Existence of Multiple Positive Solutions for a Mixed-order Three-point Boundary Value Problem with P-Laplacian

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***Abstract*—In this paper, we discuss the mixed-order three-point boundary value problem with p-Laplacian**

$$(\phi_p(D_{0+}^{\alpha}u(t)))' + a(t)f(t,u(t)) = 0, \quad 0 < t < 1, \tag{1}$$

$$D_{0+}^{\alpha}u(0) = u'(0) = 0, \quad u(1) = \gamma u(\eta), \tag{2}$$

where $\phi_p(s) = |s|^{p-2}s, p > 1, \gamma, \eta \in (0,1), 1 < \alpha \le 2$, D_{0+}^{α} is the Caputo fractional derivative. Benefiting from a fixed point theorem for operators on a cone, we establish the existence condition of multiple (at least three) positive solutions to the above mixed-order three-point boundary value problem with p-Laplacian are obtained.

***Index Terms*—Multiple positive solutions; Mixed-order three-point boundary value problem; p-Laplacian; Caputo's fractional derivative.**

I. Introduction

DIFFERENTIAL equations of fractional order have been recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control porous media, electromagnetism, etc. See [1-5]. There has been a significant development in the study of fractional differential equations in recent years, see the monographs of Kilbas et al. [6], Lakshmikantham et al. [7], Podlubny [4], Samko et al. [8], and the survey by Agarwal et al. [9].

Recently, integer order p-Laplacian boundary value problems have been widely studied owing to its importance in theory and application of mathematics and physics, see for example [10-13] and the references therein. Especially, in [10], Ji and Ge studied the existence of multiple positive solutions for Sturm-Liouville-Like four-point boundary value problem with p-Laplacian

$$(\phi_p(u'(t)))'(t) + f(t,u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) - \alpha u'(\xi) = 0, \quad u(1) + \beta u'(\eta) = 0,$$

by means of a fixed-point theorem for operators on a cone.

On the other hand, there are few articles dealing with the existence of solutions to boundary value problems for fractional differential equation with p-Laplacian operator.

In [15], T. Chen et al. investigated the existence of solutions of the boundary value problem for fractional p-Laplacian equation with the following form

$$D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha}u(t)) = f(t,u(t),D_{0+}^{\alpha}u(t)), \quad 0 < t < 1,$$

$$D_{0+}^{\alpha}u(0) = D_{0+}^{\alpha}u(1) = 0,$$

where $0 < \alpha, \beta \le 1, 1 < \alpha + \beta \le 2$, D_{0+}^{α} is the Caputo fractional derivative, and $f: [0,1] \times R^2 \to R$ is continuous.

In [16], Z. Liu et al. studied a class of BVPs for nonlinear fractional differential equations with p-Laplacian operator

$$D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha}u(t)) = f(t,u(t),D_{0+}^{\alpha}u(t)), \quad 0 < t < 1,$$

$$u(0) = \mu\int_0^1 u(s)ds + \lambda u(\xi),$$

$$D_{0+}^{\alpha}u(0) = kD_{0+}^{\alpha}u(\eta),$$

where $\phi_p(s) = |s|^{p-2}s, p > 1, 0 < \alpha, \beta \le 1, 1 < \alpha + \beta \le 2, \mu, \lambda, k \in R,\ \xi, \eta \in [0,1]$, D_{0+}^{α} denotes the Caputo fractional derivative of order α and $f: [0,1] \times R^2 \to R$ is a continuous function.

In [17], by using upper and lower solutions methods under suitable monotone conditions, the authors investigated the existence of positive solutions to the following nonlocal problem

$$D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha}u(t)))(t) + f(t,u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) = 0,\ u(1) = au(\xi),$$

$$D_{0+}^{\alpha}u(0) = 0,\ D_{0+}^{\alpha}u(1) = bD_{0+}^{\alpha}u(\eta)$$

where $\phi_p(s) = |s|^{p-2}s, p > 1, 1 < \alpha, \beta \le 2, 0 \le a, b \le 1,\ 0 < \xi < \eta < 1$.

