The Influence of Positive Feedback Control to a Single Species Stage Structure System

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***Abstract*—A single species stage structure system with positive feedback control is proposed and studied in this paper. Local and global stability property of the boundary equilibrium and the positive equilibrium are investigated, respectively. If the system without feedback control is extinct, then we show that by choosing suitable control variables, extinct species can become globally stable, or still keep the property of extinction. If the system without feedback control is global stability, then to ensure the system admits a globally stable positive equilibrium, we should also restrict the feedback control variable to the suitable interval. An example together with its numeric simulations is presented to verify the main results.**

***Index Terms*—Stage structure; Species; Local stability; Lyapunov function; Global stability**

I. INTRODUCTION

THE aim of this paper is to investigate the dynamic behaviors of the following single species stage structure system with feedback control:

$$\begin{aligned}\frac{dx_1}{dt} &= \alpha x_2 - \beta x_1 - \delta_1 x_1,\\ \frac{dx_2}{dt} &= \beta x_1 - \delta_2 x_2 - \gamma x_2^2 + d x_2 u,\\ \frac{du}{dt} &= g - eu - f x_2,\end{aligned} \tag{1.1}$$

where $\alpha, \beta, \delta_1, \delta_2, d, e, f, g$ and γ are all positive constants, $x_1(t)$ and $x_2(t)$ are the densities of the immature and mature species at time t, u is feedback control variable. The following assumptions are made in formulating the model (1.1):
1. The per capita birth rate of the immature population is $\alpha > 0$; The per capita death rate of the immature population is $\delta_1 > 0$; The per capita death rate of the mature plants is proportional to the current mature plants population with a proportionality constant $\delta_2 > 0$; $\beta > 0$ denotes the surviving rate of immaturity to reach maturity; The mature species is density dependent with the parameter $\gamma > 0$;
2. The bilinear feedback mechanism (dx_2u) is used to control the system, which can be interpreted as the stocking of the mature species.

During the last decades, many scholars investigated the dynamic behaviors of the stage structured ecosystem, see [1]-[15] and the references cited therein. Such topics as the single species stage structure system ([11], [13],[15],[16]), the stage structured predator prey system ([1],[2],[3],[7],[8],[10],[12],[14],[49],[50]), the stage structured competition system ([6]), the stage structured cooperative system ([5],[9]), the stage-structured food-chain system ([4]) are well investigated.

Recently, Khajanchi and Banerjee[17] proposed the following stage structure predator-prey model with ratio dependent functional response

$$\begin{aligned}\frac{dx_1}{dt} &= \alpha x_2(t) - \beta x_1(t) - \delta_1 x_1(t),\\ \frac{dx_2}{dt} &= \beta x_1(t) - \delta_2 x_2(t) - \gamma x_2^2(t)\\ &\quad - \frac{\eta(1-\theta)x_2(t)y(t)}{g(1-\theta)x_2(t) + hy(t)},\\ \frac{dy}{dt} &= \frac{u\eta(1-\theta)x_2(t)y(t)}{g(1-\theta)x_2(t) + hy(t)} - \delta_3 y(t).\end{aligned} \tag{1.2}$$

Here, the authors assumed that the prey species has two stage: immature(x_1) and mature (x_2), and predator species take mature prey species as its food. Obviously, if we did not consider the predator species in above system, then the prey species satisfies the following single species stage structured system.

$$\begin{aligned}\frac{dx_1}{dt} &= \alpha x_2(t) - \beta x_1(t) - \delta_1 x_1(t),\\ \frac{dx_2}{dt} &= \beta x_1(t) - \delta_2 x_2(t) - \gamma x_2^2(t).\end{aligned} \tag{1.3}$$

The system admits two possible equilibria; $O(0,0)$ and $A(x_1^*, x_2^*)$. Xiao and Lei[16] showed that if

$$\alpha\beta < \delta_2(\beta + \delta_1) \tag{1.4}$$

holds, then the boundary equilibrium $O(0,0)$ is globally asymptotically stable, which means that the species will be driven to extinction, and if

$$\alpha\beta > \delta_2(\beta + \delta_1) \tag{1.5}$$

holds, then the positive equilibrium $A(x_1^*, x_2^*)$ is globally asymptotically stable, which means the species could be exists in the long run.

On the other hand, since the pioneer work of Gopalsamy and Weng[18], the feedback control ecosystem become one of the main topics in the study of mathematics biology, see [18]-[45] and the references therein. Xiao, Tang and Chen [41] had proposed the following two species competitive system with feedback controls:

$$\begin{aligned}x_1'(t) &= x_1(t)[a_1(t) - b_{11}(t)x_1(t)\\ &\quad - b_{12}(t)x_2(t) - c_1(t)u_1(t)],\\ x_2'(t) &= x_2(t)[a_2(t) - b_{21}(t)x_1(t)\\ &\quad - b_{22}(t)x_2(t) + c_2(t)u_2(t)],\\ u_1'(t) &= -e_1(t)u_1(t) + d_1(t)x_1(t),\\ u_2'(t) &= f(t) - e_2(t)u_2(t) - d_2(t)x_2(t).\end{aligned} \tag{1.6}$$

