Super Edge-magic Total Labeling of Combination Graphs

Jingwen Li, Bimei Wang, Yanbo Gu and Shuhong Shao

***Abstract*—A graph $G(p,q)$ is said to have an edge-magic total labeling if there exists a bijective function f: $V(G) \cup E(G) \to \{1, 2, \cdots, p+q\}$, such that for any edge uv of G, $f(u)+f(v)+f(uv) = k$, k is a constant. Moreover, G is said to have a super edge-magic total labeling if $f(V(G)) = \{1, 2, \cdots, p\}$. We propose a new algorithm, based on the graph generation method, to solve the problem of super edge-magic total labeling of graphs with a large number of vertices. First, we introduce a new operation called generalized coalescence, then we generate the adjacent matrices of graphs composed of fans, circle and star. Second, we input these matrices to our proposed algorithm. Third, if a graph exist a super edge-magic total labeling, the algorithm will output the corresponding super edge-magic total labeling matrices. Otherwise, no super edge-magic total labeling exists for the graphs involved. Fourth, from the results, we conclude that regular labels are found in some of the graphs involved. Our algorithm can distinguish super edge-magic total labeling graphs from those graphs which don't have.**

***Index Terms*—super edge-magic total labeling, algorithm, graphs.**

I. INTRODUCTION

THROUGHOUT this paper, we are only concerned about simple undirected graphs (loops and multiple edges are not allowed). Let $G(p,q)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. In this paper, NSEMTL graphs means those graphs have no super edge-magic total labeling. Graph theory, which has enjoyed much of the focus of scholars [1], can be used as a powerful and analytical tool in many areas such as interconnection networks [2]. It has been divided into two main parts: graph labeling and graph coloring [3]. The graph labeling problem dates back to the mid-1960s. The concepts of edge-magic total labeling and super edge-magic total labeling were proposed by Kotzig and Rosa in 1970 [4]. Because the aforementioned labelings have very important theoretical and practical significances, they have enjoyed much focus by scholars in that context.

The super edge-magic total labeling of the C_n, caterpillars, K_n, F_n, B_n graphs has been focused on by many researchers, such as Kotzig, Rosa, Enomoto and Figueroa-Centeno, in [4], [5], [6], [7] and [8]. In [9], Figueroa-Centeno and others concluded that $2P_n$ has a super edge-magic total labeling if and only if n is not equal to 2 or 3. In [10] and [11], I.Gray pointed out that if n is an even number greater than or equal to 6, $C_3 \cup C_n$ has a super edge-magic total labeling.

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As far as we know, for the family of graphs $F_m \triangle F_n \triangle C_i \triangle S_j$, there haven't been any algorithm that can obtain its super edge-magic total labeling. Now, we design a new super edge-magic total labeling algorithm for the family of graphs $F_m \triangle F_n \triangle C_i \triangle S_j$, based on the graph generation method [12]. Some useful definitions are as follows:

Definition 1 ([5]) Let $G(p,q)$ be a finite simple connected graph, a bijection f from $V(G) \cup E(G)$ to $\{1, 2, ..., p+q\}$ is called an edge-magic total labeling (EMTL for short) of G if there exists a constant k (called the magic number of f) such that $f(u)+f(v)+f(uv) = k$ for any edge of G. An edge-magic total labeling f is called super edge-magic total labeling (SEMTL for short) if $f(V(G)) = \{1, 2, \cdots, p\}$ and $f(E(G)) = \{p+1, p+2, \cdots, p+q\}$.

Definition 2 Suppose that $V(F_n) = \{u_0, u_1, \cdots, u_n\}$, $V(F_m) = \{v_0, v_1, \cdots, v_m\}$, $V(C_i) = \{x_1, x_2, \cdots, x_i\}$, $V(S_j) = \{y_1, y_2, \cdots, y_j\}$, u_0, v_0, x_1 and y_1 are centre vertices. Let u_0, v_0, x_1 and y_1 conduct three times generalized coalescence operations, form a new composite vertex, we call it u_0, $u_0 = v_0 = x_1 = y_1$. we call this combination graph $F_n \triangle F_m \triangle C_i \triangle S_j$.

From figure 1, it is easy to show that $V(F_n \triangle F_m \triangle C_i \triangle S_j) = V(F_n) \cup V(F_m) \cup (C_i) \cup (S_j) \setminus 3v$, $|V(F_n \triangle F_m \triangle C_i \triangle S_j)| = |V(F_n)| + |V(F_m)| + |V(C_i)| + |V(S_j)| - 3$, $E(F_n \triangle F_m \triangle C_i \triangle S_j) = E(F_n) \cup E(F_m) \cup E(C_i) \cup E(S_j)$.

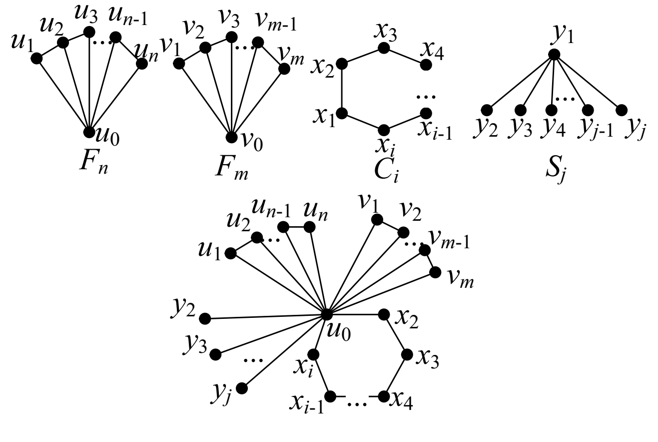

Fig. 1. Graph $F_n \triangle F_m \triangle C_i \triangle S_j$

Definition 3 Suppose there exists a triple (x,y,z) of which x, y can be used for the labelings of two arbitrarily chosen adjacent vertices of an arbitrary graph $G(p,q)$ and z can be used for the edge labeling between these two vertices. If there exists an integer k, satisfying $x+y+z = k$, $x, y \in [1,p]$, $x \neq y$, $z \in [p+1, p+q]$ and $p+q+3 \leq k \leq 2(p+q)$, we call that all these triples are the super edge-magic total labeling solution space of the graph $G(p,q)$. They denoted by $\varphi(p, q, k)$.

