)\big)^T$ be the intuitionistic fuzzy priority weight vector of the IFPR $R = (r_{ij})_{n\times n}$, where $\omega_i = (\omega_i^\mu, \omega_i^\nu)$ $(i = 1, 2, \cdots, n)$ is an IFN, which satisfies $\omega_i^\mu, \omega_i^\nu \in [0, 1]$ and $\omega_i^\mu + \omega_i^\nu \leq 1$. The intuitionistic fuzzy weight vector ω is said to be normalized if it also satisfies the following conditions [9]:

$$\sum_{j=1, j\neq i}^{n} \omega_j^\mu \leq \omega_i^\nu,\ \ \omega_i^\mu + n - 2 \geq \sum_{j=1, j\neq i}^{n} \omega_j^\nu,\ \ i = 1, 2, \cdots, n.$$

With the normalized intuitionistic fuzzy weight vector $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$, some transformation formulas have been given to construct a multiplicative consistent IFPR $P = (p_{ij})_{n\times n}$.

(I) Liao and Xu [10] proposed the following formula to establish a multiplicative consistent IFPR $P = (p_{ij})_{n\times n}$, where

$$\begin{aligned} p_{ij} &= (p_{ij}^\mu, p_{ij}^\nu) \\ &= \begin{cases} (0.5, 0.5), \quad i = j, \\ \Big(\dfrac{2\omega_i^\mu}{\omega_i^\mu - \omega_i^\nu + \omega_j^\mu - \omega_j^\nu + 2}, \\ \dfrac{2\omega_j^\mu}{\omega_i^\mu - \omega_i^\nu + \omega_j^\mu - \omega_j^\nu + 2}\Big), \quad i \neq j. \end{cases} \end{aligned} \tag{8}$$

(II) Jin et al. [15] and Lin and Wang [16] both introduced the following formula to build a multiplicative consistent IFPR $P = (p_{ij})_{n\times n}$, where

$$\begin{aligned} p_{ij} &= (p_{ij}^\mu, p_{ij}^\nu) \\ &= \begin{cases} (0.5, 0.5), & i = j, \\ \Big(\sqrt{\omega_i^\mu \omega_j^\nu}, \sqrt{\omega_i^\nu \omega_j^\mu}\Big), & i \neq j. \end{cases} \end{aligned} \tag{9}$$

Although the IFPR $P = (p_{ij})_{n\times n}$ constructed by Eq. (9) satisfies the multiplicative consistency condition Eq. (7), it still has some drawbacks. If the intuitionistic fuzzy weights reduce to the traditional fuzzy weights, i.e., $\omega_i^\mu + \omega_i^\nu = 1$ for all $i = 1, 2, \cdots, n$, then IFPR $P = (p_{ij})_{n\times n}$ constructed by Eq. (8) degenerates to a FPR $P = (p_{ij}^\mu)_{n\times n}$, this is because that $p_{ij}^\mu + p_{ij}^\nu = 1$. And in this case, $p_{ij}^\mu = \omega_i^\mu / (\omega_i^\mu + \omega_j^\mu)$, which is equivalent to the multiplicative consistency condition for a FPR given in Eq. (2). However, the IFPR $P = (p_{ij})_{n\times n}$ constructed by Eq. (9) does not have this property.

Motivated be the transformation formula provided by Liao and Xu [10], we present the following formula to generate a multiplicative consistent IFPR $P = (p_{ij})_{n\times n}$, where

$$\begin{aligned} p_{ij} &= (p_{ij}^\mu, p_{ij}^\nu) \\ &= \begin{cases} (0.5, 0.5), & i = j, \\ \Big(\dfrac{\omega_i^\mu}{2 - \omega_i^\nu - \omega_j^\nu}, \dfrac{\omega_j^\mu}{2 - \omega_i^\nu - \omega_j^\nu}\Big), & i \neq j, \end{cases} \end{aligned} \tag{10}$$

and $\omega_i^\mu, \omega_i^\nu \in [0, 1]$, $\omega_i^\mu + \omega_i^\nu \leq 1$, $\sum_{j=1, j\neq i}^{n} \omega_j^\mu \leq \omega_i^\nu$, $\omega_i^\mu + n - 2 \geq \sum_{j=1, j\neq i}^{n} \omega_j^\nu$, for all $i = 1, 2, \cdots, n$.

Theorem 6: The preference relation $P = (p_{ij})_{n\times n}$ is a multiplicative consistent IFPR, where $p_{ij} = (p_{ij}^\mu, p_{ij}^\nu)$ is defined in Eq. (10).

Proof: It is obvious that $p_{ij}^\mu = p_{ji}^\nu$ for all $i, j = 1, 2, \cdots, n$. Since $\omega_i^\mu, \omega_i^\nu \in [0, 1]$ and $\omega_i^\mu + \omega_i^\nu \leq 1$, we can easily get

$$\begin{aligned} &0 \leq \frac{\omega_i^\mu}{2 - \omega_i^\nu - \omega_j^\nu} \leq \frac{\omega_i^\mu}{\omega_i^\mu + \omega_j^\mu} \leq 1, \\ &0 \leq \frac{\omega_j^\mu}{2 - \omega_i^\nu - \omega_j^\nu} \leq \frac{\omega_j^\mu}{\omega_i^\mu + \omega_j^\mu} \leq 1, \\ &\frac{\omega_i^\mu}{2 - \omega_i^\nu - \omega_j^\nu} + \frac{\omega_j^\mu}{2 - \omega_i^\nu - \omega_j^\nu} \leq \frac{\omega_i^\mu}{\omega_i^\mu + \omega_j^\mu} + \frac{\omega_j^\mu}{\omega_i^\mu + \omega_j^\mu} = 1, \end{aligned}$$

which means $P = (p_{ij})_{n\times n}$ is an IFPR. Moreover, when $i < j < k$, we have

$$\begin{aligned} \mu_{ij} \cdot \mu_{jk} \cdot \mu_{ki} &= \frac{\omega_i^\mu}{2 - \omega_i^\nu - \omega_j^\nu} \frac{\omega_j^\mu}{2 - \omega_j^\nu - \omega_k^\nu} \frac{\omega_k^\mu}{2 - \omega_k^\nu - \omega_i^\nu}, \\ \nu_{ij} \cdot \nu_{jk} \cdot \nu_{ki} &= \frac{\omega_j^\mu}{2 - \omega_i^\nu - \omega_j^\nu} \frac{\omega_k^\mu}{2 - \omega_j^\nu - \omega_k^\nu} \frac{\omega_i^\mu}{2 - \omega_k^\nu - \omega_i^\nu}. \end{aligned}$$

Clearly, $\mu_{ij} \cdot \mu_{jk} \cdot \mu_{ki} = \nu_{ij} \cdot \nu_{jk} \cdot \nu_{ki}$. Then, according to the Definition 3 and the Theorem 4, IFPR $P = (p_{ij})_{n\times n}$ is multiplicative consistent. ■

IV. Multi-criteria intuitionistic fuzzy group decision making with individual priorities

Consider a GDM in which $X = \{x_1, x_2, \cdots, x_n\}$ is the set of alternatives, $D = \{d_1, d_2, \cdots, d_m\}$ is the set of decision makers and $C = \{c_1, c_2, \cdots, c_p\}$ is the set of criteria. Let $\lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}^T$ be the weight vector of the decision makers, such that $\lambda_k \geq 0$ for $k = 1, 2, \cdots, m$, and $\sum_{k=1}^{m} \lambda_k = 1$. Since each decision maker has his/her own opinions, he/she may believe that the importance of these criteria is not the same, and it is reasonable to assume that different experts should have different criteria weights. For the decision maker d_k, let $\delta_k = \{\delta_{k1}, \delta_{k2}, \cdots, \delta_{kp}\}^T$ be the weight vector of the criteria, such that $\delta_{kt} \geq 0$ for $t = 1, 2, \cdots, p$, and $\sum_{t=1}^{p} \delta_{kt} = 1$. Make pairwise comparisons for all alternatives, the expert d_k can establish an IFPR $R^{(kt)} = (r_{ij}^{(kt)})_{n\times n}$ with respect to each criterion c_t, where $r_{ij}^{(kt)} = (\mu_{ij}^{(kt)}, \nu_{ij}^{(kt)})$ is the preference information in IFN. However, in a practical decision making problem, it is almost impossible for decision makers to give multiplicative consistent IFPRs. And in this case, we turn to find a normalized intuitionistic fuzzy priority weight $\omega^{(kt)} = (\omega_1^{(kt)}, \omega_2^{(kt)}, \cdots, \omega_n^{(kt)})^T = \big((\omega_1^{(kt)\mu}, \omega_1^{(kt)\nu}), (\omega_2^{(kt)\mu}, \omega_2^{(kt)\nu}), \cdots, (\omega_n^{(kt)\mu}, \omega_n^{(kt)\nu})\big)^T$, such that $\Big(\frac{\omega_i^{(kt)\mu}}{2 - \omega_i^{(kt)\nu} - \omega_j^{(kt)\nu}}, \frac{\omega_j^{(kt)\mu}}{2 - \omega_i^{(kt)\nu} - \omega_j^{(kt)\nu}}\Big)$ is close to $(\mu_{ij}^{(kt)}, \nu_{ij}^{(kt)})$, which means the deviation between each IFPR $R^{(kt)} = (r_{ij}^{(kt)})_{n\times n}$ and the corresponding multiplicative consistent IFPR $P^{(kt)} = (p_{ij}^{(kt)})_{n\times n}$ established by Eq. (10) should be as small as possible. Then, we can introduce the following deviation variables:

$$\varepsilon_{ij}^{(kt)} = \frac{\omega_i^{(kt)\mu}}{2-\omega_i^{(kt)\nu}-\omega_j^{(kt)\nu}} - \mu_{ij}^{(kt)},$$
$$\xi_{ij}^{(kt)} = \frac{\omega_j^{(kt)\mu}}{2-\omega_i^{(kt)\nu}-\omega_j^{(kt)\nu}} - \nu_{ij}^{(kt)},$$
$$i \neq j,\ i,j = 1,\cdots,n,\ k = 1,\cdots,m,\ t = 1,\cdots,p.$$