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They showed that if

$$a_{1L} > b_{12M}\frac{a_{2M}}{b_{22L}},\ a_{2M} \le b_{21L}\frac{a_{1L}}{b_{11M}} \tag{1.7}$$

and

$$\begin{aligned} & a_{1L} - \frac{a_{2M}b_{12M}}{b_{22L}} \\ > \quad & \frac{c_{1M}a_{1M}d_{1M}}{e_{1L}b_{11L}} + \frac{b_{12M}c_{2M}f_M}{b_{22L}e_{2L}}, \\ & \frac{c_{2L}}{e_{2M}}\left(f_L - \frac{d_{2M}a_{2M}e_{2L} + d_{2M}c_{2M}f_M}{b_{22L}e_{2L}}\right) \\ > \quad & \frac{a_{1M}b_{21M}}{b_{11L}} - a_{2L}, \end{aligned} \tag{1.8}$$

hold, then system (1.6) is permanent. Noting that for the system without feedback controls, condition (1.7) is enough to ensure the extinction of the second species. Therefore, the results of [41] implies that by choosing a suitable feedback control mechanism, one could avoid the extinction of the species. The results of [41] is then generalized by Chen, Li and Huang [42] and Hu, Teng and Jiang[43] to the infinite delay case.

Though there are many works on feedback control ecosystem ([18]-[50]), there are still few works on stage structured ecosystem with feedback controls. Ding and Cheng[13] proposed the following single species stage-structured model with feedback control:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= \alpha x_2(t) - \gamma x_1(t) - \alpha e^{-\gamma\tau}x_2(t-\tau), \\ \frac{dx_2(t)}{dt} &= \alpha e^{-\gamma\tau}x_2(t-\tau) - \beta x_2^2(t) - cx_2(t)u(t), \\ \frac{du(t)}{dt} &= -au(t) + bx_2(t). \end{aligned} \tag{1.9}$$

In [13], it was shown that under some suitable assumption, the system admits a unique positive equilibrium which is globally attractive.

Now stimulated by the works of [11], [13], [37], [41],[42] and [43], and the fact that with the developing of modern society, many nature resources are overexploited, more and more species become endangered, without the suitable human beings protect, the species will finally be driven to extinction. Thus, one interesting issue proposed:

For system (1.3), if $O(0,0)$ is globally asymptotically stable, which implies the extinction of the species, is it possible for us choosing suitable feedback control variable, as was proposed by Xiao et al[41], to avoid the extinction of the species?

On the other hand, what would happen if we stock too much mature species to the nature? That is:

For system (1.3), if $A(x_1^*, x_2^*)$ is globally asymptotically stable, which implies the survival of the species in the long run, what would happen if the feedback control mechanism is too strong? whether the positive feedback control always has positive effect to the survival of the species?

Above two issues lead us to propose the system (1.1). The paper is arranged as follows. We will investigate the existence and locally stability property of the equilibria of system (1.1) in section 2. In section 3, by constructing some suitable Lyapunov function, we are able to investigate the global stability property of the equilibria. Section 4 presents some numerical simulations to show the feasibility of the main results. We end this paper by a briefly discussion.

II. Local stability

For the rest of the paper, we will assume that inequality (1.4) holds, that is, without the feedback control variable, the system (1.3) will be driven to extinction.

The system (1.1) always admits the boundary equilibrium $A_1\Big(0, 0, \dfrac{g}{e}\Big)$.

If

$$\Big(\delta_2 + \frac{g\gamma}{f}\Big)\Big(\beta + \delta_1\Big) > \beta\alpha > \Big(\delta_2 - \frac{dg}{e}\Big)\Big(\beta + \delta_1\Big) \tag{2.1}$$

holds, then system (1.1) admits a unique positive equilibrium $A_2(x_{1*}, x_{2*}, u_*)$, where

$$\begin{aligned} x_{1*} &= \frac{\alpha x_{2*}}{\beta + \delta_1} \\ &= \frac{\alpha\Big[e\Big(\alpha\beta - \delta_2(\beta+\delta_1)\Big) + dg(\delta_1 + \beta)\Big]}{(df + e\gamma)(\beta+\delta_1)^2}, \\ x_{2*} &= \frac{e\Big(\alpha\beta - \delta_2(\beta+\delta_1)\Big) + dg(\delta_1+\beta)}{(df + e\gamma)(\beta + \delta_1)} \\ &= \frac{\dfrac{\alpha\beta}{\beta+\delta_1} - \delta_2 + \dfrac{dg}{e}}{\gamma + d\frac{f}{e}}, \\ u_* &= \frac{g}{e} - \frac{f}{e}x_{2*} \\ &= \frac{-f\Big(\alpha\beta - \delta_2(\beta+\delta_1)\Big) + g\gamma(\delta_1+\beta)}{(df+e\gamma)(\beta+\delta_1)}. \end{aligned} \tag{2.2}$$

Obviously, x_{1*}, x_{2*} and u_* satisfies the equation

$$\begin{cases} \alpha x_{2*} - \beta x_{1*} - \delta_1 x_{1*} = 0, \\ \beta x_{1*} - \delta_2 x_{2*} - \gamma x_{2*}^2 + dx_{2*}u_* = 0, \\ g - eu_* - fx_{2*} = 0. \end{cases} \tag{2.3}$$

We shall now investigate the local stability property of the above equilibria.

