Output Feedback Stabilization via Reduced Order Observer for a Class of Feedforward Nonlinear Systems with Input Saturation

Yanling Shang, Fangzheng Gao, Guifang Qiao, Jiacai Huang and Xiaochun Zhu

***Abstract*—The problem of saturated output feedback stabilization for a class of nonlinear systems with upper-triangular structure is addressed in this paper. By constructing a reduced order observer to estimates the unmeasurable states and by skillfully using the homogeneous domination approach and the nested saturation technique, a saturated output feedback control scheme is successfully developed. It is prove that the proposed controller with appropriate design parameters can render the states of the closed-loop system globally asymptotically to zero without violation of the input constraint. A simulation example is provided to demonstrate the effectiveness of the proposed method.**

***Index Terms*—Feedforward nonlinear systems, Input saturation, Homogeneous domination approach, Reduced-order observer**

I. Introduction

During the past few decades, feedforward systems have received widely attention because they can be used to model many practical systems, such as the ball and beam system, the cart-pendulum system, the TORA system, and so forth. However, the design of globally stabilizing controller for a feedforward system has proven to constitute a challenging task due to the fact that such system is neither feedback linearizable nor stabilized by applying the frequently-used backstepping approach. To give this difficulty a solution, a number of intelligent approaches have been developed such as the nested-saturation method [1-6] and the forwarding technique [7, 8]. Thanks to these effective approaches, the state feedback stabilization problem has been well-studied recently. Nevertheless, when only part of state variables are measurable, the problem of global stabilization by output feedback is more challenging and has received little attention. As a matter of fact, the upper-triangular structure leads to an intrinsic obstacle that makes it difficult to achieve even semi-global output feedback stabilization of general feedforward systems [9].

In the existing literature, there are numerous valuable results in coping with the output feedback stabilization problem of feedforward systems under different growth conditions. For example, by imposed the restriction that the nonlinear term is a linear growth, the global output feedback stabilization for uncertain feedforward systems was first studied in [10]. Later, by employing the homogeneous domination approach introduced in [11], the linear growth condition was lifted in [9] where global output feedback stabilization was achieved for more general nonlinearities under a homogeneous growth condition, and stimulated a series of subsequent works [12-17]. However, the effect of the input constraint is omitted in the above-mentioned results.

As we all know, the actuator saturation is a common phenomenon in practical systems due to the inherent physical limitations of devices. Its existence often severely limits system performance, giving rise to undesirable inaccuracy or leading to instability [18]. Thus, it is of great significance to study the problem of saturated output feedback stabilization of feedforward nonlinear systems. Nevertheless, to the best of our knowledge, this issue has not been well-addressed in the literature.

Based on the above observations, in this paper we focus our attention to solve the problem of global stabilization for a class feedforward nonlinear systems by saturated output feedback. The major obstacle to tackle this problem lies in that, the presence of input constraint may lead to a system uncontrollable even if it is indeed controllable for the unconstrained case, that is, the common assumptions and output feedback control techniques mainly for unsaturated feedforward systems are infeasible here. Until now it still remains unanswered that under what conditions the feedforward nonlinear systems may exist saturated output feedback controller. To overcome the aforementioned difficulty, we first place a general homogeneous growth condition and design an unsaturated state feedback controller for the considered system by employing the homogeneous domination approach. Then, we impose a series of nested saturations to the developed controller and obtain a saturated state feedback controller. Moreover, different from the full order observers proposed in [19,20], in this paper we construct a reduced-order observer to estimate unmeasurable states, and obtain a saturated output feedback controller that renders that the states of the closed-loop system globally asymptotically convergence to zero.

II. Problem formulation and preliminaries

In this paper, we consider a class of feedforward nonlinear systems represented by

$$\begin{aligned} &\dot{x}_i = x_{i+1} + f_i(t, x_{i+2}, \cdots, x_n, u), \quad i = 1, \cdots, n-2 \\ &\dot{x}_{n-1} = x_n + f_{n-1}(t, u) \\ &\dot{x}_n = u \\ &y = x_1 \end{aligned} \tag{1}$$

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Yanling Shang, Fangzheng Gao, Guifang Qiao,Jiacai Huang and Xiaochun Zhu are School of Automation, Nanjing Institute of Technology, Nanjing 211167, P. R. China, hnnhsyl@126.com

where $x=(x_1,\cdots,x_n)^T \in R^n$, $u \in R$, $y \in R$ are the system state, control input and system output, respectively, and $x_2,\cdots,x_n$ are unmeasurable. The continuous functions $f_i : R \times R^{n-i} \to R$, $i=1,\cdots,n-1$ represent unknown nonlinear perturbations.