In section 2, we show the details of the algorithm. In section 3, the main results of super edge-magic total labeling are given.

II. Algorithm

According to the definition of super edge-magic total labeling, we know that C, the sum of all labels assigned to vertices and edges of a graph $G(p, q)$, is equal to

$$C = L_p + L_q = (p + q)(p + q + 1)/2 \tag{1}$$

where L_p represents the sum of all vertex labels in graph G, L_q represents the sum of all edge labels in graph G. For any edge uv of G, we have $f(u) + f(v) + f(uv) = k$. Because there are q edges in graph G, we get

$$qk = L_q + \sum_{i=1}^{p} deg(v_i)f(v_i) \tag{2}$$

where $deg(v_i)$ represents the degree of vertex v_i. Because $\sum_{i=1}^{p} deg(v_i)f(v_i) = L_p + \sum_{i=1}^{p}(deg(v_i) - 1)f(v_i)$, we derive that

$$qk = C + \sum_{i=1}^{p}(deg(v_i) - 1)f(v_i) \tag{3}$$

Since formula (3) involves only the vertex labels, we transform it to

$$qk = C + \sum_{i=1}^{p}(deg(v_i) - 1)f(v_i) + \sum_{j=1}^{q} 0 \cdot f(e_j) \tag{4}$$

Assume that

$$SUM = \sum_{i=1}^{p}(deg(v_i) - 1)f(v_i) + \sum_{j=1}^{q} 0 \cdot f(e_j) \tag{5}$$

We obtain

$$k = (C + SUM)/q \tag{6}$$

The idea behind the algorithm is as follows:

1) Generate non-isomorphism graphs of $F_n \triangle F_m \triangle C_i \triangle S_j$ and store them in the form of adjacency matrices.

2) The corresponding constant C, the range of the magic number k, the solution space $\varphi(p, q, k)$ of the super edge-magic total labeling, and the degree $deg(v_i)$ can be obtained according to these adjacency matrices.

3) Firstly, initialize vertex labels $f(v_i)$ and edge labels $f(e_j)$ according to the solution space $\varphi(p, q, k)$; Secondly, calculate the value of SUM according to formula (5). Thirdly, let's substitute SUM into formula (6), then we get the value of k. If the value of k is not a positive integer, go back to step (3) and rearrange the sequence of $f(v_i)$ and $f(e_j)$. Lastly, when the assignment satisfies the conditions $f(V(G)) = \{1, 2, \cdots, p\}$ and $f(E(G)) = \{p+1, p+2, \cdots, p+q\}$ or all the permutations are complete, the algorithm will end. If the graph is successfully labeled, the super edge-magic total labeling matrices are output. Otherwise, no super edge-magic total labeling exists for the graph involved.

The pseudo-code of super edge-magic total labeling algorithm is as follows:

Algorithm 1 Super edge-magic total labeling algorithm

Input: Adjacency matrix of $G(p, q)$;

Output: Labeled matrices of SEMTL graphs or adjacency matrices of NSEMTL graphs;

```
Begin
Calculate C(adjacency matrix);
Calculate deg(adjacency matrix);
Calculate k(adjacency matrix);
isSuccess=false;
isContinue=true;
while(isContinue)
   Calculate (SUM, deg(v_i) - 1, f(v_i), f(e_i));
   if((C + SUM)%q = 0)
      k = (C + SUM)/q;
      Apportion(k, f(v_i), f(e_i));
      if(isSuccess)
         break;
      end if
   end if
   Permutation(deg(v_i) - 1);
end while
End
```

Lemma 1: For any graph $G(p, q)$, search the whole solution space $\varphi(p, q, k)$ with super edge-magic total graph labeling algorithm. If there is a solution, graph $G(p, q)$ is super edge-magic total graph. Otherwise, it is not a super edge-magic total labeling graph.

Algorithm running environment and computer hardware configuration are as followings:
OS: Windows 7 64 bit
Processor: Intel() Core(TM)i7-7700 CPU @ 3.60 GHz
RAM: 64.0GB
Development environment: Visual Studio 2013
Development language: $C\sharp$

According to the above algorithm, the following conclusions are summarized.

III. Main Results

Theorem 1: Every combination graph $F_3 \triangle F_2 \triangle C_m \triangle S_n$ exists a super edge-magic total labeling if $3 \leq m \leq 7$, $n \geq 2$.

Proof In order to prove that graph $G(p, q)$ has super edge-magic total labeling, we only need to prove $f(V(G)) = \{1, 2, \cdots, p\}$ and $f(E(G)) = \{p + 1, p + 2, \cdots, p + q\}$. According to the value of index m, We prove theorem 1 by considering five cases now.

Case 1 when $m = 3$, $n \geq 2$, graph $F_3 \triangle F_2 \triangle C_3 \triangle S_n$ is obtained as shown in figure 2(1). The super edge-magic total labeling of graph $F_3 \triangle F_2 \triangle C_3 \triangle S_2$ is shown in figure 2(2).

If $n \geq 3$, the magic number $k \in [2n + 20, 4n + 34]$, when $k = 2n + 20$, we have

$$|V(F_3 \triangle F_2 \triangle C_3 \triangle S_n)| = 4 + 3 + 3 + n - 3 = n + 7$$

$$|E(F_3 \triangle F_2 \triangle C_3 \triangle S_n)| = 5 + 3 + 3 + (n - 1) = n + 10$$

$$|V(F_3 \triangle F_2 \triangle C_3 \triangle S_n)| + |E(F_3 \triangle F_2 \triangle C_3 \triangle S_n)| = 2n + 17$$

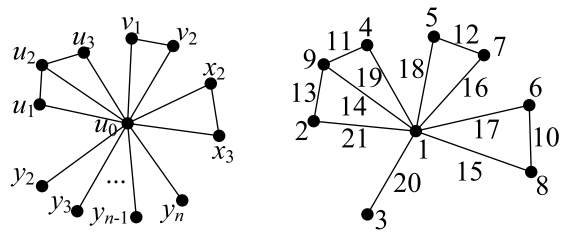

Fig. 2. (1) $F_3 \triangle F_2 \triangle C_3 \triangle S_n$ (2) $F_3 \triangle F_2 \triangle C_3 \triangle S_2$.