The variational matrix of the system (1.1) is

$$\begin{aligned} & J(x_1, x_2, u) \\ = & \begin{pmatrix} -\beta - \delta_1 & \alpha & 0 \\ \beta & -\delta_2 - 2\gamma x_2 + du & dx_2 \\ 0 & -f & -e \end{pmatrix}. \end{aligned} \tag{2.4}$$

Theorem 2.1 *Assume that*

$$\beta\alpha < \Big(\delta_2 - \frac{dg}{e}\Big)\Big(\beta + \delta_1\Big) \tag{2.5}$$

holds, then $A_1\Big(0,0,\dfrac{g}{e}\Big)$ *is locally asymptotically stable.*
Proof. Condition (2.5) implies that

$$\delta_2-\frac{dg}{e}>0,$$

and so,

$$-(\beta+\delta_1+\delta_2)+\frac{dg}{e}<0. \tag{2.6}$$

From (2.4) we could see that the Jacobian matrix of the system about the equilibrium point $A_1\Big(0,0,\dfrac{g}{e}\Big)$ is given by

$$\begin{pmatrix} -\beta-\delta_1 & \alpha & 0\\ \beta & \dfrac{dg}{e}-\delta_2 & 0\\ 0 & -f & -e \end{pmatrix}. \tag{2.7}$$

The characteristic equation of above matrix is

$$\begin{aligned}(\lambda+e)\Big[\lambda^2+\big(\beta+\delta_1+\delta_2-\tfrac{dg}{e}\big)\lambda+\beta\delta_2\\ +\delta_1\delta_2-\alpha\beta-\tfrac{dg}{e}(\beta+\delta_1)\Big]=0.\end{aligned} \tag{2.8}$$

Hence, it has one negative characteristic root $\lambda_1=-e<0$, the other two characteristic roots are determined by the equation

$$\lambda^2+\Big(\beta+\delta_1+\delta_2-\frac{dg}{e}\Big)\lambda+\beta\delta_2+\delta_1\delta_2-\alpha\beta-\frac{dg}{e}\Big(\beta+\delta_1\Big)=0. \tag{2.9}$$

Noting that under the assumption (2.5), (2.6) holds, and the two characteristic roots of equation (2.9) satisfy

$$\begin{aligned}&\lambda_2+\lambda_3=-\Big(\beta+\delta_1+\delta_2\Big)+\frac{dg}{e}<0,\\ &\lambda_2\lambda_3=\beta\delta_2+\delta_1\delta_2-\alpha\beta-\frac{dg}{e}\Big(\beta+\delta_1\Big)>0.\end{aligned} \tag{2.10}$$

hence, $\lambda_2<0,\lambda_3<0$. Above analysis shows that under the assumption of Theorem 2.1, the three characteristic roots of the matrix (2.7) are all negative, hence, $A_1\Big(0,0,\dfrac{g}{e}\Big)$ is locally asymptotically stable. This ends the proof of Theorem 2.1.

Remark 2.1 Obviously, under the assumption (1.4) holds, if $\frac{dg}{e}$ is enough small, then inequality (2.5) holds. That is, if the feedback control variable is enough small, the local stability of the boundary equilibrium $O(0,0)$ of system (1.3) is still holds.

Theorem 2.2 *Assume that*

$$\beta\alpha>\Big(\delta_2-\frac{dg}{e}\Big)\Big(\beta+\delta_1\Big) \tag{2.11}$$

holds, then $A_2(x_{1*},x_{2*},u_*)$ *is locally asymptotically stable.*
Proof. From (2.4) we could see that the Jacobian matrix of the system about the equilibrium point $A_2(x_{1*},x_{2*},u_*)$ is given by

$$\begin{pmatrix} -\beta-\delta_1 & \alpha & 0\\ \beta & -\delta_2-2\gamma x_{2*}+du_* & dx_{2*}\\ 0 & -f & -e \end{pmatrix}. \tag{2.12}$$

The characteristic equation of system (1.1) at $A_2(x_{1*},x_{2*},u_*)$ is

$$\lambda^3+B_1\lambda^2+B_2\lambda+B_3=0,$$

where

$$\begin{aligned}B_1&=-du_*+2\gamma x_{2*}+\beta+\delta_1+\delta_2+e,\\ B_2&=(\beta+\delta_1)(-du_*+2\gamma x_{2*}+\delta_2)\\ &\quad-\alpha\beta+dx_{2*}f\\ &\quad+(-du_*+2\gamma x_{2*}+\beta+\delta_1+\delta_2)e,\\ B_3&=(\beta+\delta_1)(-du_*+2\gamma x_{2*}+\delta_2)e\\ &\quad-\alpha\beta e+fdx_{2*}(\beta+\delta_1).\end{aligned} \tag{2.13}$$

From (2.2) and (2.3), we have

$$\begin{aligned}&-du_*+2\gamma x_{2*}+\delta_2\\ =&\ \beta\frac{x_{1*}}{x_{2*}}+\gamma x_{2*}\\ =&\ \frac{\alpha\beta}{\beta+\delta_1}+\gamma x_{2*}\\ >&\ 0.\end{aligned} \tag{2.14}$$