The objective of this paper is to present an output feedback control design strategy which globally stabilizes the system (1) under the following saturation constraint:

$$-u^{max} \le u \le u^{max} \tag{2}$$

where u^{max} is a priori known positive real number.

To this end, the following assumption regarding system (1) is imposed.

Assumption 1. For $i=1,\cdots,n-1$, there are constants $b>0$ and $\tau \in (-1/n,+\infty)$ such that

$$|f_i(\cdot)| \le b \sum_{j=i+2}^{n+1} |x_j|^{(r_i+\tau)/r_j}$$

where $x_{n+1}=u$, $r_1=1$, $r_{i+1}=r_i+\tau>0$, $i=1,\cdots,n$.

For simplicity, it is assumed that $\tau=-\frac{p}{q}$ with p being any even integer and q being any odd integer. Based on this, we know that $r_i \in (0,1)$ is a ratio of two positive odd integers.

In what follows, we review some useful definitions and lemmas which will serve as the basis of the coming control design and performance analysis.

Definition 1[21]**.** Consider a system

$$\dot{x}=f(x) \text{ with } f(0)=0, \quad x \in R^n \tag{3}$$

where $f : U_0 \to R^n$ is continuous with respect to x on an open neighborhood U_0 of the origin $x=0$. The equilibrium $x=0$ of the system is (locally) finite-time stable if it is Lyapunov stable and finite-time convergent in a neighborhood $U \in U_0$ of the origin. By "finite-time convergence," we mean: If, for any initial condition $x(0) \in U$, there is a settling time $T>0$, such that every solution $x(t)$ with $x(0)$ as its initial condition of (3) is well defined with $x(0) \in U \setminus \{0\}$ for $t \in [0,T)$ and satisfies $\lim_{t\to T} x(t)=0$ and $x(t)=0$ for any $t \ge T$. If $U=U_0=R^n$, the origin is a globally finite-time stable equilibrium.

Lemma 1[21]**.** Consider the nonlinear system (3). Suppose there is a C^1 function $V(x)$ defined in a neighborhood $\hat{U} \in R^n$ of the origin, real numbers $c>0$ and $0<\alpha<1$, such that

(i) $V(x)$ is positive definite on $\hat{U}$;

(ii) $\dot{V}(x)+cV^{\alpha}(x) \le 0, \quad \forall x \in \hat{U}$.

Then, the origin of system (3) is locally finite-time stable with $T \le \frac{V^{1-\alpha}(x(0))}{c(1-\alpha)}$ for initial condition $x(0)$ in some open neighborhood $U \in \hat{U}$ of the origin. If $U=R^n$ and $V(x)$ is also radially unbounded (i.e., $V(x) \to +\infty$ as $x \to +\infty$), the origin of system (3) is globally finite-time stable.

Definition 2[9]**.** Weighted Homogeneity: For fixed coordinates $(x_1,\cdots,\ x_n) \in R^n$ and real numbers $r_i>0$, $i=1,\cdots,n$,

• the dilation $\Delta_\varepsilon(x)$ is defined by $\Delta_\varepsilon(x)=(\varepsilon^{r_1}x_1,\cdots,\varepsilon^{r_n}x_n)$ for any $\varepsilon>0$, where r_i is called the weights of the coordinates. For simplicity, we define dilation weight $\Delta=(r_1,\cdots,r_n)$.

• a function $V \in (R^n,R)$ is said to be homogeneous of degree τ if there is a real number $\tau \in R$ such that $V(\Delta_\varepsilon(x))=\varepsilon^{\tau}V(x_1,\cdots,x_n)$ for any $x \in R^n \setminus \{0\}, \varepsilon>0$.

• a vector field $f \in (R^n,R^n)$ is said to be homogeneous of degree τ if there is a real number $\tau \in R$ such that $f_i(\Delta_\varepsilon(x))=\varepsilon^{\tau+r_i}f_i(x)$, for any $x \in R^n \setminus \{0\}, \varepsilon>0$, $i=1,\cdots,n$.

• a homogeneous p-norm is defined as $\|x\|_{\triangle,p}=(\sum_{i=1}^n |x_i|^{p/r_i})^{1/p}$ for all $x \in R^n$, for a constant $p \ge 1$. For simplicity, in this paper, we choose $p=2$ and write $\|x\|_{\triangle}$ for $\|x\|_{\triangle,2}$.

Lemma 2[9]**.** Suppose $V : R^n \to R$ is a homogeneous function of degree τ with respect to the dilation weight Δ. Then the following holds:

(i) $\partial V/\partial x_i$ is homogeneous of degree $\tau-r_i$ with r_i being the homogeneous weight of x_i.