We can easily check that $\varepsilon_{ij}^{(kt)} = \xi_{ji}^{(kt)}$ and $\xi_{ij}^{(kt)} = \varepsilon_{ji}^{(kt)}$. Moreover, the smaller the absolute deviation, the more exact results will be obtained. Thus, we can give the following objective function for the decision maker d_k with respect to the criterion c_t:

$$Min\ f = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(|\varepsilon_{ij}^{(kt)}| + |\xi_{ij}^{(kt)}|), \tag{11}$$
$$k = 1,2,\cdots,m,\ t = 1,2,\cdots,p.$$

Let $\varepsilon_{ij}^{+(kt)} = \frac{|\varepsilon_{ij}^{(kt)}|+\varepsilon_{ij}^{(kt)}}{2}$, $\varepsilon_{ij}^{-(kt)} = \frac{|\varepsilon_{ij}^{(kt)}|-\varepsilon_{ij}^{(kt)}}{2}$, $\xi_{ij}^{+(kt)} = \frac{|\xi_{ij}^{(kt)}|+\xi_{ij}^{(kt)}}{2}$, $\xi_{ij}^{-(kt)} = \frac{|\xi_{ij}^{(kt)}|-\xi_{ij}^{(kt)}}{2}$, then $\varepsilon_{ij}^{(kt)} = \varepsilon_{ij}^{+(kt)} - \varepsilon_{ij}^{-(kt)}$, $|\varepsilon_{ij}^{(kt)}| = \varepsilon_{ij}^{+(kt)} + \varepsilon_{ij}^{-(kt)}$, $\xi_{ij}^{(kt)} = \xi_{ij}^{+(kt)} - \xi_{ij}^{-(kt)}$, $|\xi_{ij}^{(kt)}| = \xi_{ij}^{+(kt)} + \xi_{ij}^{-(kt)}$, where $\varepsilon_{ij}^{+(kt)} \geq 0$, $\varepsilon_{ij}^{-(kt)} \geq 0$, $\xi_{ij}^{+(kt)} \geq 0$, $\xi_{ij}^{-(kt)} \geq 0$, $\varepsilon_{ij}^{+(kt)} \cdot \varepsilon_{ij}^{-(kt)} = 0$ and $\xi_{ij}^{+(kt)} \cdot \xi_{ij}^{-(kt)} = 0$. Therefore, a fractional programming Model 1 can be constructed to derive the normalized intuitionistic fuzzy priority weight vector $\omega^{(kt)}$.

Using some optimization computer packages, such as Lingo or Matlab, to solve the Model 1, we can get the optimal objective function value f and the optimal normalized intuitionistic fuzzy priority weight vector $\omega^{(kt)} = (\omega_1^{(kt)}, \omega_2^{(kt)}, \cdots, \omega_n^{(kt)})^T = \big((\omega_1^{(kt)\mu}, \omega_1^{(kt)\nu}), (\omega_2^{(kt)\mu}, \omega_2^{(kt)\nu}), \cdots, (\omega_n^{(kt)\mu}, \omega_n^{(kt)\nu})\big)^T$.

Suppose all the normalized intuitionistic fuzzy priority weight vectors $\omega^{(kt)}$ $(k = 1,2,\cdots,m,\ t = 1,2,\cdots,p)$ are obtained, using the aggregation operators in Section II (take the IFWG operator as an example), we first derive the individual intuitionistic fuzzy priority weight vectors $\omega^{(k)} = (\omega_1^{(k)}, \omega_2^{(k)}, \cdots, \omega_n^{(k)})^T = \big((\omega_1^{(k)\mu}, \omega_1^{(k)\nu}), (\omega_2^{(k)\mu}, \omega_2^{(k)\nu}), \cdots, (\omega_n^{(k)\mu}, \omega_n^{(k)\nu})\big)^T$ $(k = 1,2,\cdots,m)$, where

$$\begin{aligned}\omega_i^{(k)} &= IFWG_{\delta_k}(\omega_i^{(k1)}, \omega_i^{(k2)}, \cdots, \omega_i^{(kp)}) \\ &= \Big(\prod_{t=1}^{p}(\omega_i^{(kt)\mu})^{\delta_{kt}}, 1 - \prod_{t=1}^{p}(1-\omega_i^{(kt)\nu})^{\delta_{kt}}\Big).\end{aligned} \tag{12}$$

Then, fuse these individual intuitionistic fuzzy priority weight vectors $\omega^{(k)} = (\omega_1^{(k)}, \omega_2^{(k)}, \cdots, \omega_n^{(k)})^T$ $(k = 1,2,\cdots,m)$ into an overall intuitionistic fuzzy priority weight vector $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T = \big((\omega_1^{\mu}, \omega_1^{\nu}), (\omega_2^{\mu}, \omega_2^{\nu}), \cdots, (\omega_n^{\mu}, \omega_n^{\nu})\big)^T$, using the IFWG operator, we have

$$\begin{aligned}\omega_i &= IFWG_{\lambda}(\omega_i^{(1)}, \omega_i^{(2)}, \cdots, \omega_i^{(m)}) \\ &= \Big(\prod_{k=1}^{m}(\omega_i^{(k)\mu})^{\lambda_k}, 1 - \prod_{k=1}^{m}(1-\omega_i^{(k)\nu})^{\lambda_k}\Big).\end{aligned} \tag{13}$$

Calculate the similarity values (or accuracy values if the similarity values are equal) of the overall intuitionistic fuzzy priority weights $\omega_1, \omega_2, \cdots, \omega_n$, the ranking order of the alternatives can be given via the comparison law for IFNs.

In the following, an algorithm is proposed to handle this multi-criteria GDM problem with IFPRs, where the decision makers are taken as separate individuals.

Algorithm I:

Step 1. Make pairwise comparisons for all alternatives, decision makers construct some IFPRs $R^{(kt)} = (r_{ij}^{(kt)})_{n\times n}$ $(k = 1,2,\cdots,m;\ t = 1,2,\cdots,p)$, where $r_{ij}^{(kt)}$ $(i,j = 1,2,\cdots,n)$ is the preference information in IFN given by the expert d_k with respect to the criterion c_t. Go to the next step.

Step 2. For each IFPR $R^{(kt)} = (r_{ij}^{(kt)})_{n\times n}$, establish a fractional programming model according to the Model 1. Go to the next step.

Step 3. Solve these fractional programming models to derive the normalized intuitionistic fuzzy priority weight vectors $\omega^{(kt)} = (\omega_1^{(kt)}, \omega_2^{(kt)}, \cdots, \omega_n^{(kt)})^T$ from the IFPRs $R^{(kt)} = (r_{ij}^{(kt)})_{n\times n}$. Go to the next step.

Step 4. Synthesize the priority weight vectors $\omega^{(kt)}$ $(k = 1,2,\cdots,m;\ t = 1,2,\cdots,p)$ to get the individual intuitionistic fuzzy priority weight vectors $\omega^{(k)} = (\omega_1^{(k)}, \omega_2^{(k)}, \cdots, \omega_n^{(k)})^T$ by using the operators in Section II. Go to the next step.

Step 5. Fuse the individual intuitionistic fuzzy priority weight vectors $\omega^{(k)}$ $(k = 1,2,\cdots,m)$ into an overall intuitionistic fuzzy priority weight vector $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$. Go to the next step.

Step 6. Compare the overall intuitionistic fuzzy priority weights $\omega_1, \omega_2, \cdots, \omega_n$ to obtain the ranking order of the alternatives. Go to the next step.

Step 7. End.

Here we give a simple example concerning the selection of cars to illustrate the procedure for this multi-criteria intuitionistic fuzzy GDM problem.