Therefore, by using (2.13) and (2.14), we have

$$\begin{aligned}B_1&=-du_*+2\gamma x_{2*}+\beta+\delta_1+\delta_2+e\\ &=\frac{\alpha\beta}{\beta+\delta_1}+\gamma x_{2*}+\beta+\delta_1+e>0,\\ B_2&=(\beta+\delta_1)\Big(\frac{\alpha\beta}{\beta+\delta_1}+\gamma x_{2*}\Big)\\ &\quad-\alpha\beta+dx_{2*}f\\ &\quad+(-du_*+2\gamma x_{2*}+\beta+\delta_1+\delta_2)e\\ &\geq dx_{2*}f+(-du_*+2\gamma x_{2*}+\beta+\delta_1+\delta_2)e\\ &\geq dx_{2*}f+\Big(\frac{\alpha\beta}{\beta+\delta_1}+\gamma x_{2*}+\beta+\delta_1\Big)e\\ &>0,\\ B_3&=(\beta+\delta_1)(-du_*+2\gamma x_{2*}+\delta_2)e\\ &\quad-\alpha\beta e+fdx_{2*}(\beta+\delta_1)\\ &=(\beta+\delta_1)\Big(\frac{\alpha\beta}{\beta+\delta_1}+\gamma x_{2*}\Big)e\\ &\quad-\alpha\beta e+fdx_{2*}(\beta+\delta_1)\\ &\geq fdx_{2*}(\beta+\delta_1)\\ &>0.\end{aligned}$$

Set $\Delta=-du_*+2\gamma x_{2*}+\delta_2$, then by using (2.14), and the fact $\Delta>0$, we have

$$\begin{aligned}&B_1B_2-B_3\\ =&\Big(\Delta+\delta_1+\beta+e\Big)\Big[(\beta+\delta_1)\Delta+(\Delta+\beta+\delta_1)e\\ &-\alpha\beta+fdx_{2*}\Big]-(\beta+\delta_1)\Delta e\\ &+\alpha\beta e-fdx_{2*}(\beta+\delta_1)\end{aligned}$$

$$
\begin{aligned}
&= \Big(\Delta+\delta_1+\beta+e\Big)\Big[(\beta+\delta_1)\Big(\frac{\alpha\beta}{\beta+\delta_1}+\gamma x_{2*}\Big) \\
&\quad +(\Delta+\beta+\delta_1)e-\alpha\beta+fdx_{2*}\Big] \\
&\quad -(\beta+\delta_1)\Delta e+\alpha\beta e-fdx_{2*}(\beta+\delta_1) \\
&\geq \Big(\Delta+\delta_1+\beta+e\Big)\Big[(\beta+\delta_1)\frac{\alpha\beta}{\beta+\delta_1} \\
&\quad +(\Delta+\beta+\delta_1)e-\alpha\beta+fdx_{2*}\Big] \\
&\quad -(\beta+\delta_1)\Delta e+\alpha\beta e-fdx_{2*}(\beta+\delta_1) \\
&\geq \Big(\Delta+\delta_1+\beta+e\Big)\Big[(\Delta+\beta+\delta_1)e+fdx_{2*}\Big] \\
&\quad -(\beta+\delta_1)\Delta e+\alpha\beta e-fdx_{2*}(\beta+\delta_1) \\
&\geq \alpha\beta e>0.
\end{aligned}
$$

By Hurwitz criterion, the three characteristic roots of the matrix (2.12) are all negative, hence, $A_2(x_{1*}, x_{2*}, u_*)$ is locally asymptotically stable. This ends the proof of Theorem 2.2.

Remark 2.2. Noting that condition (2.11) is necessary condition for the system (1.1) admits a unique positive equilibrium, it then follows from Theorem 2.2 that once the system (1.1) admits the unique positive equilibrium, the equilibrium is locally asymptotically stable.

Remark 2.3. Assume that (1.4) holds, then without feedback control, the species will be driven to extinction, however, noting that under the assumption (1.4), the inequality

$$\alpha\beta < \Big(\delta_2+\frac{g\gamma}{f}\Big)\Big(\beta+\delta_1\Big)$$

always holds, hence if $\frac{dg}{e}$ is enough large, then (2.1) always holds, and the system (1.1) admits a unique positive equilibrium, also, from Theorem 2.2 this equilibrium is locally stable, which means that the species will be survival.

Remark 2.4. Assume that (1.5) holds, then without feedback control, the system (1.3) admits a unique positive equilibrium, which is globally stable. Noting that in this case, for any positive constants, inequality

$$\beta\alpha > \Big(\delta_2-\frac{dg}{e}\Big)\Big(\beta+\delta_1\Big)$$

always holds, also, if $\dfrac{g\gamma}{f}$ is enough large, then inequality

$$\Big(\delta_2+\frac{g\gamma}{f}\Big)\Big(\beta+\delta_1\Big) > \beta\alpha$$

also holds, and the system (1.1) admits a unique positive equilibrium, also, from Theorem 2.2 this equilibrium is locally stable, which means that the species will be survival.

III. Global stability

This section tries to obtain some sufficient conditions which could ensure the global asymptotical stability of the equilibria of system (1.1).