The vertex labels of graph $F_3 \triangle F_2 \triangle C_3 \triangle S_n$ are:

$$f(u_i) = \begin{cases} 1, & i = 0 \\ 2, & i = 1 \\ n+7, & i = 2 \\ 3, & i = 3 \end{cases}$$

$$f(v_i) = \begin{cases} 1, & i = 0 \\ 5, & i = 1 \\ n+6, & i = 2 \end{cases}$$

$$f(x_i) = \begin{cases} 1, & i = 1 \\ 7, & i = 2 \\ n+5, & i = 3 \end{cases}$$

$$f(y_i) = \begin{cases} 1, & i = 1 \\ 4, & i = 2 \\ 6, & i = 3 \\ i+4, & 4 \le i \le n \end{cases}$$

Let S be a set of the sum of the labels of two adjacent vertices. $S = \{3, 4, 5, 6, 7, 8\} \cup \{9, 10, \cdots, n + 12\}$. The vertex labels $f(V)$ are $\{1, 2, 3, 4, 5, 6, 7\} \cup \{8, 9, \cdots, n + 4, n + 5, n + 6, n + 7\}$. From definition 1, we know that $f(uv) = k - (f(u) + f(v))$, therefore the edge labels $f(E)$ are $\{2n + 17, 2n + 16, \cdots, 2n + 12\} \cup \{2n + 11, 2n + 10, \cdots, n + 8\}$. It is obvious that $f(V) \to [1, n + 7]$, $f(E) \to [n + 8, 2n + 17]$ and $f(V) \cap f(E) = \phi$, therefore every graph $F_3 \triangle F_2 \triangle C_3 \triangle S_n$ exists a super edge-magic total labeling if $n \ge 2$.

Case 2 when $m = 4$, $n \ge 2$, graph $F_3 \triangle F_2 \triangle C_4 \triangle S_n$ is obtained as shown in figure 3.

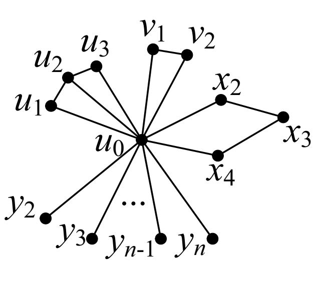

Fig. 3. $F_3 \triangle F_2 \triangle C_4 \triangle S_n$

The super edge-magic total labelings of $F_3 \triangle F_2 \triangle C_4 \triangle S_2$ and $F_3 \triangle F_2 \triangle C_4 \triangle S_3$ are shown in figures 4(1) and 4(2).

If $n \ge 4$, the magic number $k \in [2n + 22, 4n + 38]$, when $k = 2n + 22$, we have

$$|V(F_3 \triangle F_2 \triangle C_4 \triangle S_n)| = 4 + 3 + 4 + n - 3 = n + 8$$

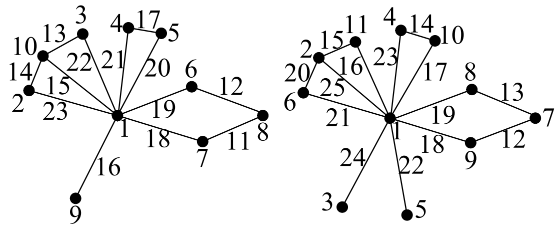

Fig. 4. (1) $F_3 \triangle F_2 \triangle C_4 \triangle S_2$ (2) $F_3 \triangle F_2 \triangle C_4 \triangle S_3$.

$$|E(F_3 \triangle F_2 \triangle C_4 \triangle S_n)| = 5 + 3 + 4 + (n - 1) = n + 11$$

$$|V(F_3 \triangle F_2 \triangle C_4 \triangle S_n)| + |E(F_3 \triangle F_2 \triangle C_4 \triangle S_n)| = 2n + 19$$

The vertex labels of graph $F_3 \triangle F_2 \triangle C_4 \triangle S_n$ are:

$$f(u_i) = \begin{cases} 1, & i = 0 \\ 4, & i = 1 \\ 2, & i = 2 \\ n+8, & i = 3 \end{cases}$$

$$f(v_i) = \begin{cases} 1, & i = 0 \\ 8, & i = 1 \\ n+5, & i = 2 \end{cases}$$

$$f(x_i) = \begin{cases} 1, & i = 1 \\ n+6, & i = 2 \\ 5, & i = 3 \\ n+7, & i = 4 \end{cases}$$

$$f(y_i) = \begin{cases} 1, & i = 1 \\ 3, & i = 2 \\ 6, & i = 3 \\ 7, & i = 4 \\ i+4, & 5 \le i \le n \end{cases}$$

In this case, $S = \{3, 4, 5, 6, 7, 8\} \cup \{9, 10, \cdots, n + 13\}$, $f(V) = \{1, 2, 3, 4, 5, 6, 7, 8\} \cup \{9, 10, \cdots, n + 4, n + 5, n + 6, n + 7, n + 8\}$, $f(E) = \{2n + 19, 2n + 18, \cdots, 2n + 14\} \cup \{2n + 13, 2n + 12, \cdots, n + 9\}$. We can see that $f(V) \to [1, n + 8]$, $f(E) \to [n + 9, 2n + 19]$ and $f(V) \cap f(E) = \phi$, therefore every graph $F_3 \triangle F_2 \triangle C_4 \triangle S_n$ exists a super edge-magic total labeling if $n \ge 2$.

Case 3 when $m = 5$, $n \ge 2$, graph $F_3 \triangle F_2 \triangle C_5 \triangle S_n$ is obtained as shown in figure 5.

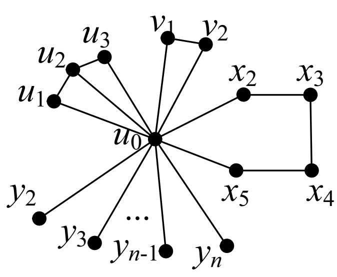

Fig. 5. $F_3 \triangle F_2 \triangle C_5 \triangle S_n$.