Guoqing Chai [18] investigated the existence and multiplicity of positive solutions for a class of boundary value problems of fractional differential equations with a p-Laplacian operator

$$D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha}u(t))) + f(t,u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) = 0,\ u(1) + \sigma D_{0+}^{\gamma}u(1) = 0,\ D_{0+}^{\alpha}u(0) = 0,$$

where $1 < \alpha \le 2, 0 < \beta \le 1,\ 0 < \gamma \le 1,\ 0 \le \alpha - \gamma - 1, \sigma$ is a positive constant number, $D_{0+}^{\alpha}, D_{0+}^{\beta}, D_{0+}^{\gamma}$ are the standard Riemann-Liouville derivatives.

No contribution exists, as far as we know, concerning the mixed-order (both Caputo's fractional-order derivative and integer-order derivative are included in the equation) three-point boundary value problem with p-Laplacian

$$(\phi_p(D_{0+}^{\alpha}u(t)))' + a(t)f(t,u(t)) = 0, \quad 0 < t < 1,$$

$$D_{0+}^{\alpha}u(0) = u'(0) = 0, \quad u(1) = \gamma u(\eta),$$

where $\phi_p(s) = |s|^{p-2}s, p > 1, \gamma, \eta \in (0,1), 1 < \alpha \le 2$, D_{0+}^{α} is the Caputo fractional derivative, the function f is assumed to satisfy certain conditions which will be specified later. To obtain the existence of multiple positive solutions to the above problem, a fixed point theorem on cones will be applied.

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II. The preliminary lemmas

The material in this section is basic in some sence. For the convenience of readers, we provide some background material in this section.

Definition 2.1[15] The fractional integral of order $\alpha > 0$ for function $y: (0, +\infty) \to R$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, \quad \alpha > 0.$$

Definition 2.2[15] The Caputo's derivative for function y is defined as

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s) ds}{(t-s)^{\alpha+1-n}}, \quad n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of real number α.

Lemma 2.1 Let $\alpha > 0$, then the fractional differential equation

$$D_{0+}^{\alpha} u(t) = 0$$

has solutions

$$\begin{aligned} u(t) &= c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}, \quad c_i \in R, \\ & i = 1, 2, \cdots, n, n = [\alpha] + 1. \end{aligned}$$

Lemma 2.2 *[15]* Let $\alpha > 0$, then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}$$

for some $c_i \in R, \quad i = 1, 2, \cdots, n, n = [\alpha] + 1$.

We shall consider the Banach space $E = C[0,1]$ equipped with standard norm

$$||u|| = \max_{0 \le t \le 1} |u(t)|.$$

The proof of existence of multiple positive solutions is based on an application of the following Theorem.

Theorem 2.3 [13] Let K be a cone in a Banach space X. Let D be an open bounded subset of X with $D_k = D \cap K \neq \emptyset$ and $\overline{D_k} \neq K$. Assume that $T: \overline{D_k} \longrightarrow K$ is a compact map such that $x \neq Tx$ for $x \in \partial D_k$. Then the following results hold:
(1) If $\|Tx\| \le \|x\|, \quad x \in \partial D_k$, then $i_k(T, D_k) = 1$.
(2) If there exists $e \in K \setminus \{0\}$ such that $x \neq Tx + \lambda e$ for all $x \in \partial D_k$ and all $\lambda > 0$, then $i_k(T, D_k) = 0$.
(3) Let U be open in X such that $\overline{U} \subset D_k$. If $i_k(T, D_k) = 1$ and $i_k(T, U_k) = 0$, then T has a fixed point in $D_k \setminus \overline{U_k}$. The same result holds if $i_k(T, D_k) = 0$ and $i_k(T, U_k) = 1$.

Lemma 2.4 Let $h(t) \in C[0,1]$. Then the following boundary value problem

$$(\phi_p(D_{0+}^{\alpha} u(t)))' + a(t) h(t) = 0, \qquad 0 < t < 1, \tag{3}$$

$$D_{0+}^{\alpha} u(0) = u'(0) = 0, \quad u(1) = \gamma u(\eta), \tag{4}$$

has a unique solution which can be expressed by

$$\begin{aligned} u(t) &= \int_0^1 G(t,s) \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds \\ &+ \tfrac{\gamma}{1-\gamma} \int_0^1 G(\eta, s) \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds, \end{aligned} \tag{5}$$

where

$$G(t,s) = \begin{cases} \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{6}$$