(ii) There is a constant c such that $V(x) \le c\|x\|_{\triangle}^{\tau}$. Moreover, if $V(x)$ is positive definite, then $\underline{c}\|x\|_{\triangle}^{\tau} \le V(x)$, where $\underline{c}$ is a constant.

Lemma 3 [9]**.** For $x \in R$, $y \in R$, and $p \ge 1$ is a constant, the following inequalities hold: (i) $|x+y|^p \le 2^{p-1}|x^p+y^p|$, $(ii)(|x|+|y|)^{1/p} \le |x|^{1/p}+|y|^{1/p} \le 2^{(p-1)/p}(|x|+|y|)^{1/p}$. Furthermore, if $p \ge 1$ is odd, then (iii) $|x-y|^p \le 2^{p-1}|x^p-y^p|$, (iv)$|x^{1/p}-y^{1/p}| \le 2^{(p-1)/p}(|x-y|)^{1/p}$, (v) $|x^p-y^p| \le p|x-y||x^{p-1}+y^{p-1}| \le c|x-y||(x-y)^{p-1}+y^{p-1}|$ for a constant $c>0$.

Lemma 4[22,23]**.**For any positive real numbers c,d and any real-valued function $\pi(x,y)>0$, the following inequality holds: $|x|^c|y|^d \le \frac{c}{c+d}\pi(x,y)|x|^{c+d}+\frac{d}{c+d}\pi^{-c/d}(x,y)|y|^{c+d}$.

III. The design of saturated output feedback controller

In this section, we give a constructive procedure for the globally stabilizer of system (1) by saturated output feedback. Before designing the controller, we first introduce the following coordinate transformation:

$$z_1=x_1, \quad z_i=\frac{x_i}{L^{\kappa_i}}, \quad i=2,\cdots,n, \quad \upsilon^{p_n}=\frac{u^{p_n}}{L^{\kappa_n+1}} \tag{4}$$

where $\kappa_i=n-1$, $i=1,\cdots,n$ and $0<L<1$ is a constant to be determined later.

Then, under the new coordinates z_i's, system (1) is transformed into:

$$\begin{aligned} \dot{z}_i &= Lz_{i+1}+\frac{f_i}{L^{\kappa_i}}, \quad i=1,\cdots,n-1 \\ \dot{z}_n &= Lv \\ y &= z_1 \end{aligned} \tag{5}$$

Noting that the transformation (4) is invertible, thus in the next, we turn to designing a saturated output feedback controller for system (5).

A. Unsaturated state feedback controller design

Step 1. Let $\rho \ge \max_{1\le i\le n+1}\{r_i\}$ is a positive number and choose the Lyapunov function $V_1=W_1=\int_0^{z_1}(s^{\rho/r_1}-0)^{(2\rho-\tau-r_1)/\rho}ds$. Clearly, the first virtual controller

$$z_2^*=-\beta_1^*\xi_1^{r_2/\rho} \tag{6}$$

with $\xi_1=z_1$ and $\beta_1^* \ge n$ being a constant, renders

$$\dot{V}_1 \le -nL\xi_1^2+L\xi_1^{(2\rho-\tau-r_1)/\rho}(z_2-z_2^*)+\frac{\partial V_1}{\partial z_1}f_1 \tag{7}$$

Step i $(i=2,\cdots,n)$**.** In this step, we can obtain the following property, whose similar proof can be found in [9] and hence is omitted here.

Proposition 1. Assume that at step $i-1$, there is a C^1, proper and positive definite Lyapunov function V_{i-1}, and a set of virtual controllers $z_1^*, \cdots, z_i^*$ defined by

$$\begin{array}{rclrcl} z_1^* &=& 0, & \xi_1 &=& z_1^{\rho/r_1} - z_1^{*\rho/r_1} \\ z_2^* &=& -\beta_1^* \xi_1^{r_2/\rho}, & \xi_2 &=& z_2^{\rho/r_2} - z_2^{*\rho/r_2} \\ \vdots & & & \vdots & & \\ z_i^* &=& -\beta_{i-1}^* \xi_{i-1}^{r_i/\rho} & \xi_i &=& z_i^{\rho/r_i} - z_i^{*\rho/r_i} \end{array} \tag{8}$$

with $\beta_j^* > 0$, $j = 1, \cdots, i-1$, being constants, such that

$$\begin{aligned} \dot{V}_{i-1} \leq -(n-i+2)L \sum_{j=1}^{i-1} \xi_j^2 + \sum_{j=1}^{i-1} \frac{\partial V_i}{\partial z_j} \frac{f_j}{L^{\kappa_j}} \\ +L\xi_i^{(2\rho-\tau-r_{i-1})/\rho}(z_i - z_i^*) \end{aligned} \tag{9}$$