The magic number $k \in [2n + 24, 4n + 42]$, when $k = 2n + 24$, we have

$$|V(F_3 \triangle F_2 \triangle C_5 \triangle S_n)| = 4 + 3 + 5 + n - 3 = n + 9$$

$$|E(F_3 \triangle F_2 \triangle C_5 \triangle S_n)| = 5+3+5+(n-1) = n+12$$

$$|V(F_3\triangle F_2\triangle C_5\triangle S_n)|+|E(F_3\triangle F_2\triangle C_5\triangle S_n)| = 2n+21$$

The vertex labels of graph $F_3 \triangle F_2 \triangle C_5 \triangle S_n$ are:

$$f(u_i) = \begin{cases} 1, & i=0 \\ 3, & i=1 \\ 2, & i=2 \\ n+6, & i=3 \end{cases}$$

$$f(v_i) = \begin{cases} 1, & i=0 \\ 5, & i=1 \\ n+9, & i=2 \end{cases}$$

$$f(x_i) = \begin{cases} 1, & i=1 \\ 6, & i=2 \\ n+7, & i=3 \\ 4, & i=4 \\ n+8, & i=5 \end{cases}$$

$$f(y_i) = \begin{cases} 1, & i=1 \\ n+5, & 2\le i\le n \end{cases}$$

In this case, $S = \{3,4,5,6,7\} \cup \{8,9,\cdots,n+14\}$, $f(V) = \{1,2,3,4,5,6\}\cup\{7,8,\cdots,n+5,n+6,n+7,n+8,n+9\}$, $f(E) = \{2n+21,2n+20,\cdots,2n+17\}\cup\{2n+16,2n+15,\cdots,n+10\}$. It is obvious that $f(V)\to[1,n+9]$, $f(E)\to[n+10,2n+21]$ and $f(V)\cap f(E)=\phi$, therefore every graph $F_3\triangle F_2\triangle C_5\triangle S_n$ exists a super edge-magic total labeling if $n\ge 2$.

Case 4 when $m=6$, $n\ge 2$, graph $F_3\triangle F_2\triangle C_6\triangle S_n$ is obtained as shown in figure 6.

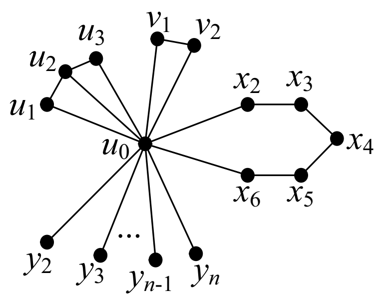

Fig. 6. $F_3 \triangle F_2 \triangle C_6 \triangle S_n$

When n is 2, 3, 4, 5 or 6, the super edge-magic total labeling of $F_3\triangle F_2\triangle C_6\triangle S_n$ are shown in figures 7.

If $n\ge 7$, the magic number $k\in[2n+26,4n+46]$, when $k=2n+26$, we have

$$|V(F_3 \triangle F_2 \triangle C_6 \triangle S_n)| = 4+3+6+n-3 = n+10$$

$$|E(F_3 \triangle F_2 \triangle C_6 \triangle S_n)| = 5+3+6+(n-1) = n+13$$

$$|V(F_3\triangle F_2\triangle C_6\triangle S_n)|+|E(F_3\triangle F_2\triangle C_6\triangle S_n)| = 2n+23$$

The vertex labels of graph $F_3 \triangle F_2 \triangle C_6 \triangle S_n$ are

$$f(u_i) = \begin{cases} 1, & i=0 \\ 4, & i=1 \\ 2, & i=2 \\ n+9, & i=3 \end{cases}$$

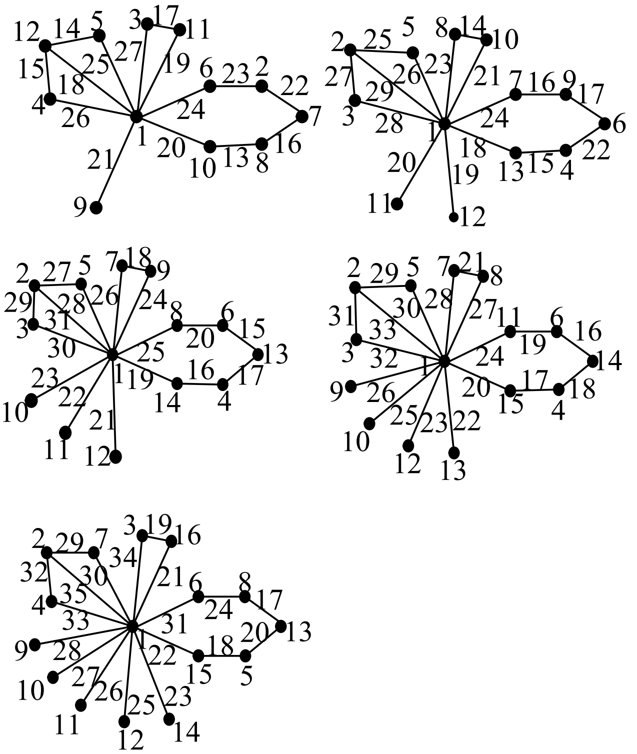

Fig. 7. The super edge-magic total labeling of $F_3\triangle F_2\triangle C_6\triangle S_n$ when n is 2, 3, 4, 5 or 6.

$$f(v_i) = \begin{cases} 1, & i=0 \\ 6, & i=1 \\ n+2, & i=2 \end{cases}$$

$$f(x_i) = \begin{cases} 1, & i=1 \\ 3, & i=2 \\ n+10, & i=3 \\ 5, & i=4 \\ n+7, & i=5 \\ 7, & i=6 \end{cases}$$

$$f(y_i) = \begin{cases} 1, & i=1 \\ n+8, & i=2 \\ n+9-i, & 3\le i\le 6 \\ n+8-i, & 7\le i\le n \end{cases}$$

In this case, $S = \{3,4,5,6,7,8\}\cup\{9,10,\cdots,n+15\}$, $f(V) = \{1,2,3,4,5,6,7\}\cup\{8,9,\cdots,n+7,n+8,n+9,n+10\}$, $f(E) = \{2n+23,2n+22,\cdots,2n+18\}\cup\{2n+17,2n+16,\cdots,n+11\}$. It is obvious that $f(V)\to[1,n+10]$, $f(E)\to[n+11,2n+23]$ and $f(V)\cap f(E)=\phi$, therefore every graph $F_3\triangle F_2\triangle C_6\triangle S_n$ exists a super edge-magic total labeling if $n\ge 2$.