Proof: Integrating both sides of equation (3) on [0,t], we have

$$\phi_p(D_{0+}^{\alpha} u(t)) - \phi_p(D_{0+}^{\alpha} u(0)) = -\int_0^t a(s) h(s) ds,$$

so

$$D_{0+}^{\alpha} u(t) = -\phi_q\Big(\int_0^t a(s) h(s) ds\Big),$$

from Lemma 2.2, it follows that

$$\begin{aligned} u(t) &= -\tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds \\ &+ A + Bt, \end{aligned}$$

since $u'(0) = 0$, we have $B = 0$. Namely

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds + A,$$

$$u(1) = -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds + A,$$

$$u(\eta) = -\frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds + A,$$

furthermore, since $u(1) = \gamma u(\eta)$, we get

$$\begin{aligned} A &= \tfrac{1}{(1-\gamma)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds \\ &- \tfrac{\gamma}{(1-\gamma)\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds. \end{aligned}$$

So,

$$\begin{aligned} u(t) &= -\tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds \\ &+ \tfrac{1}{(1-\gamma)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds \\ &- \tfrac{\gamma}{(1-\gamma)\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds, \end{aligned}$$

splitting the second integral in two parts of the form

$$\frac{1}{\Gamma(\alpha)} + \frac{\gamma}{(1-\gamma)\Gamma(\alpha)} = \frac{1}{(1-\gamma)\Gamma(\alpha)},$$

thus

$$\begin{aligned} u(t) &= -\tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds \\ &+ \tfrac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds \\ &+ \tfrac{\gamma}{(1-\gamma)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds \\ &- \tfrac{\gamma}{(1-\gamma)\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds \\ &= \int_0^t \Big(\tfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big) \\ & \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds \\ &+ \int_t^1 \tfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds \\ &+ \tfrac{\gamma}{(1-\gamma)\Gamma(\alpha)} \int_0^\eta ((1-s)^{\alpha-1} - (\eta-s)^{\alpha-1}) \\ & \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds \\ &+ \tfrac{\gamma}{(1-\gamma)\Gamma(\alpha)} \int_\eta^1 (1-s)^{\alpha-1} \phi_q\Big(\int_0^s a(\tau) h(\tau) d\tau\Big) ds. \end{aligned}$$

The proof is complete. ■

Lemma 2.5 Let $\beta \in (0,1)$ be fixed. The function $G(t,s)$ defined by (6) satisfies the following properties.

1. $0 \le G(t,s) \le G(s,s)$, for all $s \in (0,1)$,
2. $\min\limits_{0\le t\le\beta} G(t,s) \ge \frac{1-\beta^{\alpha-1}}{2} G(s,s)$, for all $s \in (0,1)$.

Proof: 1. As $1 < \alpha \le 2$ and $0 \le s \le t \le 1$, we have

$$(1-s)^{\alpha-1} > (t-s)^{\alpha-1},$$

thus $G(t,s) > 0$. Note

$$\frac{\partial G(t,s)}{\partial t} = -\frac{(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)} \le 0,$$

then $G(t,s)$ is nonincreasing as a function of t, therefore

$$G(t,s) \le G(s,s),\ \forall s \in (0,1).$$

2. For $0 \le t \le \beta$, we have

$$\min_{0\le t\le\beta} G(t,s) = G(\beta,s), \tag{7}$$

where

$$G(\beta,s) = \begin{cases} \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{(\beta-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le \beta \le 1, \\ \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le \beta \le s \le 1. \end{cases}$$

(a) If $0 \le s \le \beta \le 1$,

$$\begin{aligned} \min_{0\le t\le\beta} G(t,s) &= \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(\beta-s)^{\alpha-1}}{\Gamma(\alpha)}, \\ &= \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{\beta^{\alpha-1}(1-\frac{s}{\beta})^{\alpha-1}}{\Gamma(\alpha)}, \\ &\ge \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{\beta^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &= \frac{(1-\beta^{\alpha-1})(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &> \frac{(1-\beta^{\alpha-1})}{2}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &= \frac{(1-\beta^{\alpha-1})}{2} G(s,s), \end{aligned} \tag{8}$$