Example 1. Suppose that there is a family with three members (decision makers), the father (d_1), the mother (d_2) and the son (d_3), they want to select a new car from four candidate alternatives x_i $(i = 1,2,3,4)$. Four criteria are determined to complete this process including price (c_1), driving comfortability (c_2), hundred kilometers acceleration (c_3) and color (c_4). The weights of the decision makers are assumed to be λ_1=0.4, λ_2=0.4 and λ_3=0.2, respectively, because the money is provided by parents. Furthermore, since the father usually drives the car, he believes that the criteria c_1 and c_2 are more important than the other criteria c_3 and c_4, and he gives his weight vector of the criteria $\delta_1 = \{0.4, 0.3, 0.1, 0.2\}^T$. The mother's weight vector of the criteria is supposed to be $\delta_2 = \{0.7, 0.1, 0.1, 0.1\}^T$, because she can not drive a car and she mainly cares the price of the car. For the son, he pays attention to a car's speed, which means the weight of the criterion c_3 is the biggest, and he gives his weight vector of the criteria $\delta_3 = \{0.1, 0.2, 0.5, 0.2\}^T$. Make pairwise comparisons for these four alternatives, decision maker d_k $(k = 1,2,3)$ can establish an IFPR $R^{(kt)} = (r_{ij}^{(kt)})_{4\times 4}$ with respect to each criterion c_t $(t = 1,2,3,4)$, which is given as follows:

$$R^{(11)} = \begin{pmatrix} (0.50,0.50) & (0.45,0.15) & (0.55,0.10) & (0.50,0.05) \\ (0.15,0.45) & (0.50,0.50) & (0.70,0.10) & (0.75,0.05) \\ (0.10,0.55) & (0.10,0.70) & (0.50,0.50) & (0.30,0.30) \\ (0.05,0.50) & (0.05,0.75) & (0.30,0.30) & (0.50,0.50) \end{pmatrix},$$

Model 1 $$Min\ f=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\varepsilon_{ij}^{+(kt)}+\varepsilon_{ij}^{-(kt)}+\xi_{ij}^{+(kt)}+\xi_{ij}^{-(kt)})$$

$$s.t.\begin{cases}\frac{\omega_i^{(kt)\mu}}{2-\omega_i^{(kt)\nu}-\omega_j^{(kt)\nu}}-\mu_{ij}^{(kt)}-\varepsilon_{ij}^{+(kt)}+\varepsilon_{ij}^{-(kt)}=0, \quad i=1,2,\cdots,n-1,\ j=i+1,\cdots,n\\ \frac{\omega_j^{(kt)\mu}}{2-\omega_i^{(kt)\nu}-\omega_j^{(kt)\nu}}-\nu_{ij}^{(kt)}-\xi_{ij}^{+(kt)}+\xi_{ij}^{-(kt)}=0, \quad i=1,2,\cdots,n-1,\ j=i+1,\cdots,n\\ \omega_i^{(kt)\mu},\omega_i^{(kt)\nu}\in[0,1],\ \omega_i^{(kt)\mu}+\omega_i^{(kt)\nu}\le 1, \quad i=1,2,\cdots,n\\ \sum_{j=1,j\neq i}^{n}\omega_j^{(kt)\mu}\le\omega_i^{(kt)\nu},\ \omega_i^{(kt)\mu}+n-2\ge\sum_{j=1,j\neq i}^{n}\omega_j^{(kt)\nu}, \quad i=1,2,\cdots,n\\ \varepsilon_{ij}^{+(kt)}\ge 0,\ \varepsilon_{ij}^{-(kt)}\ge 0,\ \xi_{ij}^{+(kt)}\ge 0,\ \xi_{ij}^{-(kt)}\ge 0, \quad i=1,2,\cdots,n-1,\ j=i+1,\cdots,n\\ \varepsilon_{ij}^{+(kt)}\cdot\varepsilon_{ij}^{-(kt)}=0,\ \xi_{ij}^{+(kt)}\cdot\xi_{ij}^{-(kt)}=0, \quad i=1,2,\cdots,n-1,\ j=i+1,\cdots,n\end{cases}$$

$$R^{(12)}=\begin{pmatrix}(0.50,0.50)&(0.40,0.30)&(0.55,0.15)&(0.60,0.10)\\(0.30,0.40)&(0.50,0.50)&(0.60,0.15)&(0.70,0.05)\\(0.15,0.55)&(0.15,0.60)&(0.50,0.50)&(0.60,0.10)\\(0.10,0.60)&(0.05,0.70)&(0.10,0.60)&(0.50,0.50)\end{pmatrix},$$

$$R^{(13)}=\begin{pmatrix}(0.50,0.50)&(0.45,0.20)&(0.40,0.30)&(0.65,0.10)\\(0.20,0.45)&(0.50,0.50)&(0.40,0.25)&(0.60,0.10)\\(0.30,0.40)&(0.25,0.40)&(0.50,0.50)&(0.40,0.30)\\(0.10,0.65)&(0.10,0.60)&(0.30,0.40)&(0.50,0.50)\end{pmatrix},$$

$$R^{(14)}=\begin{pmatrix}(0.50,0.50)&(0.20,0.70)&(0.40,0.40)&(0.70,0.10)\\(0.70,0.20)&(0.50,0.50)&(0.20,0.55)&(0.70,0.20)\\(0.40,0.40)&(0.55,0.20)&(0.50,0.50)&(0.60,0.10)\\(0.10,0.70)&(0.20,0.70)&(0.10,0.60)&(0.50,0.50)\end{pmatrix},$$

$$R^{(21)}=\begin{pmatrix}(0.50,0.50)&(0.40,0.30)&(0.35,0.25)&(0.45,0.35)\\(0.30,0.40)&(0.50,0.50)&(0.55,0.10)&(0.60,0.20)\\(0.25,0.35)&(0.10,0.55)&(0.50,0.50)&(0.50,0.20)\\(0.35,0.45)&(0.20,0.60)&(0.20,0.50)&(0.50,0.50)\end{pmatrix},$$

$$R^{(22)}=\begin{pmatrix}(0.50,0.50)&(0.40,0.45)&(0.55,0.10)&(0.50,0.15)\\(0.45,0.40)&(0.50,0.50)&(0.60,0.10)&(0.55,0.15)\\(0.10,0.55)&(0.10,0.60)&(0.50,0.50)&(0.45,0.30)\\(0.15,0.50)&(0.15,0.55)&(0.30,0.45)&(0.50,0.50)\end{pmatrix},$$

$$R^{(23)}=\begin{pmatrix}(0.50,0.50)&(0.40,0.20)&(0.30,0.45)&(0.60,0.15)\\(0.20,0.40)&(0.50,0.50)&(0.55,0.25)&(0.65,0.10)\\(0.45,0.30)&(0.25,0.55)&(0.50,0.50)&(0.60,0.05)\\(0.15,0.60)&(0.10,0.65)&(0.05,0.60)&(0.50,0.50)\end{pmatrix},$$

$$R^{(24)}=\begin{pmatrix}(0.50,0.50)&(0.30,0.60)&(0.60,0.15)&(0.45,0.35)\\(0.60,0.30)&(0.50,0.50)&(0.70,0.10)&(0.60,0.20)\\(0.15,0.60)&(0.10,0.70)&(0.50,0.50)&(0.60,0.15)\\(0.35,0.45)&(0.20,0.60)&(0.15,0.60)&(0.50,0.50)\end{pmatrix},$$

$$R^{(31)}=\begin{pmatrix}(0.50,0.50)&(0.30,0.55)&(0.40,0.30)&(0.55,0.15)\\(0.55,0.30)&(0.50,0.50)&(0.70,0.15)&(0.75,0.05)\\(0.30,0.40)&(0.15,0.70)&(0.50,0.50)&(0.55,0.10)\\(0.15,0.55)&(0.05,0.75)&(0.10,0.55)&(0.50,0.50)\end{pmatrix},$$

$$R^{(32)}=\begin{pmatrix}(0.50,0.50)&(0.40,0.35)&(0.30,0.45)&(0.50,0.25)\\(0.35,0.40)&(0.50,0.50)&(0.40,0.20)&(0.55,0.10)\\(0.45,0.30)&(0.20,0.40)&(0.50,0.50)&(0.60,0.10)\\(0.25,0.50)&(0.10,0.55)&(0.10,0.60)&(0.50,0.50)\end{pmatrix},$$

$$R^{(33)}=\begin{pmatrix}(0.50,0.50)&(0.40,0.20)&(0.55,0.15)&(0.45,0.30)\\(0.20,0.40)&(0.50,0.50)&(0.50,0.25)&(0.45,0.35)\\(0.15,0.55)&(0.25,0.50)&(0.50,0.50)&(0.40,0.40)\\(0.30,0.45)&(0.35,0.45)&(0.40,0.40)&(0.50,0.50)\end{pmatrix},$$

$$R^{(34)}=\begin{pmatrix}(0.50,0.50)&(0.40,0.40)&(0.60,0.20)&(0.35,0.45)\\(0.40,0.40)&(0.50,0.50)&(0.45,0.20)&(0.55,0.15)\\(0.20,0.60)&(0.20,0.45)&(0.50,0.50)&(0.40,0.30)\\(0.45,0.35)&(0.15,0.55)&(0.30,0.40)&(0.50,0.50)\end{pmatrix}.$$

According to the Model 1, using Lingo software, we can generate the normalized intuitionistic fuzzy priority weight vector $\omega^{(kt)}$ from the IFPR $R^{(kt)}$ constructed by decision maker d_k $(k=1,2,3)$ with respect to each criterion c_t $(t=1,2,3,4)$. Taking $R^{(11)}$ as an example, we first establish a fractional programming Model 2.