Theorem 3.1 *Assume that*

$$\beta\alpha < \Big(\delta_2-\frac{dg}{e}\Big)\Big(\beta+\delta_1\Big) \tag{3.1}$$

holds, then $A_1\Big(0,0,\dfrac{g}{e}\Big)$ *is globally asymptotically stable.*

Proof. Condition (3.1) is equal to

$$\frac{\alpha\beta}{\beta+\delta_1}-\delta_2+\frac{gd}{e}<0. \tag{3.2}$$

We will prove Theorem 3.1 by constructing some suitable Lyapunov function. Let's define a Lyapunov function

$$V_1(x_1,x_2,u)=\frac{\beta}{\beta+\delta_1}x_1+x_2+\frac{d}{2f}(u-u_1)^2, \tag{3.3}$$

where

$$u_1=\frac{g}{e}. \tag{3.4}$$

One could easily see that the function V_1 is zero at the equilibrium $A_1\Big(0,0,\dfrac{g}{e}\Big)$ and is positive for all other positive values of x_1 and x_2. The time derivative of V_1 along the trajectories of (1.1) is

$$
\begin{aligned}
&D^+V_1(t) \\
&= \frac{\beta}{\beta+\delta_1}\Big(\alpha x_2-\beta x_1-\delta x_1\Big) \\
&\quad +\beta x_1-\delta_2x_2-\gamma x_2^2+dux_2 \\
&\quad +\frac{d}{f}(u-u_1)(g-eu-fx_2) \\
&= \frac{\beta}{\beta+\delta_1}\Big(\alpha x_2-\beta x_1-\delta x_1\Big) \\
&\quad +\beta x_1-\delta_2x_2-\gamma x_2^2+dux_2 \\
&\quad +\frac{d}{f}(u-u_1)(eu_1-eu-fx_2) \\
&= \Big(\frac{\alpha\beta}{\beta+\delta_1}-\delta_2\Big)x_2-\gamma x_2^2+dux_2 \\
&\quad -\frac{de}{f}(u-u_1)^2-dux_2+du_1x_2 \\
&= \Big(\frac{\alpha\beta}{\beta+\delta_1}-\delta_2+\frac{dg}{e}\Big)x_2 \\
&\quad -\gamma x_2^2-\frac{de}{f}(u-u_1)^2.
\end{aligned} \tag{3.5}
$$

It then follows from (3.2) that $D^+V_1(t) < 0$ strictly for all $x_1, x_2, u > 0$ except the boundary equilibrium $A_1\Big(0,0,\dfrac{g}{e}\Big)$, where $D^+V_1(t) = 0$. Thus, $V_1(x_1,x_2,u)$ satisfies Lyapunov's asymptotic stability theorem, and the boundary equilibrium $A_1\Big(0,0,\dfrac{g}{e}\Big)$ of system (1.1) is globally asymptotically stable.

This completes the proof of Theorem 3.1.

Remark 3.1. From Theorem 3.1, one could see that the conditions which ensure the locally asymptotically stable of the boundary equilibrium is enough to ensure its e globally asymptotically stability.

Remark 3.2. Theorem 3.1 shows that if the boundary equilibrium $O(0,0)$ in system (1.3) is globally stable, if the feedback control variable is enough small, then in system (1.1), the boundary equilibrium $A_1(0,0,\frac{g}{e})$ is also globally stable. i.e., the species still will be driven to extinction.

Theorem 3.2 *Assume that*

$$\beta\alpha > \Big(\delta_2-\frac{dg}{e}\Big)\Big(\beta+\delta_1\Big) \tag{3.6}$$

holds, then $A_2(x_{1*}, x_{2*}, u_*)$ is globally asymptotically stable.

Proof. We will prove Theorem 3.2 by constructing some suitable Lyapunov function. Let's define a Lyapunov function

$$\begin{aligned} &V_2(x_1, x_2, u)\\ =\;& \frac{\beta x_{1*}}{x_{2*}\alpha}\Big(x_1 - x_{1*} - x_{1*}\ln\frac{x_1}{x_{1*}}\Big)\\ &+\Big(x_2 - x_{2*} - x_{2*}\ln\tfrac{x_2}{x_{2*}}\Big) + \frac{d}{2f}(u-u_*)^2. \end{aligned}$$

One could easily see that the function V_2 is zero at the equilibrium $A_2(x_{1*}, x_{2*}, u_*)$ and is positive for all other positive values of x_1, x_2 and u. The time derivative of V_2 along the trajectories of (1.1) is

$$\begin{aligned} &D^+V_2(t)\\ =\;& \frac{\beta x_{1*}}{x_{2*}\alpha}\frac{x_1 - x_{1*}}{x_1}\dot{x}_1 + \frac{x_2 - x_{2*}}{x_2}\dot{x}\\ &+\frac{d}{f}(u-u_*)\dot{u}\\ =\;& \frac{\beta x_{1*}}{x_{2*}\alpha}\frac{x_1 - x_{1*}}{x_1}\Big(\alpha x_2 - (\beta+\delta_1)x_1\Big)\\ &+\frac{x_2 - x_{2*}}{x_2}\Big(\beta x_1 - \delta_2 x_2 - \gamma x_2^2 + dux_2\Big)\\ &+\frac{d}{f}(u-u_*)(g - eu - fx_2). \end{aligned} \tag{3.7}$$