(b) If $0 \le \beta \le s \le 1$,

$$\begin{aligned} \min_{0\le t\le\beta} G(t,s) &= \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &> \frac{(1-\beta^{\alpha-1})}{2}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &= \frac{(1-\beta^{\alpha-1})}{2} G(s,s), \end{aligned} \tag{9}$$

(8), (9) imply that property 2 holds. The proof is complete. ■

Define the cone K by

$$K = \{u \in C[0,1] :\ u(t) \ge 0,\ \min_{0\le t\le\beta} u(t) \ge \frac{1-\beta^{\alpha-1}}{2}\|u\|\}$$

and the operator $T :\ K \to E$ by

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)\phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds \\ &+ \tfrac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds. \end{aligned} \tag{10}$$

Remark 2.1 By Lemma 2.4, the problem (1), (2) has a solution $u(t)$ if and only if u is a fixed point of T.

Lemma 2.6 T is completely continuous and $T(K) \subset K$.

Proof: From Lemma 2.5, we have $G(t,s) \ge 0$, so $Tu(t) \ge 0$.

$$\begin{aligned} \|Tu\| &= \int_0^1 \max_{0\le t\le 1} G(t,s)\phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds \\ &+ \tfrac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds \\ &= \int_0^1 G(s,s)\phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds \\ &+ \tfrac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds \end{aligned} \tag{11}$$

$$\begin{aligned} &\min_{0\le t\le\beta} Tu(t) \\ &= \int_0^1 \min_{0\le t\le\beta} G(t,s)\phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds \\ &+ \tfrac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds \\ &\ge \tfrac{1-\beta^{\alpha-1}}{2}\int_0^1 G(s,s)\phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds \\ &+ \tfrac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds \\ &\ge \tfrac{1-\beta^{\alpha-1}}{2}\int_0^1 G(s,s)\phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds \\ &+ \tfrac{1-\beta^{\alpha-1}}{2}\tfrac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds \\ &= \tfrac{1-\beta^{\alpha-1}}{2}\Big[\int_0^1 G(s,s)\phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds \\ &+ \tfrac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds\Big] \\ &= \tfrac{1-\beta^{\alpha-1}}{2}\|Tu\|. \end{aligned} \tag{12}$$

Based on the (11), (12), it shows that $T(K) \subseteq K$. In view of the assumption of nonnegativeness and continuity of function $G(t,s)$ and $a(t)f(t,u(t))$, we conclude that $T :\ K \to K$ is continuous.

Let $\Omega \subset K$ be bounded, that is, there exists $L > 0$ such that $\|u\| \le L$ for all $u \in \Omega$. Let

$$M = \max_{0\le t\le 1,\ 0\le u\le L} |f(t,u)|,$$

then for $u \in \Omega$, from Lemma 2.4 and Lemma 2.5, we obtain

$$\begin{aligned} |Tu(t)| &= \Big|\int_0^1 G(t,s)\phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds \\ &+ \tfrac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\Big)ds\Big| \\ &\le \int_0^1 G(s,s)\phi_q\Big(\int_0^1 a(\tau)Md\tau\Big)ds \\ &+ \tfrac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\Big(\int_0^1 a(\tau)Md\tau\Big)ds \\ &\le \phi_q(M)\phi_q\Big(\int_0^1 a(\tau)d\tau\Big) \\ &\Big[\int_0^1 \tfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \tfrac{\gamma}{1-\gamma}\int_0^1 \tfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\Big] \\ &= \tfrac{\phi_q(M)}{(1-\gamma)\Gamma(\alpha)}\phi_q\Big(\int_0^1 a(\tau)d\tau\Big)\int_0^1(1-s)^{\alpha-1}ds \\ &= \tfrac{\phi_q(M)}{(1-\gamma)\Gamma(\alpha)\alpha}\phi_q\Big(\int_0^1 a(\tau)d\tau\Big) = m. \end{aligned} \tag{13}$$

Hence, $T(\Omega)$ is bounded.