Solving the Model 2 with Lingo software, we can get

$$\begin{aligned}\omega^{(11)}&=(\omega_1^{(11)},\omega_2^{(11)},\omega_3^{(11)},\omega_4^{(11)})^T\\&=\big((0.3511,0.4867),(0.3928,0.5639),\\&\qquad(0.0638,0.8749),(0.0300,0.9123)\big)^T.\end{aligned}$$

Similarly, from $R^{(12)}-R^{(34)}$, we can obtain

$$\begin{aligned}\omega^{(12)}&=\big((0.3559,0.4772),(0.3640,0.5176),\\&\qquad(0.0971,0.8758),(0.0162,0.9624)\big)^T,\\ \omega^{(13)}&=\big((0.3627,0.5384),(0.2646,0.6555),\\&\qquad(0.1653,0.6831),(0.0558,0.9035)\big)^T,\\ \omega^{(14)}&=\big((0.2873,0.6750),(0.2714,0.6978),\\&\qquad(0.2873,0.6066),(0.0479,0.9145)\big)^T,\\ \omega^{(21)}&=\big((0.2675,0.5364),(0.2590,0.6992),\\&\qquad(0.1911,0.6992),(0.0863,0.8691)\big)^T,\\ \omega^{(22)}&=\big((0.3247,0.6033),(0.3653,0.5849),\\&\qquad(0.0609,0.8063),(0.0996,0.7509)\big)^T,\\ \omega^{(23)}&=\big((0.2017,0.7479),(0.3277,0.5798),\\&\qquad(0.3025,0.5798),(0.0504,0.9160)\big)^T,\\ \omega^{(24)}&=\big((0.2270,0.7568),(0.4540,0.4865),\\&\qquad(0.0649,0.8649),(0.1535,0.7459)\big)^T,\\ \omega^{(31)}&=\big((0.2115,0.6875),(0.4958,0.4111),\\&\qquad(0.1587,0.7837),(0.0331,0.9279)\big)^T,\\ \omega^{(32)}&=\big((0.1716,0.7569),(0.3356,0.4900),\\&\qquad(0.2574,0.6711),(0.0610,0.8999)\big)^T,\\ \omega^{(33)}&=\big((0.3066,0.5388),(0.2366,0.6946),\\&\qquad(0.1183,0.8323),(0.1840,0.7798)\big)^T,\\ \omega^{(34)}&=\big((0.3234,0.6418),(0.3250,0.5458),\\&\qquad(0.1270,0.8192),(0.0952,0.8633)\big)^T.\end{aligned}$$

Next, using Eq. (12), we can derive the individual intuitionistic fuzzy priority weight vector for each decision maker

Model 2
$$
\begin{aligned}
&Min\ f = (\varepsilon_{12}^{+(11)} + \varepsilon_{12}^{-(11)} + \xi_{12}^{+(11)} + \xi_{12}^{-(11)}) + (\varepsilon_{13}^{+(11)} + \varepsilon_{13}^{-(11)} + \xi_{13}^{+(11)} + \xi_{13}^{-(11)})\\
&\quad + (\varepsilon_{14}^{+(11)} + \varepsilon_{14}^{-(11)} + \xi_{14}^{+(11)} + \xi_{14}^{-(11)}) + (\varepsilon_{23}^{+(11)} + \varepsilon_{23}^{-(11)} + \xi_{23}^{+(11)} + \xi_{23}^{-(11)})\\
&\quad + (\varepsilon_{24}^{+(11)} + \varepsilon_{24}^{-(11)} + \xi_{24}^{+(11)} + \xi_{24}^{-(11)}) + (\varepsilon_{34}^{+(11)} + \varepsilon_{34}^{-(11)} + \xi_{34}^{+(11)} + \xi_{34}^{-(11)})
\end{aligned}
$$
$$
s.t.\begin{cases}
\frac{\omega_1^{(11)\mu}}{2-\omega_1^{(11)\nu}-\omega_2^{(11)\nu}} - 0.45 - \varepsilon_{12}^{+(11)} + \varepsilon_{12}^{-(11)} = 0, \quad \frac{\omega_2^{(11)\mu}}{2-\omega_1^{(11)\nu}-\omega_2^{(11)\nu}} - 0.15 - \xi_{12}^{+(11)} + \xi_{12}^{-(11)} = 0,\\
\frac{\omega_1^{(11)\mu}}{2-\omega_1^{(11)\nu}-\omega_3^{(11)\nu}} - 0.55 - \varepsilon_{13}^{+(11)} + \varepsilon_{13}^{-(11)} = 0, \quad \frac{\omega_3^{(11)\mu}}{2-\omega_1^{(11)\nu}-\omega_3^{(11)\nu}} - 0.10 - \xi_{13}^{+(11)} + \xi_{13}^{-(11)} = 0,\\
\frac{\omega_1^{(11)\mu}}{2-\omega_1^{(11)\nu}-\omega_4^{(11)\nu}} - 0.50 - \varepsilon_{14}^{+(11)} + \varepsilon_{14}^{-(11)} = 0, \quad \frac{\omega_4^{(11)\mu}}{2-\omega_1^{(11)\nu}-\omega_4^{(11)\nu}} - 0.05 - \xi_{14}^{+(11)} + \xi_{14}^{-(11)} = 0,\\
\frac{\omega_2^{(11)\mu}}{2-\omega_2^{(11)\nu}-\omega_3^{(11)\nu}} - 0.70 - \varepsilon_{23}^{+(11)} + \varepsilon_{23}^{-(11)} = 0, \quad \frac{\omega_3^{(11)\mu}}{2-\omega_2^{(11)\nu}-\omega_3^{(11)\nu}} - 0.10 - \xi_{23}^{+(11)} + \xi_{23}^{-(11)} = 0,\\
\frac{\omega_2^{(11)\mu}}{2-\omega_2^{(11)\nu}-\omega_4^{(11)\nu}} - 0.75 - \varepsilon_{24}^{+(11)} + \varepsilon_{24}^{-(11)} = 0, \quad \frac{\omega_4^{(11)\mu}}{2-\omega_2^{(11)\nu}-\omega_4^{(11)\nu}} - 0.05 - \xi_{24}^{+(11)} + \xi_{24}^{-(11)} = 0,\\
\frac{\omega_3^{(11)\mu}}{2-\omega_3^{(11)\nu}-\omega_4^{(11)\nu}} - 0.30 - \varepsilon_{34}^{+(11)} + \varepsilon_{34}^{-(11)} = 0, \quad \frac{\omega_4^{(11)\mu}}{2-\omega_3^{(11)\nu}-\omega_4^{(11)\nu}} - 0.30 - \xi_{34}^{+(11)} + \xi_{34}^{-(11)} = 0,\\
0 \le \omega_1^{(11)\mu} \le 1, 0 \le \omega_1^{(11)\nu} \le 1, \omega_1^{(11)\mu} + \omega_1^{(11)\nu} \le 1, 0 \le \omega_2^{(11)\mu} \le 1, 0 \le \omega_2^{(11)\nu} \le 1, \omega_2^{(11)\mu} + \omega_2^{(11)\nu} \le 1,\\
0 \le \omega_3^{(11)\mu} \le 1, 0 \le \omega_3^{(11)\nu} \le 1, \omega_3^{(11)\mu} + \omega_3^{(11)\nu} \le 1, 0 \le \omega_4^{(11)\mu} \le 1, 0 \le \omega_4^{(11)\nu} \le 1, \omega_4^{(11)\mu} + \omega_4^{(11)\nu} \le 1,\\
\omega_2^{(11)\mu} + \omega_3^{(11)\mu} + \omega_4^{(11)\mu} \le \omega_1^{(11)\nu}, \omega_1^{(11)\mu} + \omega_3^{(11)\mu} + \omega_4^{(11)\mu} \le \omega_2^{(11)\nu},\\
\omega_1^{(11)\mu} + \omega_2^{(11)\mu} + \omega_4^{(11)\mu} \le \omega_3^{(11)\nu}, \omega_1^{(11)\mu} + \omega_2^{(11)\mu} + \omega_3^{(11)\mu} \le \omega_4^{(11)\nu},\\
\omega_1^{(11)\mu} + 2 \ge \omega_2^{(11)\nu} + \omega_3^{(11)\nu} + \omega_4^{(11)\nu}, \omega_2^{(11)\mu} + 2 \ge \omega_1^{(11)\nu} + \omega_3^{(11)\nu} + \omega_4^{(11)\nu},\\
\omega_3^{(11)\mu} + 2 \ge \omega_1^{(11)\nu} + \omega_2^{(11)\nu} + \omega_4^{(11)\nu}, \omega_4^{(11)\mu} + 2 \ge \omega_1^{(11)\nu} + \omega_2^{(11)\nu} + \omega_3^{(11)\nu},\\
\varepsilon_{12}^{+(11)} \ge 0, \varepsilon_{12}^{-(11)} \ge 0, \xi_{12}^{+(11)} \ge 0, \xi_{12}^{-(11)} \ge 0, \varepsilon_{13}^{+(11)} \ge 0, \varepsilon_{13}^{-(11)} \ge 0, \xi_{13}^{+(11)} \ge 0, \xi_{13}^{-(11)} \ge 0,\\
\varepsilon_{14}^{+(11)} \ge 0, \varepsilon_{14}^{-(11)} \ge 0, \xi_{14}^{+(11)} \ge 0, \xi_{14}^{-(11)} \ge 0, \varepsilon_{23}^{+(11)} \ge 0, \varepsilon_{23}^{-(11)} \ge 0, \xi_{23}^{+(11)} \ge 0, \xi_{23}^{-(11)} \ge 0,\\
\varepsilon_{24}^{+(11)} \ge 0, \varepsilon_{24}^{-(11)} \ge 0, \xi_{24}^{+(11)} \ge 0, \xi_{24}^{-(11)} \ge 0, \varepsilon_{34}^{+(11)} \ge 0, \varepsilon_{34}^{-(11)} \ge 0, \xi_{34}^{+(11)} \ge 0, \xi_{34}^{-(11)} \ge 0,\\
\varepsilon_{12}^{+(11)}\varepsilon_{12}^{-(11)} = 0, \xi_{12}^{+(11)}\xi_{12}^{-(11)} = 0, \varepsilon_{13}^{+(11)}\varepsilon_{13}^{-(11)} = 0, \xi_{13}^{+(11)}\xi_{13}^{-(11)} = 0, \varepsilon_{14}^{+(11)}\varepsilon_{14}^{-(11)} = 0, \xi_{14}^{+(11)}\xi_{14}^{-(11)} = 0,\\
\varepsilon_{23}^{+(11)}\varepsilon_{23}^{-(11)} = 0, \xi_{23}^{+(11)}\xi_{23}^{-(11)} = 0, \varepsilon_{24}^{+(11)}\varepsilon_{24}^{-(11)} = 0, \xi_{24}^{+(11)}\xi_{24}^{-(11)} = 0, \varepsilon_{34}^{+(11)}\varepsilon_{34}^{-(11)} = 0, \xi_{34}^{+(11)}\xi_{34}^{-(11)} = 0.
\end{cases}
$$