Noting that from the relationship of x_{1*}, x_{2*} and u_* (see (3.2)), we have

$$\begin{aligned} &\alpha x_2 - (\beta+\delta_1)x_1\\ =\;& \frac{\alpha}{x_{1*}}\Big(-x_2(x_1 - x_{1*}) + x_1(x_2 - x_{2*})\Big), \end{aligned} \tag{3.8}$$

also, from (3.1) and (3.2), we have

$$\begin{aligned} &\beta x_1 - \delta_2 x_2 - \gamma x_2^2 + dux_2\\ =\;& \frac{\beta}{x_{2*}}\Big(x_1x_{2*} - x_2x_{1*}\Big) + \beta x_2\frac{x_{1*}}{x_{2*}}\\ &-\delta_2x_2 - \gamma x_2^2 + dux_2\\ =\;& \frac{\beta}{x_{2*}}\Big(x_1x_{2*} - x_1x_2 + x_1x_2 - x_2x_{1*}\Big)\\ &+\Big(\frac{\alpha\beta}{\beta+\delta_1} - \delta_2\Big)x_2 - \gamma x_2^2 + dux_2\\ =\;& \frac{\beta}{x_{2*}}\Big(x_1x_{2*} - x_1x_2 + x_1x_2 - x_2x_{1*}\Big)\\ &+\Big[\Big(\gamma + d\frac{f}{e}\Big)x_{2*} - \frac{dg}{e}\Big]x_2 - \gamma x_2^2 + dux_2\\ =\;& \frac{\beta}{x_{2*}}\Big(x_1(x_{2*} - x_2) + x_2(x_1 - x_{1*})\Big)\\ &+\gamma x_2\big(x_{2*} - x_2\big) + dx_2\big(u - u_*\big), \end{aligned} \tag{3.9}$$

from the third equation of (3.2), we have

$$\begin{aligned} &g - eu - fx_2\\ =\;& -eu - fx_2 + eu_* + fx_{2*}\\ =\;& -e(u-u_*) - f(x_2 - x_{2*}). \end{aligned} \tag{3.10}$$

Applying (3.8)-(3.10) to (3.7) leads to

$$\begin{aligned} &D^+V_2(t)\\ =\;& \frac{\beta x_{1*}}{x_{2*}\alpha}\frac{x_1 - x_{1*}}{x_1}\frac{\alpha}{x_{1*}}\Big(-x_2(x_1 - x_{1*})\\ &+x_1(x_2 - x_{2*})\Big)\\ &+\frac{x_2 - x_{2*}}{x_2}\frac{\beta}{x_{2*}}\Big(x_1(x_{2*} - x_2)\\ &+x_2(x_1 - x_{1*})\Big)\\ &+\gamma x_2\frac{x_2 - x_{2*}}{x_2}\Big(x_{2*} - x_2\Big)\\ &+k_2dx_2\frac{x_2 - x_{2*}}{x_2}\Big(u - u_*\Big)\\ &+\frac{d}{f}(u-u_*)\Big(-e(u-u_*)\\ &-f(x_2 - x_{2*})\Big)\\ =\;& -\frac{\beta x_2}{x_1x_{2*}}(x_1 - x_{1*})^2\\ &+\frac{2\beta}{x_{2*}}(x_1 - x_{1*})(x_2 - x_{2*})\\ &-\frac{\beta x_1}{x_2x_{2*}}(x_2 - x_{2*})^2\\ &-\gamma\Big(x_2 - x_{2*}\Big)^2 - \frac{de}{f}(u-u_*)^2\\ =\;& -\frac{\beta}{x_{2*}}\Big[\sqrt{\frac{x_2}{x_1}}(x_1 - x_{1*})\\ &-\sqrt{\tfrac{x_1}{x_2}}(x_2 - x_{2*})\Big]^2\\ &-\gamma\Big(x_2 - x_{2*}\Big)^2 - \frac{de}{f}(u-u_*)^2. \end{aligned} \tag{3.11}$$

Hence, $D^+V_2(t) < 0$ strictly for all $x_1, x_2, u > 0$ except the positive equilibrium $A_2(x_{1*}, x_{2*}, u_*)$, where $D^+V_2(t) = 0$. Thus, $V_2(x_1, x_2, u)$ satisfies Lyapunov's asymptotic stability theorem, and the positive equilibrium $A_2(x_{1*}, x_{2*}, u_*)$ of system (1.1) is globally asymptotically stable.

This completes the proof of Theorem 3.2.

Remark 3.3. From Theorem 3.2, one could see that the conditions which ensure the locally asymptotically stable of the positive equilibrium is enough to ensure its e globally asymptotically stability. Also, noting that condition (3.6) is necessary to ensure the existence of the positive equilibrium. Hence, we can draw the conclusion: Once the system (1.1) admits the positive equilibrium, it is globally asymptotically stable.

Remark 3.4. As was shown in Remark 2.3-2.4. If the species in system (1.3) is extinct, then by choosing suitable feedback control variable, the system (1.1) may admits a unique positive equilibrium, Theorem 3.2 shows that this equilibrium is globally stable. Therefore, by choosing suitable feedback control variable, the species could be avoid to extinction. If the species in system (1.3) is permanent, then one should also choosing the suitable feedback control variable to maintain the persistent property of the species.

IV. Numeric simulations

Now let's consider the following example.

