Dynamic Behaviors of May Cooperative Model with Density Dependent Birth Rate

Zhenliang Zhu, Fengde Chen, Liyun Lai and Zhong Li

***Abstract*—By incorporating the density dependent birth rate to the traditional May type cooperative system, we propose the following cooperative system**

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{b_{11}}{b_{12}+b_{13}x}-b_{14}-a_{11}x-\frac{a_{12}x}{y+k_1}\Big),\\ \frac{dy}{dt} &= y\Big(\frac{b_{21}}{b_{22}+b_{23}y}-b_{24}-a_{22}y-\frac{a_{21}y}{x+k_2}\Big),\end{aligned}$$

where $b_{ij}, i=1,2, j=1,2,3,4, a_{11}, a_{12}$, a_{21}, a_{22}, k_1 and k_2 are all positive constants. By applying the comparison principle, sufficient conditions which ensure the extinction, partial survival and permanence of the system are obtained, respectively. By applying the Dulac criterion, a set of sufficient conditions which ensure the existence of a unique globally asymptotically stable equilibrium is obtained. Numeric simulations are carried out to show the feasibility of the main results.

***Index Terms*—Global attractivity;Cooperative model; Density dependent birth rate.**

I. Introduction

THE aim of this paper is to investigate the dynamic behaviors of the following May cooperative model incorporating density dependent birth rate

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{b_{11}}{b_{12}+b_{13}x}-b_{14}-a_{11}x-\frac{a_{12}x}{y+k_1}\Big),\\ \frac{dy}{dt} &= y\Big(\frac{b_{21}}{b_{22}+b_{23}y}-b_{24}-a_{22}y-\frac{a_{21}y}{x+k_2}\Big),\end{aligned}\tag{1.1}$$

where $b_{ij}, i=1,2, j=1,2,3,4, a_{11}, a_{12}, a_{21}$, a_{22}, k_1 and k_2 are all positive constants. $x(t), y(t)$ are the density of the first and second species at time t, respectively. Here we make the following assumptions:

(a) $\dfrac{b_{11}}{b_{12}+b_{13}x}$ is the birth rate of the first species, which is density dependent, b_{14} is the death rate of the first species;

(c) $\dfrac{b_{21}}{b_{22}+b_{23}y}$ is the birth rate of the second species, b_{24} is the death rate of the second species;

(e) The relationship between two species is cooperative, i. e., both species has positive effect to the other species, and this effect is described by increasing the environment carrying capacity of the corresponding species.

During the last decade, many scholars investigated the dynamic behaviors of the mutualism model([1]-[36]). Some scholars ([1], [2], [3], [4], [10], [11], [12], [13], [14], [16], [17], [18], [20], [22], [26], [32], [33]) investigated the stability property of the positive equilibrium, some scholars ([5], [6], [7], [8], [9], [15]) focused their attention to the the persistent property of the system, some scholars ([20], [21], [23], [24]) investigated the positive periodic solution of the system. Other topics such as the extinction of the species ([5], [25], [28]), the influence of harvesting([11], [13], [14], [26], [29]), the influence of feedback control variables([1], [5], [6], [8], [9], [10], [22]) and the influence of stage structure([2], [4]) were also extensively investigated. Among those works, May cooperative system, as one of the basic cooperative system, was studied by many scholars ([2], [7], [11], [12], [14], [15], [18]).

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May [19] suggested the following set of equations to describe a pair of mutualist:

$$\begin{aligned}\frac{dx_1}{dt} &= rx_1\Big[1-\frac{x_1}{K_1+\alpha x_2}\Big],\\ \frac{dx_2}{dt} &= rx_2\Big[1-\frac{x_2}{K_2+\beta x_1}\Big],\end{aligned}\tag{1.2}$$

where x_1, x_2 are the densities of the species, respectively. r, K_i, α, β, $i=1,2$ are positive constants. The system will "run away", in the sense that both populations growing unboundedly large if $\alpha\beta \geq 1$. To overcome this drawback, May further considered the density restriction of the species and proposed the following system:

$$\begin{aligned}\dot{x} &= r_1x\Big[1-\frac{x}{K_1+\alpha_1 y}-\varepsilon_1 x\Big],\\ \dot{y} &= r_2y\Big[1-\frac{y}{K_2+\alpha_2 x}-\varepsilon_2 y\Big],\end{aligned}\tag{1.3}$$

where r_i, K_i, α_i, ε_i, $i=1,2$ are positive constants. May showed that system (1.3) has a unique positive equilibrium which is globally stable.