d_k $(k = 1, 2, 3)$, which is given by

$$
\begin{aligned}
\omega^{(1)} &= \big((0.3398, 0.5339), (0.3428, 0.5920),\\
&\quad (0.1075, 0.8277), (0.0291, 0.9317)\big)^T,\\
\omega^{(2)} &= \big((0.2608, 0.5974), (0.2903, 0.6611),\\
&\quad (0.1602, 0.7253), (0.0879, 0.8573)\big)^T,\\
\omega^{(3)} &= \big((0.2659, 0.6290), (0.2911, 0.6088),\\
&\quad (0.1444, 0.8002), (0.1089, 0.8471)\big)^T.
\end{aligned}
$$

Finally, from Eq. (13), the overall intuitionistic fuzzy priority weight vector is derived as

$$
\begin{aligned}
\omega &= (\omega_1, \omega_2, \omega_3, \omega_4)^T\\
&= \big((0.2910, 0.5800), (0.3104, 0.6244),\\
&\quad (0.1338, 0.7861), (0.0590, 0.8922)\big)^T.
\end{aligned}
$$

Since $L(\omega_1)$=0.3720, $L(\omega_2)$=0.3526, $L(\omega_3)$=0.1980, $L(\omega_4)$=0.1028, which means $L(\omega_1) > L(\omega_2) > L(\omega_3) > L(\omega_4)$, then from Zhang and Xu's ranking method [27], we have $\omega_1 > \omega_2 > \omega_3 > \omega_4$. Thus, the family's best choice is the first car x_1.

V. Multi-criteria intuitionistic fuzzy group decision making with overall priority

In the above section, we have studied a multi-criteria GDM problem, where the decision maker acts as separate individual. In this section, we continue to consider the same GDM in which $X = \{x_1, x_2, \cdots, x_n\}$ is the set of alternatives, $D = \{d_1, d_2, \cdots, d_m\}$ is the set of decision makers and $C = \{c_1, c_2, \cdots, c_p\}$ is the set of criteria. In this case, the decision makers are taken as a group, and the overall intuitionistic fuzzy priority weight vector $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T = \big((\omega_1^\mu, \omega_1^\nu), (\omega_2^\mu, \omega_2^\nu), \cdots, (\omega_n^\mu, \omega_n^\nu)\big)^T$ can be obtained directly from a building fractional programming model.

In a practical decision making problem, it is hard for decision makers to give absolutely identical IFPRs. However, if the intuitionistic fuzzy decision making has a solution, i.e., the underlying overall intuitionistic fuzzy priority weights $\omega_1, \omega_2, \cdots, \omega_n$ exist, then a multiplicative consistent IFPR $P = (p_{ij})_{n\times n}$ can always be established from Eq. (10). It is natural and expected that the deviation between each IFPR $R^{(kt)} = (r_{ij}^{(kt)})_{n\times n}$ given by the decision maker d_k $(k = 1, 2, \cdots, m)$ with respect to the criterion c_t $(t = 1, 2, \cdots, p)$ and the corresponding multiplicative consistent IFPR $P = (p_{ij})_{n\times n}$ should be as small as possible. Since the weight of the decision maker d_k is λ_k and the weight of the criterion c_t is δ_{kt} with respect to d_k, the overall deviation between the group's intuitionistic fuzzy preference values and the constructed multiplicative consistent IFPR $P = (p_{ij})_{n\times n}$ can be given as

$$
f = \sum_{k=1}^{m} \lambda_k \sum_{t=1}^{p} \delta_{kt} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (|\hat{\varepsilon}_{ij}^{(kt)}| + |\hat{\xi}_{ij}^{(kt)}|), \tag{14}
$$

where

$$
\hat{\varepsilon}_{ij}^{(kt)} = \frac{\omega_i^\mu}{2 - \omega_i^\nu - \omega_j^\nu} - \mu_{ij}^{(kt)}, \hat{\xi}_{ij}^{(kt)} = \frac{\omega_j^\mu}{2 - \omega_i^\nu - \omega_j^\nu} - \nu_{ij}^{(kt)}.
$$

It can be checked that Eq. (14) is equivalent to

$$f=\sum_{k=1}^{m}\lambda_k\sum_{t=1}^{p}\delta_{kt}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\hat{\varepsilon}_{ij}^{+(kt)}+\hat{\varepsilon}_{ij}^{-(kt)}+\hat{\xi}_{ij}^{+(kt)}+\hat{\xi}_{ij}^{-(kt)}),$$

where $\hat{\varepsilon}_{ij}^{+(kt)}=\frac{|\hat{\varepsilon}_{ij}^{(kt)}|+\hat{\varepsilon}_{ij}^{(kt)}}{2}$, $\hat{\varepsilon}_{ij}^{-(kt)}=\frac{|\hat{\varepsilon}_{ij}^{(kt)}|-\hat{\varepsilon}_{ij}^{(kt)}}{2}$, $\hat{\xi}_{ij}^{+(kt)}=\frac{|\hat{\xi}_{ij}^{(kt)}|+\hat{\xi}_{ij}^{(kt)}}{2}$, $\hat{\xi}_{ij}^{-(kt)}=\frac{|\hat{\xi}_{ij}^{(kt)}|-\hat{\xi}_{ij}^{(kt)}}{2}$. Therefore, a fractional programming Model 3 can be established for this multi-criteria intuitionistic fuzzy GDM problem.