Example 4.1. Let's consider the following single species stage structure system with feedback control:

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 - x_1 - x_1, \\ \frac{dx_2}{dt} &= x_1 - x_2 - x_2^2 + dx_2 u, \\ \frac{du}{dt} &= g - u - x. \end{aligned} \tag{4.1}$$

Here we choose $\alpha = \beta = \delta_1 = \delta_2 = \gamma = e = f = 1$, d, g is determined later. For the system without feedback control, the system degenerate to

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 - x_1 - x_1, \\ \frac{dx_2}{dt} &= x_1 - x_2 - x_2^2. \end{aligned} \tag{4.2}$$

Since

$$\alpha\beta = 1 < 2 = \delta_2\big(\delta + \beta_1\big), \tag{4.3}$$

the boundary equilibrium $O(0,0)$ of system (4.2) is globally asymptotically stable. Also, one could easily check that if $dg < \frac{1}{2}$, then

$$\alpha\beta = 1 < \Big(\delta_2 - \frac{dg}{e}\Big)(\delta_1 + \beta). \tag{4.4}$$

Therefore, condition of Theorem 3.1 is satisfied, and $A_1(0, 0, \frac{g}{e})$ of system (4.1) is globally asymptotically stable. As an example, now let's take $d = g = \frac{1}{2}$, then $A_1(0, 0, \frac{1}{2})$ is globally asymptotically stable, Fig. 2-4 support those findings. On the other hand, one could easily check that if $dg > \frac{1}{2}$, then

$$\alpha\beta = 1 > \Big(\delta_2 - \frac{dg}{e}\Big)(\delta_1 + \beta). \tag{4.5}$$

Therefore, condition of Theorem 3.2 satisfied, and $A_2(x_{1*}, x_{2*}, u_*)$ of system (4.1) is globally asymptotically stable. As an example, now let's take $d = 1, g = \frac{3}{2}$, then $A_2(\frac{1}{4}, \frac{1}{2}, 1)$ is globally asymptotically stable, Fig. 5-7 support those findings.

V. Conclusion

During the past decade, many scholars investigated the dynamic behaviors of the ecosystem with feedback control system (see [16]-[43]). However, there are few work on stage structured system with feedback control([13]). In [13], Ding and Cheng proposed the single species stage structured system with feedback control (system (1.9)), the system always admits a unique positive equilibrium. To ensure the positive equilibrium be global asymptotically stable, the authors needed some additional condition. Their result indicates that the harvesting process of human being has no influence on the persistent property of the species, since the system always admits a unique positive equilibrium which is locally stable.

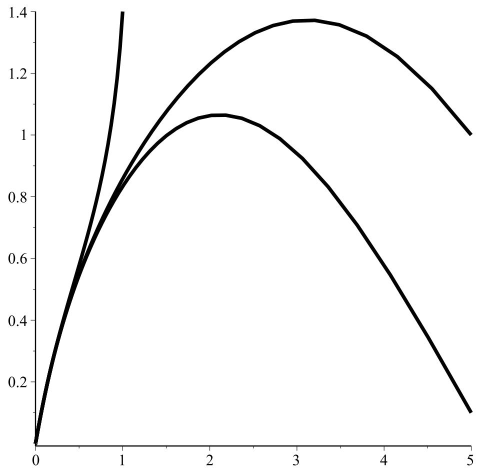

Fig. 1. Phase portraits of system (4.2), the initial conditions $(x_1(0), x_2(0))$= $(5, 1)$,$(5, 0.1)$ and $(1, 1.4)$, respectively.

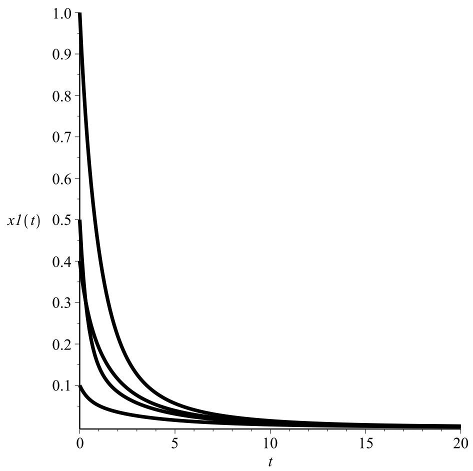

Fig. 2. Dynamics behaviors of the first component $x_1(t)$ of the system (4.1), here we take $d = g = \frac{1}{2}$, the initial conditions $(x_1(0), x_2(0), u(0)) = (0.5, 0.1, 0.2), (0.1, 0.1, 0.1), (1, 1, 1)$ and $(0.4, 0.4, 0.4)$, respectively.

Since with the development of the human beings, more and more species becomes endangered, and without the suitable help of human beings, those species will finally be driven to extinction. This stimulated us to propose the single species stage structured system with positive feedback control (system (1.1)). Here, by means of positive feedback control, we means that the stocking of the species.

Our results show that if the positive feedback control is limited, such that inequality (2.5) holds, then despite the stocking (the help of human beings), the species still will be driven to extinction. However, if the positive feedback control is enough large, such that inequality (2.11) holds, then such kind of mechanism is very useful, and the species will finally living in long run. However, if the original system (system (1.3)) admits the unique positive equilibrium which is globally stable, then to ensure the system (1.1) admits a unique positive equilibrium, we also need to restrict the feedback control variable to some suitable area. Such an finding may help us to choose some suitable mechanism to

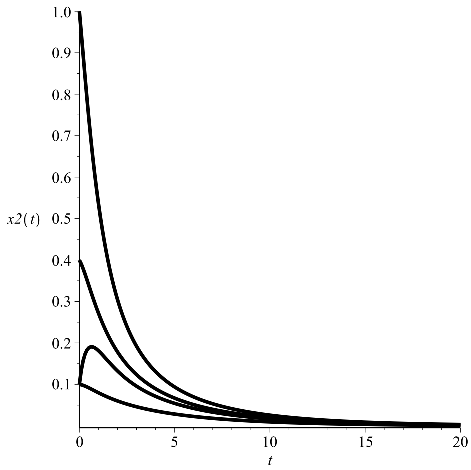

Fig. 3. Dynamics behaviors of the second component $x_2(t)$ of the system (4.1), here we take $d = g = \frac{1}{2}$, the initial conditions $(x_1(0), x_2(0), u(0)) = (0.5, 0.1, 0.2), (0.1, 0.1, 0.1), (1, 1, 1)$ and $(0.4, 0.4, 0.4)$, respectively.