Case 5 when $m=7$, $n\ge 2$, graph $F_3\triangle F_2\triangle C_7\triangle S_n$ is obtained as shown in figure 8.

The magic number $k\in[2n+28,4n+50]$, when $k=2n+29$, we have

$$|V(F_3 \triangle F_2 \triangle C_7 \triangle S_n)| = 4+3+7+n-3 = n+11$$

$$|E(F_3 \triangle F_2 \triangle C_7 \triangle S_n)| = 5+3+7+(n-1) = n+14$$

$$|V(F_3\triangle F_2\triangle C_7\triangle S_n)|+|E(F_3\triangle F_2\triangle C_7\triangle S_n)| = 2n+25$$

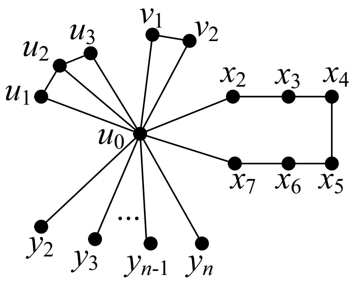

Fig. 8. $F_3 \triangle F_2 \triangle C_7 \triangle S_n$.

The vertex labels of graph $F_3 \triangle F_2 \triangle C_7 \triangle S_n$ are:

$$f(u_i)=\begin{cases} 1, & i=0\\ 4, & i=1\\ 3, & i=2\\ n+7, & i=3 \end{cases}$$

$$f(v_i)=\begin{cases} 1, & i=0\\ 8, & i=1\\ n+8, & i=2 \end{cases}$$

$$f(x_i)=\begin{cases} 1, & i=1\\ 5, & i=2\\ n+9, & i=3\\ 6, & i=4\\ n+11, & i=5\\ 2, & i=6\\ n+10, & i=7 \end{cases}$$

$$f(y_i)=\begin{cases} 1, & i=1\\ 7, & i=2\\ i+6, & 3\le i\le n \end{cases}$$

In this case, $S=\{4,5,6,7,8,9\}\cup\{10,11,\cdots,n+17\}$, $f(V)=\{1,2,3,4,5,6,7,8\}\cup\{9,10,\cdots,n+6,n+7,n+8,n+9,n+10,n+11\}$, $f(E)=\{2n+25,2n+24,\cdots,2n+20\}\cup\{2n+19,2n+18,\cdots,n+12\}$. It is obvious that $f(V)\to[1,n+11]$, $f(E)\to[n+12,2n+25]$ and $f(V)\cap f(E)=\phi$, therefore every graph $F_3 \triangle F_2 \triangle C_7 \triangle S_n$ exists a super edge-magic total labeling if $n\ge 2$.

All cases imply that $3\le m\le 7$, $n\ge 2$, so the proof of theorem 1 is complete.

Theorem 2: Every combination graph $F_3\triangle F_3\triangle C_m\triangle S_n$ exists a super edge-magic total labeling if $3\le m\le 7$, $n\ge 2$.

Proof The method used to prove theorem 2 is similar to the method used to prove theorem 1. According to the value of index m, we now prove theorem 2 by considering five cases.

Case 1 when $m=3$, $n\ge 2$, graph $F_3\triangle F_3\triangle C_3\triangle S_n$ is obtained as shown in figure 9(1). The super edge-magic total labeling of $F_3 \triangle F_3 \triangle C_3 \triangle S_2$ is shown in figure 9(2).

If $n\ge 3$, the magic number $k\in[2n+23,4n+40]$, when $k=2n+23$, we have

$$|V(F_3 \triangle F_3 \triangle C_3 \triangle S_n)|=4+4+3+n-3=n+8$$

$$|E(F_3 \triangle F_3 \triangle C_3 \triangle S_n)|=5+5+3+(n-1)=n+12$$

$$|V(F_3\triangle F_3\triangle C_3\triangle S_n)|+|E(F_3\triangle F_3\triangle C_3\triangle S_n)|=2n+20$$

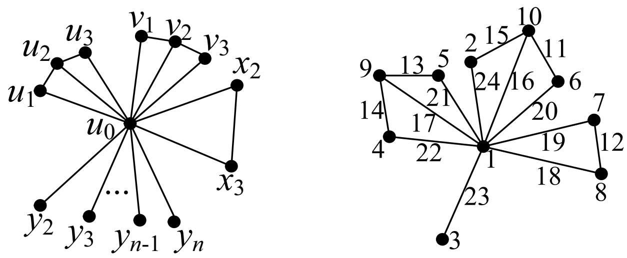

Fig. 9. (1) $F_3 \triangle F_3 \triangle C_3 \triangle S_n$ (2) $F_3 \triangle F_3 \triangle C_3 \triangle S_2$.

The vertex labels of graph $F_3 \triangle F_3 \triangle C_3 \triangle S_n$ are:

$$f(u_i)=\begin{cases} 1, & i=0\\ 5, & i=1\\ n+7, & i=2\\ 6, & i=3 \end{cases}$$

$$f(v_i)=\begin{cases} 1, & i=0\\ 2, & i=1\\ n+8, & i=2\\ 3, & i=3 \end{cases}$$

$$f(x_i)=\begin{cases} 1, & i=1\\ 8, & i=2\\ n+6, & i=3 \end{cases}$$

$$f(y_i)=\begin{cases} 1, & i=1\\ 4, & i=2\\ 7, & i=3\\ i+5, & 4\le i\le n \end{cases}$$

In this case, $S=\{3,4,5,6,7,8,9\}\cup\{10,11\cdots,n+14\}$, $f(V)=\{1,2,3,4,5,6,7,8\}\cup\{9,10,\cdots,n+5,n+6,n+7,n+8\}$, $f(E)=\{2n+20,2n+19,\cdots,2n+14\}\cup\{2n+13,2n+12,\cdots,n+9\}$. It is obvious that $f(V)\to[1,n+8]$, $f(E)\to[n+9,2n+20]$ and $f(V)\cap f(E)=\phi$, therefore every graph $F_3 \triangle F_3 \triangle C_3 \triangle S_n$ exists a super edge-magic total labeling if $n\ge 2$.