Then the ith Lyapunov function defined by

$$V_i = V_{i-1} + \int_{z_i^*}^{z_i} (s^{\rho/r_i} - z_i^{*\rho/r_i})^{(2\rho-\tau-r_i)/\rho} ds \tag{10}$$

is C^1, proper and positive definite, and there exists the C^0 virtual controller $z_{i+1}^* = -\beta_i^* \xi_i^{r_{i+1}/\rho}$ such that

$$\begin{aligned} \dot{V}_i \leq -(n-i+1)L \sum_{j=1}^{i} \xi_j^2 + \sum_{j=1}^{i} \frac{\partial V_i}{\partial z_j} \frac{f_j}{L^{k_j}} \\ +L\xi_i^{(2\rho-\tau-r_i)/\rho}(z_{i+1} - z_{i+1}^*) \end{aligned} \tag{11}$$

where $\beta_i > 0$ is a constant.

Hence at step n, choosing

$$V_n = \sum_{i=1}^{n} \int_{z_i^*}^{z_i} \left(s^{\rho/r_i} - z_i^{*\rho/r_i}\right)^{(2\rho-\tau-r_i)/\rho} ds \tag{12}$$

and

$$\begin{aligned} z_{n+1}^* &= -\beta_n^* \xi_n^{r_{n+1}/\rho} \\ &= -\beta_n^* \Big(z_n^{\rho/r_n} + \beta_{n-1}^{*\rho/r_n} \Big(z_{n-1}^{\rho/r_{n-1}} + \cdots \\ &\qquad + \beta_2^{*\rho/r_3} \Big(z_2^{\rho/r_2} + \beta_1^{*\rho/r_2} z_1\Big)\Big)\Big)^{r_{n+1}/\rho} \\ &= -\beta_n^* \Big(\bar{\beta}_n^* z_n^{\rho/r_n} + \bar{\beta}_{n-1}^* z_{n-1}^{\rho/r_{n-1}} \\ &\qquad + \cdots + \bar{\beta}_1^* z_1^{\rho/r_1}\Big)^{r_{n+1}/\rho} \end{aligned} \tag{13}$$

where

$$\bar{\beta}_i^* = \begin{cases} \bar{\beta}_{n-1}^{*\rho/r_n} \cdots \bar{\beta}_i^{*\rho/r_{i+1}}, & i = 1, \cdots, n-1 \\ 1, & i = n \end{cases} \tag{14}$$

from Proposition 1, we arrive at

$$\begin{aligned} \dot{V}_n \leq -L \sum_{j=1}^{n} \xi_j^2 + \sum_{j=1}^{n-1} \frac{\partial V_n}{\partial z_j} \frac{f_j}{L^{\kappa_j}} \\ +L\xi_n^{(2\rho-\tau-r_n)/\rho}(v - z_{n+1}^*) \end{aligned} \tag{15}$$

Consequently, the following result is obtained.

Lemma 5. For the nonlinear system (5) under Assumption 1, the unsaturated state feedback controller $v = z_{n+1}^*$ in (13) renders the origin of the closed-loop system is globally asymptotically stable.

B. Saturated state feedback controller design

In this subsection, a saturated state feedback controller is designed to solve the global stabilization problem for system (5). By the combined saturation technique, we impose a series of nested saturations to the controller $v = z_{n+1}^*$ in (13) and obtain a saturated controller as following form

$$v_{ssf} = v_n(Z_n) = -\beta_n \sigma^{r_{n+1}/\rho}\Big(z_n^{\rho/r_n} - v_{n-1}^{\rho/r_n}(Z_{n-1})\Big) \tag{16}$$

where $v_0 = 0$, $v_i(Z_i) = -\beta_i \sigma^{r_{i+1}/\rho}(z_i^{\rho/r_i} - v_{i-1}^{\rho/r_i}(Z_{i-1}))$, $Z_i = (z_1, \cdots, z_i)$, $i = 1, \cdots, n$,

$$\sigma(x) = \begin{cases} \varepsilon sign(x), & |x| > \varepsilon \\ x, & |x| \leq \varepsilon \end{cases}$$

for a small constant $\varepsilon > 0$ to be determined later, and the gains β_i's are selected as

$$\begin{aligned} &\beta_1 > max\Big\{\beta_1^*, 2^{1+r_2}\Big\} \\ &\beta_i > max\Big\{\beta_i^*, 2^{r_{i+1}/\rho}\Big(4(1+\beta_{i-1})\alpha_{i-1}(\cdot) + 2\Big)\Big\} \\ &\quad i = 2, \cdots, n \end{aligned} \tag{17}$$