Xie, Chen and Xue[12] further studied the influence of harvesting to the above system, indeed, they studied the dynamic behaviors of the following system

$$\begin{aligned}\dot{x} &= x\Big(r_1-b_1x-\frac{a_1x}{y+k_1}\Big)-Eqx,\\ \dot{y} &= y\Big(r_2-b_2y-\frac{a_2y}{x+k_2}\Big).\end{aligned}\tag{1.4}$$

They showed that $r_1 > Eq$ is enough to ensure the system (1.4) admits a unique globally attractive positive equilibrium. Recently, Chen, Wu and Xie[15] argued that the discrete time models governed by difference equations are more appropriate than the continuous ones. Corresponding to system (1.4), they further proposed a discrete cooperative model incorporating harvesting, they also investigated the stability property of the interior equilibrium. Lei[14] studied the dynamic behaviors of the following non-selective harvesting May cooperative system incorporating partial closure for the populations

$$\begin{aligned}\dot{x} &= x\Big(r_1 - b_1 x - \frac{a_1 x}{y + k_1}\Big) - Eq_1 mx,\\ \dot{y} &= y\Big(r_2 - b_2 y - \frac{a_2 y}{x + k_2}\Big) - Eq_2 my.\end{aligned} \tag{1.5}$$

Han and Chen[11] incorporated the feedback control variables to May cooperative system, this leads to

$$\begin{aligned}\frac{dx_1}{dt} &= r_1 x_1\Big[1 - b_1 x_1 - \frac{x_1}{\alpha x_2 + k_1} - c_1 u_1\Big],\\ \frac{dx_2}{dt} &= r_2 x_2\Big[1 - b_2 x_2 - \frac{x_2}{\beta x_1 + k_2} - c_2 u_2\Big],\\ \frac{du_1}{dt} &= -\eta_1 u_1 + e_1 x_1,\\ \frac{du_2}{dt} &= -\eta_2 u_2 + e_2 x_2.\end{aligned} \tag{1.6}$$

For the case $b_i \neq 0,\ i = 1, 2$, they showed that the feedback control variables have no influence on the global stability of the unique positive equilibrium of the system. For the case $b_i \equiv 0$, they showed that the system admits a unique globally attractive positive equilibrium, which means that the unbounded system becomes bounded, and feedback control variables have the stable effect to the system.

Chen, Xie and Chen[2] proposed the following stage structured May cooperative system

$$\begin{aligned}\dot{x}_1(t) &= b_1 e^{-d_{11}\tau_1} x_1(t-\tau_1) - d_{12} x_1(t)\\ &\quad - \frac{a_{11} x_1^2(t)}{c_1 + f_1 x_2(t)} - a_{12} x_1^2(t),\\ \dot{y}_1(t) &= b_1 x_1(t) - d_{11} y_1(t)\\ &\quad - b_1 e^{-d_{11}\tau_1} x_1(t-\tau_1),\\ \dot{x}_2(t) &= b_2 e^{-d_{22}\tau_2} x_2(t-\tau_2) - d_{21} x_2(t)\\ &\quad - \frac{a_{22} x_2^2(t)}{c_2 + f_2 x_1(t)} - a_{21} x_2^2(t),\\ \dot{y}_2(t) &= b_2 x_2(t) - d_{22} y_2(t)\\ &\quad - b_2 e^{-d_{22}\tau_2} x_2(t-\tau_2).\end{aligned} \tag{1.7}$$

Their study showed that the stage structure and the death rate of mature species are two of the most important reasons that cause global attractivity and extinction of the species, however cooperate has no influence on the persistent property of the model.