On the other hand, let $u \in \Omega,\ t_1,\ t_2 \in [0,1]$ with $t_1 < t_2$, then

$$|Tu(t_2)-Tu(t_1)| \le \phi_q(M)\int_0^1 |G(t_2,s)-G(t_1,s)| \phi_q\left(\int_0^s a(\tau)d\tau\right)ds.$$

The continuity of G implies that the right side of the above inequality tends to zero if $t_2 \to t_1$. Therefore, by applying the Arzela-Ascoli Theorem, we have T is completely continuous. ■

Let

$$\xi = \frac{1-\beta^{\alpha-1}}{2}\xi_1$$

$$\xi_1 = \frac{\frac{1-\beta^{\alpha-1}}{2}\left(\int_0^\beta G(s,s)\phi_q\left(\int_0^s a(\tau)d\tau\right)ds\right)\alpha(1-\gamma)\Gamma(\alpha)}{\phi_q\left(\int_0^1 a(\tau)d\tau\right)} + \frac{\frac{\gamma}{1-\gamma}\left(\int_0^\beta G(\eta,s)\phi_q\left(\int_0^s a(\tau)d\tau\right)ds\right)\alpha(1-\gamma)\Gamma(\alpha)}{\phi_q\left(\int_0^1 a(\tau)d\tau\right)}$$

$$K_\rho = \{u\in K:\ \|u\|<\rho\},$$

$$\Omega_\rho = \{u\in K:\ \min_{0\le t\le\beta} u(t) < \xi\rho\} = \{u:\ u\in C[0,1],\ u(t)\ge 0,\ \xi\|u\| \le \min_{0\le t\le\beta} u(t) < \xi\rho\}.$$

Lemma 2.7 Ω_ρ has the following properties:

(a) Ω_ρ *is open relative to* K.
(b) $K_{\xi\rho}\subset\Omega_\rho\subset K_\rho$.
(c) $u\in\partial\Omega_\rho$ *if and only if* $\min_{0\le t\le\beta} u(t)=\xi\rho$.
(d) *If* $u\in\partial\Omega_\rho$, *then* $\xi\rho\le u(t)\le\rho$ *for* $t\in[0,\beta]$.

Now for convenience we introduce the following notations. Let

$$f_{\xi\rho}^{\rho} = \min\left\{\frac{f(t,u)}{\phi_p(\rho)}:\ \ t\in[0,\beta],\ u\in[\xi\rho,\rho]\right\},$$
$$f_0^\rho = \max\left\{\frac{f(t,u)}{\phi_p(\rho)}:\ \ t\in[0,1],\ u\in[0,\rho]\right\},$$
$$f^\alpha = \lim_{u\to\alpha}\max\left\{\frac{f(t,u)}{\phi_p(u)}:\ \ t\in[0,1]\right\},$$
$$f_\alpha = \lim_{u\to\alpha}\min\left\{\frac{f(t,u)}{\phi_p(u)}:\ \ t\in[0,\beta]\right\}$$
$$(\alpha:=\infty,\ or\ 0^+),$$
$$n = \frac{\alpha\Gamma(\alpha)(1-\gamma)}{\phi_q\left(\int_0^1 a(\tau)d\tau\right)},$$
$$N = \left[\frac{1-\beta^{\alpha-1}}{2}\left(\int_0^\beta G(s,s)\phi_q\left(\int_0^s a(\tau)d\tau\right)ds + \frac{\gamma}{1-\gamma}\left(\int_0^\beta G(\eta,s)\phi_q\left(\int_0^s a(\tau)d\tau\right)ds\right)\right)\right]^{-1}.$$

Remark 2.2 It is easy to see that $0<n, N<+\infty$ and $N\xi = N\frac{1-\beta^{\alpha-1}}{2}\xi_1 = \frac{1-\beta^{\alpha-1}}{2}n < n$.

Lemma 2.8 If f satisfies the condition:

$$f_0^\rho < \phi_p(n), \tag{14}$$

then $i_k(T,K_\rho)=1$.