Case 2 when $m=4$, $n\ge 2$, graph $F_3\triangle F_3\triangle C_4\triangle S_n$ is obtained as shown in figure 10(1). The super edge-magic total labeling of $F_3\triangle F_3\triangle C_4\triangle S_2$ is shown in figure 10(2).

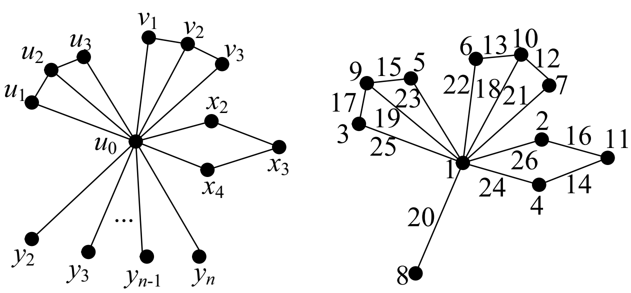

Fig. 10. (1) $F_3 \triangle F_3 \triangle C_4 \triangle S_n$ (2) $F_3 \triangle F_3 \triangle C_4 \triangle S_2$.

If $n\ge 3$, the magic number $k\in[2n+25,4n+44]$, when $k=2n+25$, we have

$$|V(F_3 \triangle F_3 \triangle C_4 \triangle S_n)|=4+4+4+n-3=n+9$$

$|E(F_3 \triangle F_3 \triangle C_4 \triangle S_n)| = 5 + 5 + 4 + (n-1) = n + 13$

$|V(F_3 \triangle F_3 \triangle C_4 \triangle S_n)| + |E(F_3 \triangle F_3 \triangle C_4 \triangle S_n)| = 2n+22$

The vertex labels of graph $F_3 \triangle F_3 \triangle C_4 \triangle S_n$ are:

$$f(u_i) = \begin{cases} 1, & i = 0 \\ 7, & i = 1 \\ 2, & i = 2 \\ n+9, & i = 3 \end{cases}$$

$$f(v_i) = \begin{cases} 1, & i = 0 \\ 4, & i = 1 \\ n+8, & i = 2 \\ 5, & i = 3 \end{cases}$$

$$f(x_i) = \begin{cases} 1, & i = 1 \\ n+6, & i = 2 \\ 8, & i = 3 \\ n+7, & i = 4 \end{cases}$$

$$f(y_i) = \begin{cases} 1, & i = 1 \\ 3, & i = 2 \\ 6, & i = 3 \\ i+5, & 4 \le i \le n \end{cases}$$

In this case, $S = \{3,4,5,6,7,8,9\} \cup \{10, 11 \cdots, n+15\}$, $f(V) = \{1,2,3,4,5,6,7,8,9\} \cup \{10, 11, \cdots, n+5, n+6, n+7, n+8, n+9\}$, $f(E) = \{2n+22, 2n+21, \cdots, 2n+16\} \cup \{2n+15, 2n+14, \cdots, n+10\}$. It is obvious that $f(V) \to [1, n+9]$, $f(E) \to [n+10, 2n+22]$ and $f(V) \cap f(E) = \phi$, therefore every graph $F_3 \triangle F_3 \triangle C_4 \triangle S_n$ exists a super edge-magic total labeling if $n \ge 2$.

Case 3 when $m = 5$, $n \ge 2$, graph $F_3 \triangle F_3 \triangle C_5 \triangle S_n$ is obtained as shown in figure 11.

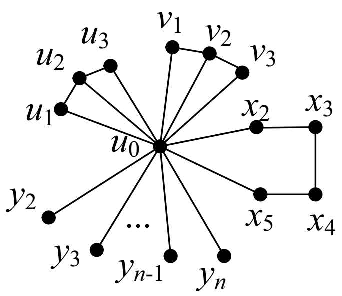

Fig. 11. $F_3 \triangle F_3 \triangle C_5 \triangle S_n$

The super edge-magic total labeling of $F_3 \triangle F_3 \triangle C_5 \triangle S_2$ and $F_3 \triangle F_3 \triangle C_5 \triangle S_3$ are shown in figure 12.

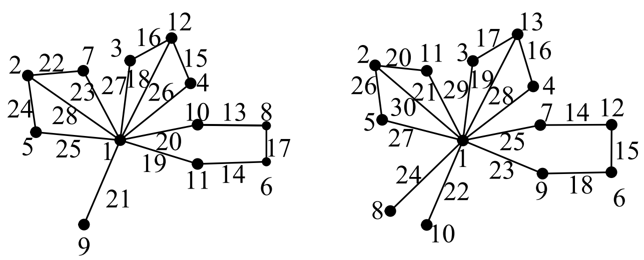

Fig. 12. The super edge-magic total labeling of $F_3 \triangle F_3 \triangle C_5 \triangle S_n$ when n is 2 or 3.