with

$$\begin{aligned} &\alpha_1(\beta_1) = \rho\beta_1^{\rho/r_2}(1+\beta_1) \\ &\alpha_j(\beta_1, \cdots, \beta_j) = \frac{\rho\beta_j^{\rho/r_{j+1}}}{r_j}(1+\beta_{j-1})^{\rho/r_j - 1}(1+\beta_j) \\ &\qquad + \beta_j^{\rho/r_{j+1}} \alpha_{j-1}(\cdot), \quad j = 2, \cdots, n-1 \end{aligned} \tag{18}$$

Remark 1. From (16) and the definition of saturation function $\sigma(\cdot)$, it can clearly be seen that the controller $v_{ssf} = v_n(Z_n)$ is bounded by a constant $\beta_n \varepsilon^{r_{n+1}}$, which means that the bound of controller (16) can be arbitrarily small by choosing appropriate design constant ε.

We begin our the main result of this subsection by introducing an important lemma, whose similar proof can be found in [2].

Lemma 6. Consider the system (5) with saturated controller (16). For $i = 1, \cdots, n-1$, under the condition $|z_j| \leq \varepsilon^{r_j/\rho}(1+\beta_{j-1})$, $j = i+1, \cdots n+1$, there exist a series of functions $\alpha_i(\beta_1, \cdots, \beta_i)$ defined as (18) and a constant $0 < \varepsilon_1 < 1$ such that for any $0 < \varepsilon \leq \varepsilon_1$, $\bar{t} \geq \underline{t}$, the following inequalities hold:

$$\left|\frac{f_i}{L^{\kappa_i}}\right| \leq L\varepsilon^{r_{i+1}/\rho} \tag{19}$$

$$\left|v_i^{\rho/r_{i+1}}(Z_i(\bar{t})) - v_i^{\rho/r_{i+1}}(Z_i(\underline{t}))\right| \leq L\alpha_i(\cdot)\varepsilon^{(\rho+\tau)/\rho}(\bar{t} - \underline{t}) \tag{20}$$

With the help of Lemmas 5 and 6, we are ready to state the main result of this subsection.

Theorem 1. For the nonlinear system (5) under Assumption 1, the saturated state feedback controller (16) renders that the origin of the closed-loop system is globally asymptotically stable.

C. Reduced order observer and main result

Since $z_2, \cdots, z_n$ are not available for feedback, the controller (16) is not implementable. To estimate the unmeasurable states, we construct a homogeneous observer

$$\begin{aligned} &\dot{\eta}_i = -Ll_{i-1}\hat{z}_i \\ &\hat{z}_i = (\eta_i + l_{i-1}\hat{z}_{i-1})^{r_i/r_{i-1}}, \quad i = 2, \cdots, n \end{aligned} \tag{21}$$

where $\hat{z}_1 = z_1$. Based on (16), we design a saturated output feedback controller

$$v_{sof}(\hat{z}) = v_n(\hat{Z}_n) = -\beta_n \sigma^{r_{n+1}/\rho}\Big(\hat{z}_n^{\rho/r_n} - v_{n-1}^{\rho/r_n}(\hat{Z}_{n-1})\Big) \tag{22}$$

where $v_0 = 0$, $v_i(\hat{Z}_i) = -\beta_i \sigma^{r_{i+1}/\rho}(\hat{z}_i^{\rho/r_i} - v_{i-1}^{\rho/r_i}(\hat{Z}_{i-1}))$, $\hat{Z}_i = (\hat{z}_1, \cdots, \hat{z}_i)$, $i = 1, \cdots, n$ and β_i's are determined by (17).

Remark 2. From (22) and the definition of saturation function $\sigma(\cdot)$, one can easily verify the following inequality holds:

$$|v_{sof}(\hat{z})| \le \bar{v}(\hat{z}) \triangleq \beta_n \Big(\bar{\beta}_n |\hat{z}_n|^{\rho/r_n} + \bar{\beta}_{n-1}|\hat{z}_{n-1}|^{\rho/r_{n-1}} + \cdots + \bar{\beta}_1 |\hat{z}_1|\Big)^{r_{n+1}/\rho} \tag{23}$$

Remark 3. Similarly to [17], using the certainty equivalence principle together with (13), one can also obtain an implementable unsaturated output feedback controller for system (5) as follows

$$v_{uof}(\hat{z}) = -\beta_n \Big(\bar{\beta}_n \hat{z}_n^{\rho/r_n} + \bar{\beta}_{n-1}\hat{z}_{n-1}^{\rho/r_{n-1}} + \cdots + \bar{\beta}_1 \hat{z}_1^{\rho/r_1}\Big)^{r_{n+1}/\rho} \tag{24}$$