Proof: Based on the (10) and (14), we conclude for $u(t)\in\partial K_\rho$,

$$\begin{aligned}
\|Tu\| &= \int_0^1 G(s,s)\phi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\right)ds \\
&+ \frac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\right)ds \\
&\le \int_0^1 G(s,s)\phi_q\left(\int_0^1 a(\tau)f(\tau,u(\tau))d\tau\right)ds \\
&+ \frac{\gamma}{1-\gamma}\int_0^1 G(s,s)\phi_q\left(\int_0^1 a(\tau)f(\tau,u(\tau))d\tau\right)ds \\
&< \phi_q(\phi_p(\rho)\phi_p(n))\left(\int_0^1 G(s,s)\phi_q\left(\int_0^1 a(\tau)d\tau\right)ds\right. \\
&\left.+ \frac{\gamma}{1-\gamma}\int_0^1 G(s,s)\phi_q\left(\int_0^1 a(\tau)d\tau\right)ds\right) \\
&= n\rho\left(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \frac{\gamma}{1-\gamma}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right) \\
&\phi_q\left(\int_0^1 a(\tau)d\tau\right) \\
&= n\rho\frac{1}{\alpha\Gamma(\alpha)(1-\gamma)}\phi_q\left(\int_0^1 a(\tau)d\tau\right) = \rho = \|u\|.
\end{aligned}$$

Which yields $\|Tu\|<\|u\|$ for $u(t)\in\partial K_\rho$. By Theorem 2.3 (1), we have $i_k(T,K_\rho)=1$. ■

Lemma 2.9 If f satisfies the condition:

$$f_{\xi\rho}^\rho > \phi_p(N\xi), \tag{15}$$

then $i_k(T,\Omega_\rho)=0$.

Proof: Let $e(t)\equiv 1$ for $t\in[0,1]$. Then $e\in\partial K_1$, we claim that

$$u\ne Tu+\lambda e,\quad u\in\partial\Omega_\rho,\quad \lambda\ge 0.$$

In fact, if not, there exist $u_0\in\partial\Omega_\rho$ and $\lambda_0\ge 0$ such that $u_0 = Tu_0+\lambda_0 e$. Based on the lemma 2.5 and (15), we have that for $t\in[0,\beta]$,

$$\begin{aligned}
u_0(t) &= Tu_0(t)+\lambda_0 e(t) \ge \frac{1-\beta^{\alpha-1}}{2}\|Tu_0\|+\lambda_0 \\
&= \frac{1-\beta^{\alpha-1}}{2}\left[\int_0^1 G(s,s)\phi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\right)ds\right. \\
&\left.+ \frac{\gamma}{1-\gamma}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\right)ds\right]+\lambda_0 \\
&\ge \frac{1-\beta^{\alpha-1}}{2}\left[\int_0^\beta G(s,s)\phi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\right)ds\right. \\
&\left.+ \frac{\gamma}{1-\gamma}\int_0^\beta G(\eta,s)\phi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\right)ds\right]+\lambda_0 \\
&> \frac{1-\beta^{\alpha-1}}{2}\phi_q(\phi_p(\rho)\phi_p(N\xi)) \\
&\left[\int_0^\beta G(s,s)\phi_q\left(\int_0^s a(\tau)d\tau\right)ds\right. \\
&\left.+ \frac{\gamma}{1-\gamma}\int_0^\beta G(\eta,s)\phi_q\left(\int_0^s a(\tau)d\tau\right)ds\right]+\lambda_0 \\
&> \frac{1-\beta^{\alpha-1}}{2}N\xi\rho\left[\int_0^\beta G(s,s)\phi_q\left(\int_0^s a(\tau)d\tau\right)ds\right. \\
&\left.+ \frac{\gamma}{1-\gamma}\int_0^\beta G(\eta,s)\phi_q\left(\int_0^s a(\tau)d\tau\right)ds\right]+\lambda_0 \\
&= \xi\rho+\lambda_0.
\end{aligned}$$

Which yields $\xi\rho > \xi\rho+\lambda_0$. which is a contradiction. Hence, by Theorem II (2), it follows that $i_k(T,\Omega_\rho)=0$. ■

III. Main Result

We now give our results on the existence of multiple positive solutions of BVP (1) and (2).