If $n \ge 4$, the magic number $k \in [2n+27, 4n+48]$, when $k = 2n+27$, we have

$|V(F_3 \triangle F_3 \triangle C_5 \triangle S_n)| = 4 + 4 + 5 + n - 3 = n + 10$

$|E(F_3 \triangle F_3 \triangle C_5 \triangle S_n)| = 5 + 5 + 5 + (n-1) = n + 14$

$|V(F_3 \triangle F_3 \triangle C_5 \triangle S_n)| + |E(F_3 \triangle F_3 \triangle C_5 \triangle S_n)| = 2n+24$

The vertex labels of graph $F_3 \triangle F_3 \triangle C_5 \triangle S_n$ are:

$$f(u_i) = \begin{cases} 1, & i = 0 \\ 6, & i = 1 \\ 2, & i = 2 \\ n+6, & i = 3 \end{cases}$$

$$f(v_i) = \begin{cases} 1, & i = 0 \\ n+9, & i = 1 \\ 3, & i = 2 \\ n+10, & i = 3 \end{cases}$$

$$f(x_i) = \begin{cases} 1, & i = 1 \\ 9, & i = 2 \\ n+7, & i = 3 \\ 7, & i = 4 \\ n+8, & i = 5 \end{cases}$$

$$f(y_i) = \begin{cases} 1, & i = 1 \\ 4, & i = 2 \\ 5, & i = 3 \\ 8, & i = 4 \\ i+5, & 5 \le i \le n \end{cases}$$

In this case, $S = \{3,4,5,6,7,8,9,10\} \cup \{11, 12 \cdots, n+16\}$, $f(V) = \{1,2,3,4,5,6,7,8,9\} \cup \{10, 11, \cdots, n+5, n+6, n+7, n+8, n+9, n+10\}$, $f(E) = \{2n+24, 2n+23, \cdots, 2n+17\} \cup \{2n+16, 2n+15, \cdots, n+11\}$. It is obvious that $f(V) \to [1, n+10]$, $f(E) \to [n+11, 2n+24]$ and $f(V) \cap f(E) = \phi$, therefore every graph $F_3 \triangle F_3 \triangle C_5 \triangle S_n$ exists a super edge-magic total labeling if $n \ge 2$.

Case 4 when $m = 6$, $n \ge 2$, graph $F_3 \triangle F_3 \triangle C_6 \triangle S_n$ is obtained as shown in figure 13.

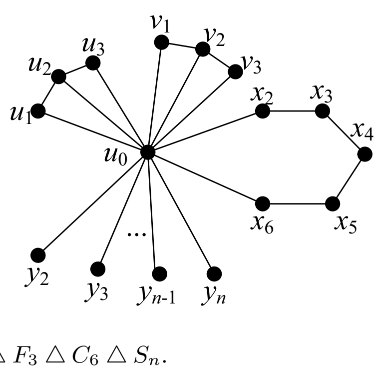

Fig. 13. $F_3 \triangle F_3 \triangle C_6 \triangle S_n$.

The magic number $k \in [2n+29, 4n+52]$, when $k = 2n+29$, we have

$|V(F_3 \triangle F_3 \triangle C_6 \triangle S_n)| = 4 + 4 + 6 + n - 3 = n + 11$

$|E(F_3 \triangle F_3 \triangle C_6 \triangle S_n)| = 5 + 5 + 6 + (n-1) = n + 15$

$|V(F_3 \triangle F_3 \triangle C_6 \triangle S_n)| + |E(F_3 \triangle F_3 \triangle C_6 \triangle S_n)| = 2n+26$

The vertex labels of graph $F_3 \triangle F_3 \triangle C_6 \triangle S_n$ are:

$$f(u_i)=\begin{cases}1, & i=0\\ 4, & i=1\\ 2, & i=2\\ n+7, & i=3\end{cases}$$

$$f(v_i)=\begin{cases}1, & i=0\\ 6, & i=1\\ 3, & i=2\\ n+11, & i=3\end{cases}$$

$$f(x_i)=\begin{cases}1, & i=1\\ n+9, & i=2\\ 8, & i=3\\ n+8, & i=4\\ 5, & i=5\\ n+10, & i=6\end{cases}$$

$$f(y_i)=\begin{cases}1, & i=1\\ 7, & i=2\\ i+6, & 3\le i\le n\end{cases}$$

In this case, $S=\{3,4,5,6,7,8,9\}\cup\{10,11\cdots,n+17\}$, $f(V)=\{1,2,3,4,5,6,7,8\}\cup\{9,10,\cdots,n+6,n+7,n+8,n+9,n+10,n+11\}$, $f(E)=\{2n+26,2n+25,\cdots,2n+20\}\cup\{2n+19,2n+18,\cdots,n+12\}$. It is obvious that $f(V)\to[1,n+11]$, $f(E)\to[n+12,2n+26]$ and $f(V)\cap f(E)=\phi$, therefore every graph $F_3 \triangle F_3 \triangle C_6 \triangle S_n$ exists a super edge-magic total labeling if $n\ge 2$.

Case 5 when $m=7, n\ge 2$, graph $F_3\triangle F_3\triangle C_7\triangle S_n$ is obtained as shown in figure 14.

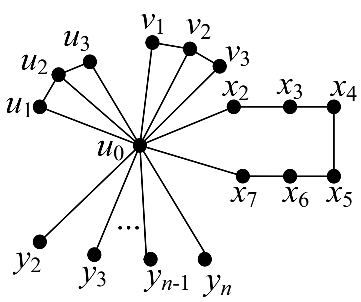

Fig. 14. $F_3 \triangle F_3 \triangle C_7 \triangle S_n$

When n is 2, 3, 4, 5, 6 or 7, the super edge-magic total labeling of $F_3 \triangle F_3 \triangle C_7 \triangle S_n$ are shown in figure 15.

If $n\ge 8$, the magic number $k\in[2n+31,4n+56]$, when $k=2n+31$, we have

$$|V(F_3 \triangle F_3 \triangle C_7 \triangle S_n)|=4+4+7+n-3=n+12$$

$$|E(F_3 \triangle F_3 \triangle C_7 \triangle S_n)|=5+5+7+(n-1)=n+16$$

$$|V(F_3\triangle F_3\triangle C_7\triangle S_n)|+|E(F_3\triangle F_3\triangle C_7\triangle S_n)|=2n+28$$