Define the estimate errors $e_i = (z_i^{p_{i-1}} - \hat{z}_i^{p_{i-1}})^{\rho/(r_i p_{i-1})}$, $i = 2, \cdots, n$, and choose the Lyapunov function

$$U_i = \int_{\gamma_i^{(2\rho-\tau-r_{i-1})/r_{i-1}}}^{z_i^{(2\rho-\tau-r_{i-1})/r_i}} (s^{r_{i-1}/(2\rho-\tau-r_{i-1})} - \gamma_i) ds \tag{25}$$

where $\gamma_i = \eta_i + l_{i-1} z_{i-1}$. Then, for $i = 2, \cdots, n$, it follows from (21) and (25) that

$$\begin{aligned}\dot{U}_i =& \frac{\partial U_i}{\partial z_i}\Big(L z_{i+1} + \frac{f_i}{L^{\kappa_i}}\Big) + \frac{\partial U_i}{\partial z_{i-1}}\Big(L z_i + \frac{f_{i-1}}{L^{\kappa_{i-1}}}\Big) - L\frac{\partial U_i}{\partial \eta_i} l_{i-1}\hat{z}_i \\ =& \frac{2\rho-\tau-r_{i-1}}{r_i} L z_i^{(2\rho-\tau-r_{i-1}-r_i)/r_i}(z_i^{r_{i-1}/r_i} - \gamma_i) z_{i+1} \\ & -L l_{i-1} e_i^{r_i}\Big(z_i^{(2\rho-\tau-r_{i-1})/r_i} - \hat{z}_i^{(2\rho-\tau-r_{i-1})/r_i}\Big) \\ & -L l_{i-1} e_i^{r_i}\Big(\hat{z}_i^{(2\rho-\tau-r_{i-1})/r_i} - \gamma_i^{(2\rho-\tau-r_{i-1})/r_{i-1}}\Big) \\ & +\frac{\partial U_i}{\partial z_i}\frac{f_i}{L^{\kappa_i}} + \frac{\partial U_i}{\partial z_{i-1}}\frac{f_{i-1}}{L^{\kappa_{i-1}}}\end{aligned} \tag{26}$$

where $z_{n+1} = v_{sof}(\hat{z})$. The following propositions give the proper estimations of some terms of the right-hand side of (26) whose proofs can be achieved by lemmas 3 and 4.

Proposition 2. There exists a positive constant λ_i such that

$$\begin{aligned}& -l_{i-1} e_i^{r_i}\Big(z_i^{(2\rho-\tau-r_{i-1})/r_i} - \hat{z}_i^{(2\rho-\tau-r_{i-1})/r_i}\Big) \\ & \le -l_{i-1}\lambda_i e_i^2\end{aligned} \tag{27}$$

Proposition 3. For $i = 2, \cdots, n-1$, there holds

$$\begin{aligned}& \frac{2\rho-\tau-r_{i-1}}{r_i} z_i^{(2\rho-\tau-r_{i-1}-r_i)/r_i}(z_i^{r_{i-1}/r_i} - \gamma_i) z_{i+1} \\ & \le \frac{1}{12}\sum_{j=i-1}^{i+1} \xi_j^2 + m_i e_i^2 + g_i(l_{i-1}) e_{i-1}^2\end{aligned} \tag{28}$$

where g_i is a continuous function of l_{i-1} and $m_i > 0$ is a constant.

Proposition 4. For the saturated output feedback controller $v_{sof}(\hat{z})$, we obtain

$$\begin{aligned}& \frac{2\rho-\tau-r_{n-1}}{r_n} z_n^{(2\rho-\tau-r_{n-1}-r_n)/r_n}(z_n^{r_{n-1}/r_n} - \gamma_n) v_{sof} \\ & \le \frac{1}{8}\sum_{j=1}^{n} \xi_j^2 + c\sum_{i=2}^{n} e_i^2 + g_n(l_{n-1}) e_{n-1}^2\end{aligned} \tag{29}$$

where g_n is a continuous function of l_{n-1} and $c > 0$ is a constant.

Proposition 5. For $i = 3, \cdots, n$, there holds

$$\begin{aligned}& -l_{i-1} e_i^{r_i}\Big(\hat{z}_i^{(2\rho-\tau-r_{i-1})/r_i} - \gamma_i^{(2\rho-\tau-r_{i-1})/r_{i-1}}\Big) \\ & \le \frac{1}{16}(\xi_{i-1}^2 + \xi_i^2) + e_i^2 + h_i(l_{i-1}) e_{i-1}^2\end{aligned} \tag{30}$$

where h_i is a continuous function of l_{i-1}.