Zhao, Qin and Chen [7] proposed a May cooperative system with strong and weak cooperative partners as follows:

$$\begin{aligned}&\frac{dH_1}{dt} = r_1 H_1\Big(1 - \frac{H_1}{a_1 + b_1 P} - c_1 H_1 - \frac{\alpha H_2}{r_1}\Big),\\ &\frac{dH_2}{dt} = H_2\big(\alpha H_1 + d - e H_2\big),\\ &\frac{dP}{dt} = r_2 P\Big(1 - \frac{P}{a_2 + b_2 H_1} - c_2 P\Big),\end{aligned} \tag{1.8}$$

where $r_i, a_i, b_i, c_i, d, alpha, i = 1, 2$ are positive constants. The authors investigated the permanence and non-permanence of the system, the existence and global stability of the positive equilibrium point and boundary equilibrium point, respectively.

It brings to our attention that in system (1.2)-(1.6), if we do not consider the relationship of the two species, then the equations for both species reduces to the traditional Logistic equation. For example, the first species in (1.3) takes the form

$$\dot{x} = r_1 x\Big[1 - \frac{x}{K_1} - \varepsilon_1 x\Big], \tag{1.9}$$

where r_1 is the intrinsic growth rate, which is equal to the birth rate minus death rate of the species x. Recently, several scholars ([28]-[30]) argued that the density dependent birth rate of the species is more suitable. For example, Chen, Xue, Lin and Xie[28] proposed the following commensalism model with density dependent birth rate:

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{b_{11}}{b_{12} + b_{13} x} - b_{14} - a_{11} x + a_{12} y\Big),\\ \frac{dy}{dt} &= y\Big(\frac{b_{21}}{b_{22} + b_{23} y} - b_{24} - a_{22} y\Big).\end{aligned} \tag{1.10}$$

Their study showed that the system may admit four nonnegative equilibria, and under some suitable assumptions, all of the four equilibria maybe globally asymptotically stable, such a property is quite different to the traditional Lotka-Volterra commensalism model. Noting that all of the three papers ([28]-[30]) are concerned with the commensalism model, to the best of our knowledge, to this day, still no scholars propose and study the cooperative system with density dependent birth rate. This motivated us to propose and study the dynamic behaviors of the system (1.1). We must point out that there are some essential difference between system (1.7) and (1.1). In system (1.7), since the second and fourth equations are independent of the other species, its dynamic behaviors is easily to investigate. One could refer to [29],[30] for more detail analysis on this aspect. However, in system (1.1), since both equations are dependent on x and y, it's not an easy thing to deal with the stability property of the positive equilibrium. We also point out that Chen, Xie and Chen[2], Xie Chen and Xue[12], Xie, Chen, Yang and Xue[13] had used the iterative method to investigate the stability property of the mutualism model, however, this method could not be applied to system (1.1), since here we could not express the upper bound or lower bound of the solution clearly, and could not use the method of [2], [12], [13] directly.

The aim of this paper is to investigate the dynamic behaviors of the system (1.1) and to find out the influence of the nonlinear density birth rate. The rest of the paper is arranged as follows. In section 2, we investigate the extinction or partial survival of the system; We then investigate the persistent property in section 3; In section 4, by applying the Dulac criterion, we obtain a set of sufficient conditions which ensure the global asymptotic stability of the positive equilibrium. Section 5 presents some numerical simulations to show the feasibility of the main results. We end this paper by a briefly discussion.

II. Extinction of system (1.1)

We first introduce a lemma, which is useful in the prove of the main result.

Consider the following equation

$$\frac{dy}{dt} = y\Big(\frac{a_{21}}{a_{22} + a_{23} y} - a_{24} - b_2 y\Big) \tag{2.1}$$

with $y(0) = y_0 > 0 \in [0, +\infty)$.

Lemma 2.1 [29] *The following statements on (2.1) hold.*
(1) Assume that $a_{21} > a_{22}a_{24}$, then the unique positive equilibrium y^* of system (2.1) is globally attractive in $(0, +\infty)$;

(2)Assume that $a_{21} \leq a_{22}a_{24}$, then the equilibrium $y = 0$ of system (2.1) is globally attractive in $(0, +\infty)$.