Theorem 3.1 Assume the following condition (H_1) holds:
(H_1) There exist $\rho_1,\ \rho_2,\ \rho_3 \in (0,\infty)$, with $\rho_1 < \xi\rho_2 < \rho_2 < \rho_3$ such that

$$f_0^{\rho_1} < \phi_p(n), \quad f_{\xi\rho_2}^{\rho_2} > \phi_p(N\xi), \quad f_0^{\rho_3} \le \phi_p(n).$$

Then system (1) and (2) has three positive solutions in K.
Assume the following condition (H_2) holds:
(H_2) There exist $\rho_1,\ \rho_2,\ \rho_3 \in (0,\infty)$, with $\rho_1 < \rho_2 < \xi\rho_3$ such that

$$f_{\xi\rho_1}^{\rho_1} > \phi_p(N\xi), \quad f_0^{\rho_2} < \phi_p(n), \quad f_{\xi\rho_3}^{\rho_3} \ge \phi_p(N\xi).$$

Then problem (1) and (2) has two positive solutions in K.

Proof: From lemma 2.8 and 2.9, we know $i_k(T, K_{\rho_1}) = 1,\ i_k(T, \Omega_{\rho_2}) = 0,\ i_k(T, K_{\rho_3}) = 1$. For $\rho_1 < \xi\rho_2$, lemma 2.7 (b) implies $K_{\rho_1} \subset K_{\xi\rho_2} \subset \Omega_{\rho_2}$. Theorem 2.3 implies T has two fixed points $u_1 \in K_{\rho_1}$ and $u_2 \in \Omega_{\rho_2}\backslash\overline{K_{\rho_1}}$. Similarly, we can prove that T has the third fixed point $u_3 \in K_{\rho_3}\backslash\overline{\Omega_{\rho_2}}$. The proof of (H_2) is similar to that in (H_1), we omit it here. ■

Theorem 3.1 can be generalized to obtain many positive solutions, we also omit it here.

As a special case of Theorem 3.1, we obtain the following result.

Corollary 3.2 Assume there exists $\rho \in (0,\infty)$, such that the following condition (H_3) holds:

$$(H_3) \qquad 0 \le f^0 < \phi_p(n),\ \ f_{\xi\rho}^{\rho} > \phi_p(N\xi),\ \ 0 \le f^\infty < \phi_p(n).$$

Then system (1) and (2) has three positive solutions in K.
Assume if there exists $\rho \in (0,\infty)$, such that the following condition (H_4) holds:

$$(H_4) \quad \phi_p(N) < f_0 \le \infty,\ \ f_0^{\rho} < \phi_p(n),\ \phi_p(N) < f_\infty \le \infty.$$

Then problem (1) and (2) has two positive solutions in K.

Proof: We show that (H_3) implies (H_1). It is easy to verify that $0 \le f^0 < \phi_p(n)$ implies that there exists $\rho_1 \in (0,\xi\rho)$ such that $f_0^{\rho_1} < \phi_p(n)$. Let $k \in (f^\infty, \phi_p(n))$. Then there exists $\delta > \rho$ such that $\max_{t\in[0,1]} f(t,u) \le k\phi_p(u)$ for $u \in [\delta,\infty)$ since $0 \le f^\infty < \phi_p(n)$. Let

$$\begin{aligned} \Phi \quad &= \max\left\{ \max_{t\in[0,1]} f(t,u) : \quad 0 \le u \le \delta \right\} \\ \text{and} \quad &\rho_3 > \phi_q\left(\frac{\Phi}{\phi_p(n)-k}\right). \end{aligned}$$

Then we have

$$\begin{aligned} \max_{t\in[0,1]} f(t,u) \quad &\le k\phi_p(u) + \Phi \le k\phi_p(\rho_3) + \Phi \\ &< \phi_p(n)\phi_p(\rho_3) \quad \text{for} \quad u \in [0,\rho_3]. \end{aligned}$$

This implies that $f_0^{\rho_3} \le \phi_p(n)$ and (H_1) holds. Similarly (H_4) implies (H_2). ■

By an argument similar to that of Theorem 3.1 we obtain the following results.

Theorem 3.3 Assume one of the following conditions holds:
(H_5) There exist $\rho_1,\ \rho_2 \in (0,\infty)$ with $\rho_1 < \xi\rho_2$ such that $f_0^{\rho_1} \le \phi_p(n), \quad f_{\xi\rho_2}^{\rho_2} \ge \phi_p(N\xi)$,
(H_6) There exist $\rho_1,\ \rho_2 \in (0,\infty)$ with $\rho_1 < \rho_2$ such that $f_{\xi\rho_1}^{\rho_1} \ge \phi_p(N\xi), \quad f_0^{\rho_2} \le \phi_p(n)$.
Then problem (1) and (2) has a positive solution in K.