Let $U = \sum_{i=2}^n U_i$. Using the estimates (27), (28), (29) and (30), the time derivative of U satisfies

$$\begin{aligned}\dot{U} =& \frac{1}{2}\sum_{i=1}^n \xi_i^2 + \Big(-l_1\lambda_2 + m_2 + c + g_3(l_2) + h_3(l_2)\Big) e_2^2 \\ & + \sum_{i=3}^{n-1}\Big(-l_{i-1}\lambda_i + m_i + 1 + c + g_{i+1}(l_i) + h_{i+1}(l_i)\Big) e_i^2 \\ & + (-l_{n-1}\lambda_n + 1 + c) e_n^2\end{aligned} \tag{31}$$

By (13), (22), (23) and (24), we can estimate $\xi_i^{2\rho-\tau-r_n}(v_{sof}^{p_n} - z_{n+1}^{*p_n})$ by the following proposition.

Proposition 6. There exists a positive constant μ such that

$$\xi_i^{2\rho-\tau-r_n}(v_{sof} - z_{n+1}^*) \le \frac{1}{4}\sum_{i=1}^n \xi_i^2 + \mu\sum_{i=2}^n e_i^2 \tag{32}$$

With the help of Proposition 6, defining $\Gamma = V_n + U$, combining (15) and (31), and recursively choosing

$$\begin{aligned}& l_{n-1} \ge \lambda_n^{-1}\Big(\frac{1}{4} + 1 + c + \mu\Big) \\ & l_{i-1} \ge \lambda_i^{-1}\Big(\frac{1}{4} + m_i + 1 + c + \mu + g_{i+1}(l_i) + h_{i+1}(l_i)\Big) \\ & \quad i = n-1, \cdots, 3 \\ & l_1 \ge \lambda_2^{-1}\Big(\frac{1}{4} + m_2 + c + \mu + g_3(l_2) + h_3(l_2)\Big)\end{aligned} \tag{33}$$

we obtain

$$\dot{\Gamma} \le -\frac{L}{4}\sum_{i=1}^n \xi_i^2 - \frac{L}{4}\sum_{i=2}^n e_i^2 + \sum_{j=1}^{n-1}\Big|\frac{\partial \Gamma}{\partial z_j}\Big|\Big|\frac{f_j}{L^{\kappa_j}}\Big| \tag{34}$$

The main result of the paper can be summarized into the following theorem:

Theorem 2. For the high-order feedforward nonlinear systems (1) under Assumption 1, the saturated output feedback controller $u = L^{\kappa_n+1} v_{sof}$ in (4), (21) and (22), renders that the origin of the closed-loop system is globally asymptotically stable.

Proof. From the construction of Γ, it can be verified that Γ is positive definite and proper with respect to $Z = (z_1, \cdots, z_n, \eta_2, \cdots, \eta_n)^T$. Denoting the dilation weight

$$\Delta = (\underbrace{r_1, \cdots, r_n}_{for\ z_1,\cdots,z_n}, \underbrace{r_1, \cdots, r_{n-1}}_{for\ \eta_2\cdots,\eta_n}) \tag{35}$$

from Definition 2, it can be shown that Γ is homogeneous of degree $2\rho-\tau$ with respect to Δ. Furthermore, by Lemma 2, there is a constant m_1, such that

$$\Gamma \le m_1 \|Z\|_\Delta^{2\rho-\tau} \tag{36}$$

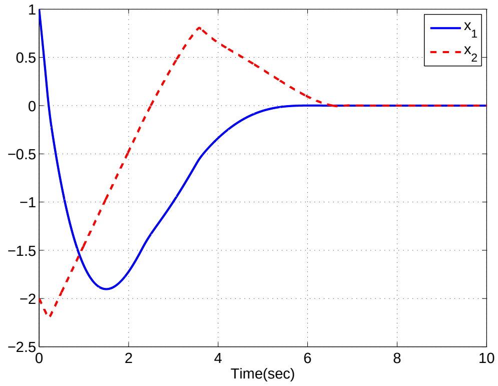

Fig. 1. System states.