Since $\frac{\omega_i^{\mu}}{2-\omega_i^{\nu}-\omega_j^{\nu}}-\mu_{ij}^{(kt)}-\hat{\varepsilon}_{ij}^{+(kt)}+\hat{\varepsilon}_{ij}^{-(kt)}=0$ $(i=1,2,\cdots,n-1,\ j=i+1\cdots,n,\ t=1,2,\cdots,p,\ k=1,2,\cdots,m)$, and $\sum_{t=1}^{p}\delta_{kt}=1$, $\sum_{k=1}^{m}\lambda_k=1$, we can get

$$\frac{\omega_i^{\mu}}{2-\omega_i^{\nu}-\omega_j^{\nu}}-\sum_{k=1}^{m}\lambda_k\sum_{t=1}^{p}\delta_{kt}\mu_{ij}^{(kt)}-\sum_{k=1}^{m}\lambda_k\sum_{t=1}^{p}\delta_{kt}\hat{\varepsilon}_{ij}^{+(kt)}+\sum_{k=1}^{m}\lambda_k\sum_{t=1}^{p}\delta_{kt}\hat{\varepsilon}_{ij}^{-(kt)}=0.$$

Similarly, we can also have

$$\frac{\omega_j^{\mu}}{2-\omega_i^{\nu}-\omega_j^{\nu}}-\sum_{k=1}^{m}\lambda_k\sum_{t=1}^{p}\delta_{kt}\nu_{ij}^{(kt)}-\sum_{k=1}^{m}\lambda_k\sum_{t=1}^{p}\delta_{kt}\hat{\xi}_{ij}^{+(kt)}+\sum_{k=1}^{m}\lambda_k\sum_{t=1}^{p}\delta_{kt}\hat{\xi}_{ij}^{-(kt)}=0.$$

Let $\hat{\varepsilon}_{ij}^{+}=\sum_{k=1}^{m}\lambda_k\sum_{t=1}^{p}\delta_{kt}\hat{\varepsilon}_{ij}^{+(kt)}$, $\hat{\varepsilon}_{ij}^{-}=\sum_{k=1}^{m}\lambda_k\sum_{t=1}^{p}\delta_{kt}\hat{\varepsilon}_{ij}^{-(kt)}$, $\hat{\xi}_{ij}^{+}=\sum_{k=1}^{m}\lambda_k\sum_{t=1}^{p}\delta_{kt}\hat{\xi}_{ij}^{+(kt)}$ and $\hat{\xi}_{ij}^{-}=\sum_{k=1}^{m}\lambda_k\sum_{t=1}^{p}\delta_{kt}\hat{\xi}_{ij}^{-(kt)}$, then the Model 3 can be transformed into a fractional programming Model 4.

Using Lingo or Matlab to solve Model 4, the normalized overall intuitionistic fuzzy priority weight vector $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T=\big((\omega_1^{\mu},\omega_1^{\nu}),(\omega_2^{\mu},\omega_2^{\nu}),\cdots,(\omega_n^{\mu},\omega_n^{\nu})\big)^T$ can be generated without using the aggregation operator. Then the ranking order of the alternatives can be given via the comparison law for IFNs.

For the convenience of application, the procedure of this method can be clarified as the following algorithm.

Algorithm II:

Step 1. See step 1 in Algorithm I.

Step 2. Establish a fractional programming model according to the Model 4. Go to the next step.

Step 3. Solve the constructed fractional programming model to derive the normalized overall intuitionistic fuzzy priority weight vector $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$. Go to the next step.

Step 4. Compare the overall intuitionistic fuzzy priority weights $\omega_1,\omega_2,\cdots,\omega_n$ to obtain the ranking order of the alternatives. Go to the next step.

Step 5. End.

Example 2. Below we use Algorithm II to solve the Example 1 again, and a fractional programming Model 5 can be generated from the Model 4.

Solving Model 5 with Lingo software, the normalized overall intuitionistic fuzzy priority weight vector is given by

$$\begin{aligned}\omega&=(\omega_1,\omega_2,\omega_3,\omega_4)^T\\&=\big((0.3125,0.5261),(0.3000,0.6563),\\&\quad(0.1525,0.7902),(0.0736,0.8660)\big)^T.\end{aligned}$$

Since $L(\omega_1)=0.4080>L(\omega_2)=0.3293>L(\omega_3)=0.1984>L(\omega_4)=0.1264$, then from Zhang and Xu's ranking method [27], we have $\omega_1>\omega_2>\omega_3>\omega_4$, which means the family's best choice is still the first car x_1.

VI. Comparative Analysis

To verify the validity of the developed GDM approach with IFPRs, we use different transformation formulas to solve the same example again, the overall priority weight vectors and the ranking orders are presented as shown in Table I, where the decision makers are taken as a group.

From Table I, we can see that the same ranking order $x_1>x_2>x_3>x_4$ can be obtained by our transformation formula and Liao and Xu's formula [10], which validate the effectiveness of the proposed Algorithm II, because these two formulas are similar. However, if we use the transformation formula Eq. (9), different ranking order $x_2>x_1>x_3>x_4$ is derived, and the family's best choice becomes the second car x_2. In the Section III, we have showed that the Eq. (9) used to construct a multiplicative consistent IFPR is unreasonable, which means the theory established by Eq. (9) will be inappropriate and the final result $x_2>x_1>x_3>x_4$ is not convincing.

In this paper, we assume that different experts have different criteria weights. In order to illustrate the effect of criteria weights, we suppose the weights of criteria to be $\delta_1=\delta_2=\delta_3=\{0.7,0.1,0.1,0.1\}^T$ in the Example 1, and the overall priority weight vectors and ranking orders are obtained as shown in Table II.

From Table II, we can see that the final ranking order becomes $x_2>x_1>x_3>x_4$ for $\delta_i=\{0.7,0.1,0.1,0.1\}^T$ $(i=1,2,3)$, which is different from the ranking order $x_1>x_2>x_3>x_4$ derived in our previous two examples. Thus, in a real decision making problem, one can not always suppose that the experts have same criteria weights, and it is reasonable and useful to assume that different decision makers should have different criteria weights. If we compare the priority weights obtained by the two methods, we can find that the results of them are similar. However, since the operators may loss some information in the process of aggregation, the Algorithm II will produce a more convincing result than the Algorithm I. Moreover, we can not guarantee the final overall intuitionistic fuzzy priority weight vector synthesized in Algorithm I is still normalized, while the Model 4 always generate a normalized intuitionistic fuzzy priority weight vector. The drawback of Algorithm II is that it can only handle the case where the decision makers are taken as a group and it can not deal with the first situation when the experts are seen as separate individuals.

VII. Conclusion

In this paper, decision makers establish some IFPRs to express their preferences by comparing each pair of alternatives with respect to each criterion, and the weights of criteria can be different for different experts, then we propose two algorithms for multi-criteria GDM based on the multiplicative consistency of IFPR. In the first scenario, decision makers' individual intuitionistic fuzzy priority weights with respect to each criterion are obtained from some fractional programming models, and using the aggregation operator, the

Model 3 $$Min\ f=\sum_{k=1}^{m}\lambda_k\sum_{t=1}^{p}\delta_{kt}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\hat{\varepsilon}_{ij}^{+(kt)}+\hat{\varepsilon}_{ij}^{-(kt)}+\hat{\xi}_{ij}^{+(kt)}+\hat{\xi}_{ij}^{-(kt)})$$

$$s.t.\begin{cases}\frac{\omega_i^\mu}{2-\omega_i^\nu-\omega_j^\nu}-\mu_{ij}^{(kt)}-\hat{\varepsilon}_{ij}^{+(kt)}+\hat{\varepsilon}_{ij}^{-(kt)}=0,\quad i=1,2,\cdots,n-1,\ j=i+1,\cdots,n,\ t=1,2,\cdots,p,\ k=1,2,\cdots,m\\ \frac{\omega_j^\mu}{2-\omega_i^\nu-\omega_j^\nu}-\nu_{ij}^{(kt)}-\hat{\xi}_{ij}^{+(kt)}+\hat{\xi}_{ij}^{-(kt)}=0,\quad i=1,2,\cdots,n-1,\ j=i+1,\cdots,n,\ t=1,2,\cdots,p,\ k=1,2,\cdots,m\\ \omega_i^\mu,\omega_i^\nu\in[0,1],\ \omega_i^\mu+\omega_i^\nu\le 1,\quad i=1,2,\cdots,n\\ \sum_{j=1,j\neq i}^{n}\omega_j^\mu\le\omega_i^\nu,\ \omega_i^\mu+n-2\ge\sum_{j=1,j\neq i}^{n}\omega_j^\nu,\quad i=1,2,\cdots,n\\ \hat{\varepsilon}_{ij}^{+(kt)}\ge 0,\ \hat{\varepsilon}_{ij}^{-(kt)}\ge 0,\ \hat{\xi}_{ij}^{+(kt)}\ge 0,\ \hat{\xi}_{ij}^{-(kt)}\ge 0,\quad i=1,\cdots,n-1,\ j=i+1,\cdots,n,\ t=1,\cdots,p,\ k=1,\cdots,m\\ \hat{\varepsilon}_{ij}^{+(kt)}\cdot\hat{\varepsilon}_{ij}^{-(kt)}=0,\ \hat{\xi}_{ij}^{+(kt)}\cdot\hat{\xi}_{ij}^{-(kt)}=0,\ i=1,2,\cdots,n-1,\ j=i+1,\cdots,n,\ t=1,2,\cdots,p,\ k=1,2,\cdots,m\end{cases}$$