Consider the following equation

$$\frac{du}{dt} = u\Big(\frac{b_{11}}{b_{12} + b_{13}u} - b_{14} - a_{11}u - \frac{a_{12}u}{\varepsilon + k_1}\Big) \qquad (2.2)$$

with $u(0) = u_0 > 0 \in [0, +\infty)$.

Lemma 2.2. *Assume that $b_{11} > b_{12}b_{14}$ hold, then the unique positive equilibrium $u^*(\varepsilon)$ of system (2.2) is globally attractive in $(0, +\infty)$, where*

$$u^*(\varepsilon) = \frac{-A_2(\varepsilon) + \sqrt{(A_2(\varepsilon))^2 - 4A_1(\varepsilon)A_3(\varepsilon)}}{2A_1(\varepsilon)}, \qquad (2.3)$$

here

$$\begin{aligned} A_1(\varepsilon) &= a_{11}b_{13}k_1 + a_{12}b_{13} + a_{11}b_{13}\varepsilon, \\ A_2(\varepsilon) &= a_{11}b_{12}k_1 + a_{11}b_{12}\varepsilon + b_{13}b_{14}k_1 \\ &\quad + b_{13}b_{14}\varepsilon + a_{12}b_{12}, \\ A_3(\varepsilon) &= b_{12}b_{14}k_1 - b_{11}k_1 + b_{12}b_{24}\varepsilon - b_{11}\varepsilon. \end{aligned} \qquad (2.4)$$

Obviously, $u^*(\varepsilon)$ is the continuous function of ε and

$$u^*(\varepsilon) \to x^* \ \text{as}\ \varepsilon \to 0, \qquad (2.5)$$

where x^* is defined by (2.12).

Consider the following equation

$$\frac{dv}{dt} = v\Big(\frac{b_{21}}{b_{22} + b_{23}v} - b_{24} - a_{22}v - \frac{a_{21}v}{\varepsilon + k_2}\Big), \qquad (2.6)$$

with $v(0) = v_0 > 0 \in [0, +\infty)$.

Lemma 2.3. *Assume that $b_{21} > b_{22}b_{24}$ hold, then the unique positive equilibrium $v^*(\varepsilon)$ of system (2.6) is globally attractive in $(0, +\infty)$, where*

$$v^*(\varepsilon) = \frac{-B_2(\varepsilon) + \sqrt{(B_2(\varepsilon))^2 - 4B_1(\varepsilon)B_3(\varepsilon)}}{2B_1(\varepsilon)}, \qquad (2.7)$$

here

$$\begin{aligned} B_1(\varepsilon) &= a_{22}b_{23}k_2 + a_{21}b_{23} + a_{22}b_{23}\varepsilon, \\ B_2(\varepsilon) &= a_{22}b_{22}k_2 + b_{23}b_{24}k_2 + a_{21}b_{22} \\ &\quad + a_{22}b_{22}\varepsilon + b_{23}b_{24}\varepsilon, \\ B_3(\varepsilon) &= b_{22}b_{24}k_2 - b_{21}k_2 + b_{22}b_{24}\varepsilon - b_{21}\varepsilon. \end{aligned} \qquad (2.8)$$

Also, $v^*(\varepsilon)$ is the continuous function of ε and

$$v^*(\varepsilon) \to y^* \ \text{as}\ \varepsilon \to 0, \qquad (2.9)$$

where y^* is defined by (2.17).

The proof of Lemma 2.2 and 2.3 is similarly to that of the proof of Lemma 2.1, one could refer to [29] for more detail, here we omit it.