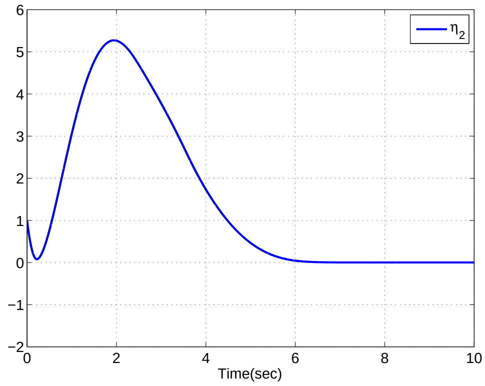

Fig. 2. Observer state.

where $m_1 > 0$ and $\|Z\|_\Delta = \sqrt{(\sum_{i=1}^{n}|z_i|^{2/r_i}) + (\sum_{i=2}^{n}|\eta_i|^{2/r_i})}$. Similarly, since $\sum_{i=1}^{n}\xi_i^2 + \sum_{i=2}^{n}e_i^2$ is homogeneous of degree 2ρ, by Lemma 2.2, there is a constant m_2 such that

$$\dot{\Gamma} \le -m_2 L\|Z\|_\Delta^{2\rho} + \sum_{j=1}^{n-1}\left|\frac{\partial\Gamma}{\partial z_j}\right|\left|\frac{f_j}{L^{\kappa_j}}\right| \tag{37}$$

Noting (23), we can find a constant k such that

$$\dot{\Gamma} \le -k\Gamma^{2\rho/(2\rho-\tau)} \tag{38}$$

Therefore, the closed-loop system (5)with (21) and (22) is globally asymptotically stable. Furthermore, by noting that (4) is an equivalent transformation, the closed-loop system consisting of (1), $u^{p_n} = L^{\kappa_n+1}v^{p_n}$ in (4), (21) and (22), has the same properties as the system (5) with (21) and (22). Thus, the proof is completed.

Remark 4. By noting the fact that $0 < L < 1$ and $\kappa_n+1 > 0$, it is easily observed from Remark 1 that the control law $u(t)$ is bounded by a constant $\beta_n\varepsilon^{r_{n+1}}$, that is, by choosing design parameters ε and β_n as $\beta_n\varepsilon^{r_{n+1}} < u^{max}$, $|u(t)| \le u^{max}$ can be guaranteed.

IV. Simulation Example

Consider the following feedforward system

$$\begin{aligned} \dot{x}_1 &= x_2 + u^3 \\ \dot{x}_2 &= u \\ y &= x_1 \end{aligned} \tag{39}$$

with the requirement of $|u| \le u^{max} = 1$. Choosing $\tau = -\frac{2}{5} \in (-1, 0)$, we have $r_1 = 1$, $r_2 = \frac{3}{5}$ and $r_3 = \frac{1}{5}$. It is obvious that Assumption 1 holds with $b = 1$. Therefore, by Theorem 2, we can explicitly construct a saturated output feedback controller for this example. Specifically, we can choose

$$\begin{aligned} &\dot{\eta}_2 = -Ll_1\hat{z}_2, \quad \hat{z}_2 = (\eta_2 + l_1 y)^{3/5} \\ &u = -L^3\beta_2\sigma^{1/5}\left(\hat{z}_2^{5/3} + \beta_1^{5/3}\sigma(y)\right) \end{aligned} \tag{40}$$

with appropriate positive constants l_1, β_1, β_2, ε and a small enough gain L such that the output feedback controller (40) renders the system (39) globally asymptotically (finite-time) stable.

In the simulation, by choosing the design parameters as $\beta_1 = 1.2$, $\beta_2 = 1.4$, $l_1 = 3$, $\varepsilon = 0.6$, $L = 0.85$ and the

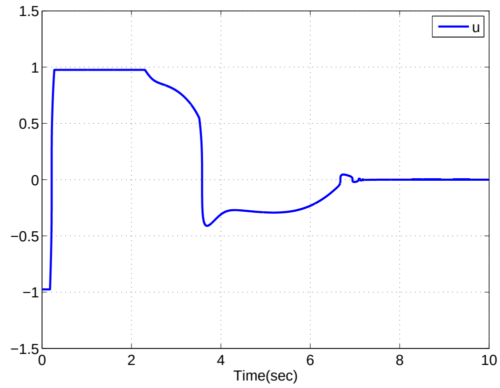

Fig. 3. Control input.

initial condition as $(x_1(0), x_2(0), \eta_2(0))= (1, -2, 1)$, Figs. 1-3 are obtained to exhibit the responses of the closed-loop system, from which the validity of the proposed method is demonstrated.

V. Conclusion

This paper has solved the problem of saturated stabilization by output feedback for a class of feedforward nonlinear systems. With the help of the homogeneous domination approach and the nested saturation technique, a constructive design procedure for reduced order observer-based output feedback control is given, which can guarantee that the closed-loop system states are globally asymptotically regulated to zero and the amplitude of the control signal is bounded.

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