Model 4 $$Min\ f=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\hat{\varepsilon}_{ij}^{+}+\hat{\varepsilon}_{ij}^{-}+\hat{\xi}_{ij}^{+}+\hat{\xi}_{ij}^{-})$$

$$s.t.\begin{cases}\frac{\omega_i^\mu}{2-\omega_i^\nu-\omega_j^\nu}-\sum_{k=1}^{m}\lambda_k\sum_{t=1}^{p}\delta_{kt}\mu_{ij}^{(kt)}-\hat{\varepsilon}_{ij}^{+}+\hat{\varepsilon}_{ij}^{-}=0,\quad i=1,2,\cdots,n-1,\ j=i+1,\cdots,n\\ \frac{\omega_j^\mu}{2-\omega_i^\nu-\omega_j^\nu}-\sum_{k=1}^{m}\lambda_k\sum_{t=1}^{p}\delta_{kt}\nu_{ij}^{(kt)}-\hat{\xi}_{ij}^{+}+\hat{\xi}_{ij}^{-}=0,\quad i=1,2,\cdots,n-1,\ j=i+1,\cdots,n\\ \omega_i^\mu,\omega_i^\nu\in[0,1],\ \omega_i^\mu+\omega_i^\nu\le 1,\quad i=1,2,\cdots,n\\ \sum_{j=1,j\neq i}^{n}\omega_j^\mu\le\omega_i^\nu,\ \omega_i^\mu+n-2\ge\sum_{j=1,j\neq i}^{n}\omega_j^\nu,\quad i=1,2,\cdots,n\\ \hat{\varepsilon}_{ij}^{+}\ge 0,\ \hat{\varepsilon}_{ij}^{-}\ge 0,\ \hat{\xi}_{ij}^{+}\ge 0,\ \hat{\xi}_{ij}^{-}\ge 0,\quad i=1,2,\cdots,n-1,\ j=i+1,\cdots,n\end{cases}$$

Model 5 $$\begin{aligned}Min\quad f=&(\hat{\varepsilon}_{12}^{+}+\hat{\varepsilon}_{12}^{-}+\hat{\xi}_{12}^{+}+\hat{\xi}_{12}^{-})+(\hat{\varepsilon}_{13}^{+}+\hat{\varepsilon}_{13}^{-}+\hat{\xi}_{13}^{+}+\hat{\xi}_{13}^{-})+(\hat{\varepsilon}_{14}^{+}+\hat{\varepsilon}_{14}^{-}+\hat{\xi}_{14}^{+}+\hat{\xi}_{14}^{-})\\&+(\hat{\varepsilon}_{23}^{+}+\hat{\varepsilon}_{23}^{-}+\hat{\xi}_{23}^{+}+\hat{\xi}_{23}^{-})+(\hat{\varepsilon}_{24}^{+}+\hat{\varepsilon}_{24}^{-}+\hat{\xi}_{24}^{+}+\hat{\xi}_{24}^{-})+(\hat{\varepsilon}_{34}^{+}+\hat{\varepsilon}_{34}^{-}+\hat{\xi}_{34}^{+}+\hat{\xi}_{34}^{-})\end{aligned}$$

$$s.t.\begin{cases}\frac{\omega_1^\mu}{2-\omega_1^\nu-\omega_2^\nu}-0.388-\hat{\varepsilon}_{12}^{+}+\hat{\varepsilon}_{12}^{-}=0,\quad \frac{\omega_2^\mu}{2-\omega_1^\nu-\omega_2^\nu}-0.319-\hat{\xi}_{12}^{+}+\hat{\xi}_{12}^{-}=0,\\ \frac{\omega_1^\mu}{2-\omega_1^\nu-\omega_3^\nu}-0.457-\hat{\varepsilon}_{13}^{+}+\hat{\varepsilon}_{13}^{-}=0,\quad \frac{\omega_3^\mu}{2-\omega_1^\nu-\omega_3^\nu}-0.223-\hat{\xi}_{13}^{+}+\hat{\xi}_{13}^{-}=0,\\ \frac{\omega_1^\mu}{2-\omega_1^\nu-\omega_4^\nu}-0.512-\hat{\varepsilon}_{14}^{+}+\hat{\varepsilon}_{14}^{-}=0,\quad \frac{\omega_4^\mu}{2-\omega_1^\nu-\omega_4^\nu}-0.217-\hat{\xi}_{14}^{+}+\hat{\xi}_{14}^{-}=0,\\ \frac{\omega_2^\mu}{2-\omega_2^\nu-\omega_3^\nu}-0.542-\hat{\varepsilon}_{23}^{+}+\hat{\varepsilon}_{23}^{-}=0,\quad \frac{\omega_3^\mu}{2-\omega_2^\nu-\omega_3^\nu}-0.178-\hat{\xi}_{23}^{+}+\hat{\xi}_{23}^{-}=0,\\ \frac{\omega_2^\mu}{2-\omega_2^\nu-\omega_4^\nu}-0.628-\hat{\varepsilon}_{24}^{+}+\hat{\varepsilon}_{24}^{-}=0,\quad \frac{\omega_4^\mu}{2-\omega_2^\nu-\omega_4^\nu}-0.154-\hat{\xi}_{24}^{+}+\hat{\xi}_{24}^{-}=0,\\ \frac{\omega_3^\mu}{2-\omega_3^\nu-\omega_4^\nu}-0.481-\hat{\varepsilon}_{34}^{+}+\hat{\varepsilon}_{34}^{-}=0,\quad \frac{\omega_4^\mu}{2-\omega_3^\nu-\omega_4^\nu}-0.214-\hat{\xi}_{34}^{+}+\hat{\xi}_{34}^{-}=0,\\ 0\le\omega_1^\mu\le 1,0\le\omega_1^\nu\le 1,0\le\omega_2^\mu\le 1,0\le\omega_2^\nu\le 1,0\le\omega_3^\mu\le 1,0\le\omega_3^\nu\le 1,0\le\omega_4^\mu\le 1,0\le\omega_4^\nu\le 1,\\ \omega_1^\mu+\omega_1^\nu\le 1,\omega_2^\mu+\omega_2^\nu\le 1,\omega_3^\mu+\omega_3^\nu\le 1,\omega_4^\mu+\omega_4^\nu\le 1,\\ \omega_1^\mu+\omega_2^\mu+\omega_3^\mu\le\omega_4^\nu,\omega_1^\mu+\omega_2^\mu+\omega_4^\mu\le\omega_3^\nu,\omega_1^\mu+\omega_3^\mu+\omega_4^\mu\le\omega_2^\nu,\omega_2^\mu+\omega_3^\mu+\omega_4^\mu\le\omega_1^\nu,\\ \omega_1^\mu+2\ge\omega_2^\nu+\omega_3^\nu+\omega_4^\nu,\omega_2^\mu+2\ge\omega_1^\nu+\omega_3^\nu+\omega_4^\nu,\omega_3^\mu+2\ge\omega_1^\nu+\omega_2^\nu+\omega_4^\nu,\omega_4^\mu+2\ge\omega_1^\nu+\omega_2^\nu+\omega_3^\nu,\\ \hat{\varepsilon}_{12}^{+}\ge 0,\hat{\varepsilon}_{12}^{-}\ge 0,\hat{\xi}_{12}^{+}\ge 0,\hat{\xi}_{12}^{-}\ge 0,\hat{\varepsilon}_{13}^{+}\ge 0,\hat{\varepsilon}_{13}^{-}\ge 0,\hat{\xi}_{13}^{+}\ge 0,\hat{\xi}_{13}^{-}\ge 0,\hat{\varepsilon}_{14}^{+}\ge 0,\hat{\varepsilon}_{14}^{-}\ge 0,\hat{\xi}_{14}^{+}\ge 0,\hat{\xi}_{14}^{-}\ge 0,\\ \hat{\varepsilon}_{23}^{+}\ge 0,\hat{\varepsilon}_{23}^{-}\ge 0,\hat{\xi}_{23}^{+}\ge 0,\hat{\xi}_{23}^{-}\ge 0,\hat{\varepsilon}_{24}^{+}\ge 0,\hat{\varepsilon}_{24}^{-}\ge 0,\hat{\xi}_{24}^{+}\ge 0,\hat{\xi}_{24}^{-}\ge 0,\hat{\varepsilon}_{34}^{+}\ge 0,\hat{\varepsilon}_{34}^{-}\ge 0,\hat{\xi}_{34}^{+}\ge 0,\hat{\xi}_{34}^{-}\ge 0.\end{cases}$$

TABLE I

OVERALL PRIORITY WEIGHT VECTORS AND RANKING ORDERS WITH DIFFERENT TRANSFORMATION FORMULAS.