Corollary 3.4 Assume one of the following conditions holds:
(H_7) $0 \le f^0 < \phi_p(n), \quad \phi_p(N) < f_\infty \le \infty$,
(H_8) $0 \le f^\infty < \phi_p(n), \quad \phi_p(N) < f_0 \le \infty$.
Then problem (1) and (2) has a positive solution in K.

REFERENCES

[1] W.G. Glockle, T.F. Nonnenmacher, "A fractional calculus approach of self-similar protein dynamics," *Biophys J.*, vol. 68, pp. 46-53, 1995.

[2] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[3] F. Metzler, W. Schick, H.G. Kilian, T.F. Nonnenmacher, , "Relaxation in filled polymers: a fractional calculus approach ," *J Chem Phys.*, vol. 103, pp. 7180-7186, 1995.

[4] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[5] I. Podlubny, , "Geometric and physical interpretation of fractional integration and fractional differentiation," *Fract Calc Appl Anal.*, vol. 5, pp. 367-386, 2002.

[6] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, In North-Holland Mathematics Studies, Amsterdam, 2006.

[7] V. Lakshmikantham, S. Leela, J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.

[8] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivaes Theory and Applications*, Gordon and Breach Science Publisher, Yverdon, 1993.

[9] R.P. Agarwal, M. Benchohra, S. Hamani, "A survey on existence result for boundary value problems of nonlinear fractional differential equations and inclusion," *Acta Appl Math.*, vol. 109, pp. 973-1033, 2010.

[10] D.H. Ji, W.G. Ge, "Existence of multiple positive solutions for Sturm-Liouville-like four-point boundary value problem with p-Laplacian," *Nonl. Anal.*, vol. 68, pp. 2638-2646, 2008.

[11] D.H. Ji, Y.T. Yang, W.G. Ge, "Triple positive pseudo-symmetric solutions to a four-point boundary value problem with p-Laplacian," *Appl. Math. Lett.*, vol. 21, pp. 268-274, 2008.

[12] D.H. Ji, W.G. Ge, "Positive solution for boundary value problems with p-Laplacian in Banach spaces," *Bound Value Probl.*, vol. 51, pp. 1-6, 2012.

[13] Y.Y. Wang, C.M. Hou, "Existence of multiple positive solutions for one- dimensional p-Laplacian," *J. Math. Anal. Appl.*, vol. 315, pp. 144-153, 2006.

[14] B. Ahmad, S. Sivasundaram, "On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order," *Appl. Math. Comput.*, vol. 217, pp. 480-487, 2010.

[15] T. Chen, W. Liu, Z.G. Hu, "A boundary value problem for fractional differential equation with p-Laplacian operator at resonance," *Nonlinear Anal.*, vol. 75, pp. 3210-3217, 2012.

[16] Z.H. Liu, L. Lu, "A class of BVPs for nonlinear fractional differential equations with p-Laplacian operator, Electron," *J. Qual. Theo.*, vol. 70, pp. 1-16, 2012.

[17] J.H. Wang, H.J. Xiang, "Upper and Lower Solutions Method for a Class of Singular Fractional Boundary Value Problems with p-Laplacian Operator," *Abstr. Appl. Anal.*, vol. 2010, pp. 1-12, 2010.

[18] G.Q. Chai, "Positive solutions for boundary value problem of fractional differential equation with p-Laplacian operator," *Bound Value Probl.*, vol. 2012, pp. 1-20, 2012.

[19] L.L. Wang, "Dynamic Output Feedback Guaranteed Cost Control for Linear Nominal Impulsive Systems," *Engineering Letters*, vol. 26, no. 4, pp. 405-409, 2018.

[20] G. Zhou and M. Zeng, "Existence and multiplicity of solutions for p-Laplacian Equations without the AR condition," *IAENG International Journal of Applied Mathematics*, vol.47, no.2, pp.233-237, 2017.