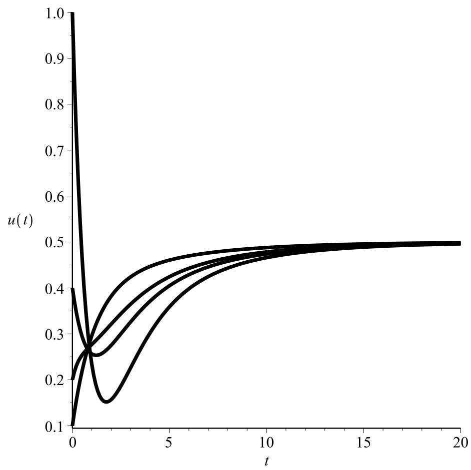

Fig. 4. Dynamics behaviors of the third component $u(t)$ of the system (4.1), here we take $d = g = \frac{1}{2}$, the initial conditions $(x_1(0), x_2(0), u(0)) = (0.5, 0.1, 0.2), (0.1, 0.1, 0.1), (1, 1, 1)$ and $(0.4, 0.4, 0.4)$, respectively.

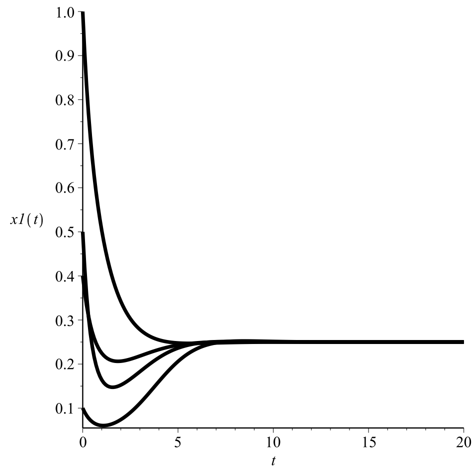

Fig. 5. Dynamics behaviors of the first component $x_1(t)$ of the system (4.1), here we take $d = 1, g = \frac{3}{2}$, the initial conditions $(x_1(0), x_2(0), u(0)) = (0.5, 0.1, 0.2), (0.1, 0.1, 0.1), (1, 1, 1)$ and $(0.4, 0.4, 0.4)$, respectively.

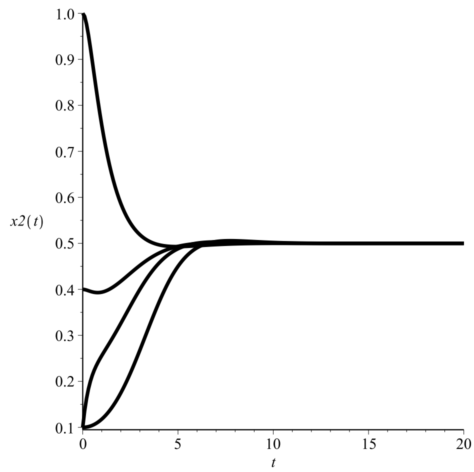

Fig. 6. Dynamics behaviors of the second component $x_2(t)$ of the system (4.1), here we take $d = 1, g = \frac{3}{2}$, the initial conditions $(x_1(0), x_2(0), u(0)) = (0.5, 0.1, 0.2), (0.1, 0.1, 0.1), (1, 1, 1)$ and $(0.4, 0.4, 0.4)$, respectively.

avoid the extinction of the endangered species.

VI. Declarations

Competing interests

The author declare that there is no conflict of interests.

Funding

The research was supported by First-class disciplines of Anhui Province(0044619001) and Key program of Anhui Provincial Education Department (KJ2019A0618) and the Natural Science Foundation of Anhui Province(1808085MG224).

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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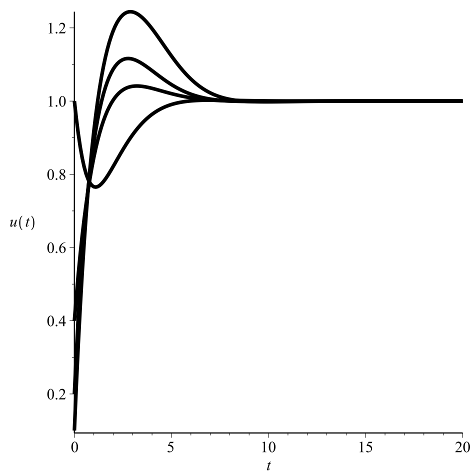

Fig. 7. Dynamics behaviors of the third component $u(t)$ of the system (4.1), here we take $d = 1, g = \frac{3}{2}$, the initial conditions $(x_1(0), x_2(0), u(0)) = (0.5, 0.1, 0.2), (0.1, 0.1, 0.1), (1, 1, 1)$ and $(0.4, 0.4, 0.4)$, respectively.

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