Transformation formula	Method	Overall priority weight vector $(\omega_1,\omega_2,\omega_3,\omega_4)^T$	Ranking order
Formula Eq. (8) in [10]	Algorithm II	$((0.2690, 0.4529), (0.2583, 0.6664), (0.1313, 0.7700), (0.0633, 0.8326))^T$	$x_1 > x_2 > x_3 > x_4$
Formula Eq. (9) in [15], [16]	Algorithm II	$((0.2792, 0.5452), (0.3928, 0.4317), (0.0912, 0.7479), (0.0612, 0.9388))^T$	$x_2 > x_1 > x_3 > x_4$
Formula Eq. (10) in this paper	Algorithm II	$((0.3125, 0.5261), (0.3000, 0.6563), (0.1525, 0.7902), (0.0736, 0.8660))^T$	$x_1 > x_2 > x_3 > x_4$

TABLE II

OVERALL PRIORITY WEIGHT VECTORS AND RANKING ORDERS WITH DIFFERENT CRITERIA WEIGHT VECTORS.

Method	Weight vector δ_i $(i=1,2,3)$	Overall priority weight vector $(\omega_1,\omega_2,\omega_3,\omega_4)^T$	Ranking order
Algorithm I	$\delta_1,\delta_2,\delta_3$ in Example 1	$((0.2910, 0.5800), (0.3104, 0.6244), (0.1338, 0.7861), (0.0590, 0.8922))^T$	$x_1 > x_2 > x_3 > x_4$
Algorithm II	$\delta_1,\delta_2,\delta_3$ in Example 1	$((0.3125, 0.5261), (0.3000, 0.6563), (0.1525, 0.7902), (0.0736, 0.8660))^T$	$x_1 > x_2 > x_3 > x_4$
Algorithm I	$\delta_i=\{0.7, 0.1, 0.1, 0.1\}^T$	$((0.2833, 0.5850), (0.3418, 0.5983), (0.1240, 0.7924), (0.0513, 0.8964))^T$	$x_2 > x_1 > x_3 > x_4$
Algorithm II	$\delta_i=\{0.7, 0.1, 0.1, 0.1\}^T$	$((0.2859, 0.5522), (0.3498, 0.5885), (0.1388, 0.8111), (0.0636, 0.8862))^T$	$x_2 > x_1 > x_3 > x_4$

overall priority weight vector can be synthesized. While for the second case, the decision makers are taken as a group, and the normalized overall intuitionistic fuzzy priority weight vector can be derived directly from one building fractional programming model. Finally, we have applied the proposed two methods to select an optimal car for a family.

Although this paper develops two techniques for multi-criteria GDM with IFPRs, there are still some problems that need further study. In a real decision making problem, it is almost impossible for decision makers to give multiplicative consistent IFPRs. Thus, how to repair the inconsistent IFPRs to be of acceptable consistency becomes a critical problem. For a GDM problem, how to measure the consensus degree and how to reach the consensus are also important. Moreover, in our model, the weights of decision makers and the weights of criteria are determined beforehand, how to get the corresponding objective weights can also be analyzed in the future.

REFERENCES

[1] R. Urena, F. Chiclana, J. A. Morente-Molinera, and E. Herrera-Viedma, "Managing incomplete preference relations in decision making: A review and future trends," *Information Sciences*, vol. 302, pp. 14-32, 2015.

[2] F. Chiclana, E. Herrera-Viedma, F. Herrera, and S. Alonso, "Induced ordered weighted geometric operators and their use in the aggregation of multiplicative preference relations," *International Journal of Intelligent Systems*, vol. 19, no. 3, pp. 233-255, 2004.

[3] F. Y. Meng and X. H. Chen, "A new method for group decision making with incomplete fuzzy preference relations," *Knowledge-Based Systems*, vol. 73, pp. 111-123, 2015.

[4] E. Barrenechea, J. Fernandez, M. Pagola, F. Chiclana, and H. Bustince, "Construction of interval-valued fuzzy preference relations from ignorance functions and fuzzy preference relations. Application to decision making," *Knowledge-Based Systems*, vol. 58, pp. 33-44, 2014.

[5] J. Wu and F. Chiclana, "Visual information feedback mechanism and attitudinal prioritisation method for group decision making with triangular fuzzy complementary preference relations," *Information Sciences*, vol. 279, pp. 716-734, 2014.

[6] E. Szmidt and J. Kacprzyk, "A consensus-reaching process under intuitionistic fuzzy preference relations," *International Journal of Intelligent Systems*, vol. 18, no. 7, pp. 837-852, 2003.

[7] Z. S. Xu and X. Q. Cai, "Group decision making with incomplete interval-valued intuitionistic preference relations," *Group Decision and Negotiation*, vol. 24, pp. 193-215, 2015.

[8] S. P. Wan, G. L. Xu, and J. Y. Dong, "A novel method for group decision making with interval-valued Atanassov intuitionistic fuzzy preference relations," *Information Sciences*, vol. 372, pp. 53-71, 2016.

[9] Z. J. Wang, "Derivation of intuitionistic fuzzy weights based on intuitionistic fuzzy preference relations," *Applied Mathematical Modelling*, vol. 37, no. 9, pp. 6377-6388, 2013.

[10] H. C. Liao and Z. S. Xu, "Priorities of intuitionistic fuzzy preference relation based on multiplicative consistency," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 6, pp. 1669-1681, 2014.

[11] H. C. Liao and Z. S. Xu, "Some algorithms for group decision making with intuitionistic fuzzy preference information," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 22, no. 4, pp. 505-529, 2014.

[12] H. C. Liao, Z. S. Xu, X. J. Zeng, and J. M. Merigo, "Framework of group decision making with intuitionistic fuzzy preference information," *IEEE Transactions on Fuzzy Systems*, vol. 23, no. 4, pp. 1211-1226, 2015.

[13] Z. S. Xu and H. C. Liao, "A survey of approaches to decision making with intuitionistic fuzzy preference relations," *Knowledge-Based Systems*, vol. 80, pp. 131-142, 2015.

[14] G. L. Xu, S. P. Wan, F. Wang, J. Y. Dong, and Y. F. Zeng, "Mathematical programming methods for consistency and consensus in group decision making with intuitionistic fuzzy preference relations," *Knowledge-Based Systems*, vol. 98, pp. 30-43, 2016.

[15] F. F. Jin, Z. W. Ni, H. Y. Chen, and Y. P. Li, "Approaches to group decision making with intuitionistic fuzzy preference relations based on multiplicative consistency," *Knowledge-Based Systems*, vol. 97, pp. 48-59, 2016.

[16] Y. Lin and Y. M. Wang, "Group decision making with consistency of intuitionistic fuzzy preference relations under uncertainty," *IEEE/CAA Journal of Automatica Sinica*, vol. 5, no. 3, pp. 741-748, 2018.

[17] M. Supriya, K. Sangeeta, and G. K. Patra, "Trustworthy cloud service provider selection using multi criteria decision making methods," *Engineering Letters*, vol. 24, no. 1, pp. 1-10, 2016.

[18] D. S. Xu, Y. R. Hong, and H. Qiao, "Multiple attribute decision making based on neutrosophic sets in venture capital," *Engineering Letters*, vol. 26, no. 4, pp. 441-446, 2018.

[19] Z. L. Yue, "TOPSIS-based group decision-making methodology in intuitionistic fuzzy setting," *Information Sciences*, vol. 277, pp. 141-153, 2014.

[20] Y. B. Zhang, K. W. Li, and Z. J. Wang, "Prioritization and aggregation of intuitionistic preference relations: a multiplicative-transitivity-based transformation from intuitionistic judgment data to priority weights," *Group Decision and Negotiation*, vol. 26, pp. 409-436, 2017.

[21] H. Garg and K. Kumar, "Improved possibility degree method for ranking intuitionistic fuzzy numbers and their application in multiattribute decision-making," *Granular Computing*, vol. 4, pp. 237-247, 2019.

[22] Z. S. Xu and H. C. Liao, "Intuitionistic fuzzy analytic hierarchy process," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 4, pp. 749-761, 2014.

[23] Z. S. Xu, X. Q. Cai, and E. Szmidt, "Algorithms for estimating missing elements of incomplete intuitionistic preference relations," *International Journal of Intelligent Systems*, vol. 26, pp. 787-813, 2011.

[24] T. Tanino, "Fuzzy preference orderings in group decision making," *Fuzzy Sets and Systems*, vol. 12, no. 2, pp. 117-131, 1984.

[25] Z. S. Xu, "A survey of preference relations," *International Journal of General Systems*, vol. 36, no. 2, pp. 179-203, 2007.

[26] Z. S. Xu, "Intuitionistic preference relations and their application in group decision making," *Information Sciences*, vol. 177, pp. 2363-2379, 2007.

[27] X. M. Zhang and Z. S. Xu, "A new method for ranking intuitionistic fuzzy values and its application in multi-attribute decision making," *Fuzzy Optimization and Decision Making*, vol. 11, no. 2, pp. 135-146, 2012.

[28] Z. S. Xu, "Intuitionistic fuzzy aggregation operators," *IEEE Transactions on Fuzzy Systems*, vol. 15, no. 6, pp. 1179-1187, 2007.

[29] Z. S. Xu and R. R. Yager, "Some geometric aggregation operators based on intuitionistic fuzzy sets," *International Journal of General Systems*, vol. 35, no. 4, pp. 417-433, 2006.