The vertex labels of graph $F_3 \triangle F_3 \triangle C_7 \triangle S_n$ are:

$$f(u_i)=\begin{cases}1, & i=0\\ 4, & i=1\\ 2, & i=2\\ n+10, & i=3\end{cases}$$

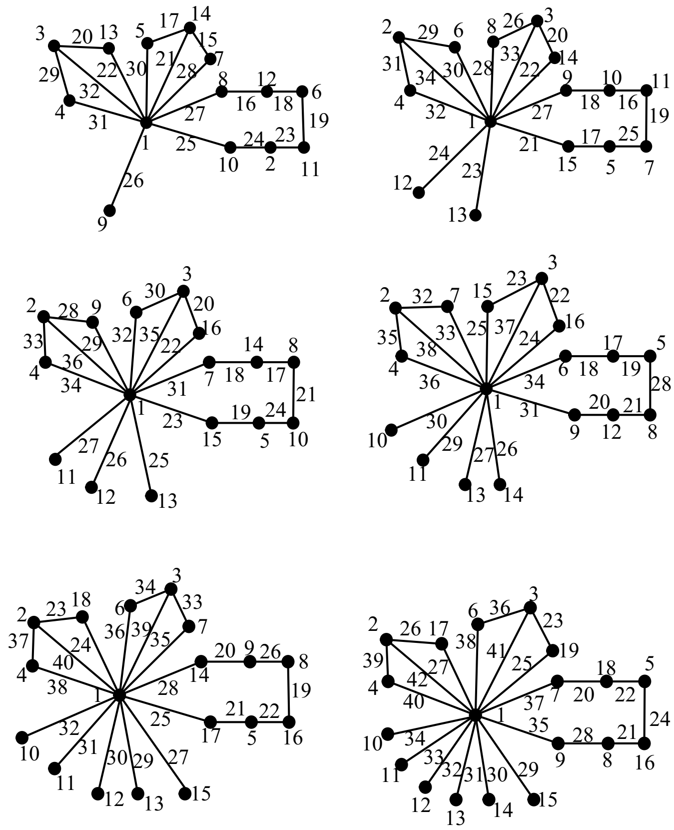

Fig. 15. The super edge-magic total labeling of $F_3\triangle F_3\triangle C_7\triangle S_n$ when n is 2, 3, 4, 5, 6 or 7.

$$f(v_i)=\begin{cases}1, & i=0\\ 6, & i=1\\ 3, & i=2\\ n+12, & i=3\end{cases}$$

$$f(x_i)=\begin{cases}1, & i=1\\ 7, & i=2\\ n+11, & i=3\\ 5, & i=4\\ n+9, & i=5\\ 8, & i=6\\ n+2, & i=7\end{cases}$$

$$f(y_i)=\begin{cases}1, & i=1\\ n+10-i, & 2\le i\le 7\\ n+9-i, & 8\le i\le n\end{cases}$$

In this case, $S=\{3,4,5,6,7,8,9\}\cup\{10,11\cdots,n+18\}$, $f(V)=\{1,2,3,4,5,6,7,8\}\cup\{9,10,\cdots,n+9,n+10,n+11,n+12\}$, $f(E)=\{2n+28,2n+27,\cdots,2n+22\}\cup\{2n+21,2n+20,\cdots,n+13\}$. It is obvious that $f(V)\to[1,n+12]$, $f(E)\to[n+13,2n+28]$ and $f(V)\cap f(E)=\phi$, therefore every graph $F_3 \triangle F_3 \triangle C_7 \triangle S_n$ exists a super edge-magic total labeling if $n\ge 2$.

All cases imply that $3\le m\le 7$, $n\ge 2$, so the proof of the theorem 2 is complete.

IV. Conclusion

In this paper, we have proposed a new algorithm, based on the graph generation method, to solve the super edge-magic total labeling of composite graphs $F_m \triangle F_n \triangle C_i \triangle S_j$ that satisfy certain criteria. However, it is very very difficult to solve the labeling problem using traditional manual method. Finally, we conclude that combining computer algorithm with the method of induction to solve the labeling problem proved to be very effective.

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REFERENCES

[1] Ruo-Wei Hung, Horng-Dar Chen, and Sian-Cing Zeng, "The Hamiltonicity and Hamiltonian Connectivity of Some Shaped Supergrid Graphs," *IAENG International Journal of Computer Science*, vol. 44, no.4, pp. 432-444, 2017.

[2] Fu-Hsing Wang and Cheng-Ju Hsu, "Rainbow Colorings on Pyramid Networks," *IAENG International Journal of Computer Science*, vol. 46, no.3, pp. 439-444, 2019.

[3] Chen-xu Yang, Xing-chao Deng, and Rui-fang Shao, "On R-hued Coloring of Some Perfect and Circulant Graphs," *IAENG International Journal of Applied Mathematics*, vol. 49, no.4, pp. 421-426, 2019.

[4] A. Kotzig and A. Rosa, "Magic valuations of finite graphs," *Math. Bull.*, vol. 13, pp. 451-461, Jul. 1970.

[5] H. Enomoto, A. S. Llado, T. Nakamigawa, and G. Ringel, "Super edge-magic graphs," *SUT J. Math.*, vol. 34, pp. 105-109, 1998.

[6] A. Kotzig and A. Rosa, "Magic valuations of complete graphs," *Centre de Recherches Mathematiques, Universite de Montreal*, CRM-175, 1972.

[7] R. Figueroa-Centeno, R. Ichishima, and F. Muntaner-Batle,"On super edge-magic graphs," *Ars Combin.*, vol. 64, pp. 81-95, 2002.

[8] Y. Yuansheng, X. Yue, X. Xirong, and M. Xinhong, "Super edge magic labelings of book graphs Bn," *Ars Combin.*, vol. 93, pp. 431-438, 2009.

[9] R. Figueroa-Centeno, R. Ichishima, and F. Muntaner-Batle, "On edge-magic labelings of certain disjoint unions of graphs," *Australas. J.Combin.*, vol. 32, pp. 225-242, 2005.

[10] R. Figueroa-Centeno, R. Ichishima, F. Muntaner-Batle and A. Oshima, "A magical approach to some labeling conjectures," *Discussiones Math. Graph Theory*, vol. 31, pp. 79-113, 2011.

[11] I. G ray, "New Construction Methods for Vertex-magic Total Labelings of Graphs," *Ph.D. thesis*, University of Newcastle, 2006.

[12] McKay B D, Piperno A. "Practical graph isomorphism, II," *Journal of Symbolic Computation*, vol. 60, pp. 94-112, 2014.