Now we are in the position of considering the existence and stability property of the boundary equilibria of system (1.1). The equilibrium of system (1.1) is determined by the equation

$$\begin{aligned} x\Big(\frac{b_{11}}{b_{12} + b_{13}x} - b_{14} - a_{11}x - \frac{a_{12}x}{y + k_1}\Big) &= 0, \\ y\Big(\frac{b_{21}}{b_{22} + b_{23}y} - b_{24} - a_{22}y - \frac{a_{21}y}{x + k_2}\Big) &= 0. \end{aligned} \qquad (2.10)$$

The system (1.1) always admits a boundary equilibrium $A_1(0, 0)$. Assume that

$$\frac{b_{11}}{b_{12}} > b_{14} \qquad (2.11)$$

holds, then

$$\frac{b_{11}}{b_{12} + b_{13}x} - b_{14} - a_{11}x - \frac{a_{12}x}{k_1} = 0 \qquad (2.12)$$

admits a unique positive solution x^*, where

$$x^* = \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1}, \qquad (2.13)$$

here

$$\begin{aligned} A_1 &= a_{11}b_{13}k_1 + a_{12}b_{13}, \\ A_2 &= a_{11}b_{12}k_1 + b_{12}b_{13}k_1 + a_{12}b_{12}, \\ A_3 &= b_{12}b_{14}k_1 - b_{11}k_1. \end{aligned} \qquad (2.14)$$

Assume that (2.11) and

$$\frac{b_{21}}{b_{22}} < b_{24} \qquad (2.15)$$

hold, then system (1.1) admits the nonnegative boundary equilibrium $A_2(x^*, 0)$. Assume that

$$\frac{b_{21}}{b_{22}} > b_{24} \qquad (2.16)$$

hold, then

$$\frac{b_{21}}{b_{22} + b_{23}y} - b_{24} - a_{22}y - \frac{a_{21}y}{k_2} = 0$$

admits a unique positive solution y^*, where

$$y^* = \frac{-B_2 + \sqrt{B_2^2 - 4B_1B_3}}{2B_1}, \qquad (2.17)$$

here

$$\begin{aligned} B_1 &= a_{22}b_{23}k_2 + a_{21}b_{23}, \\ B_2 &= a_{22}b_{22}k_2 + b_{23}b_{24}k_2 + a_{21}b_{22}, \\ B_3 &= b_{22}b_{24}k_2 - b_{21}k_2. \end{aligned} \qquad (2.18)$$

Assume that (2.16) and

$$\frac{b_{11}}{b_{12}} < b_{14} \qquad (2.19)$$

hold, then system (1.1) admits the nonnegative boundary equilibrium $A_3(0, y^*)$.

Concerned with the extinction of the species in system (1.1), we have the following result.

Theorem 2.1.

(1) Assume that (2.15) and (2.19) hold, then

$$\lim_{t\to+\infty} x(t) = 0, \ \lim_{t\to+\infty} y(t) = 0, \qquad (2.20)$$

i.e., the boundary equilibrium $A_1(0,0)$ is globally asymptotically stable;
(2) Assume that (2.11) and (2.15) hold, then

$$\lim_{t\to+\infty} x(t) = x^*, \ \lim_{t\to+\infty} y(t) = 0, \qquad (2.21)$$

i.e., the boundary equilibrium $A_2(x^*,0)$ is globally asymptotically stable;
(3) Assume that (2.16) and (2.19) hold, then

$$\lim_{t\to+\infty} x(t) = 0, \ \lim_{t\to+\infty} y(t) = y^*, \qquad (2.22)$$

i.e., the boundary equilibrium $A_3(0,y^*)$ is globally asymptotically stable.

Proof.
(1) It follows from the first equation of system (1.1) that

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x - \frac{a_{12}x}{y+k_1}\Big)\\ &\le x\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x\Big).\end{aligned} \qquad (2.23)$$

Now let us consider the equation

$$\frac{du}{dt} = u\Big(\frac{b_{11}}{b_{12}+b_{13}u} - b_{14} - a_{11}u\Big). \qquad (2.24)$$

It follows from (2.19) and Lemma 2.1 that $u=0$ of system (2.24) is globally attractive, i.e.,

$$\lim_{t\to+\infty} u(t) = 0. \qquad (2.25)$$

Applying the caparison principle of differential equation to (2.23) and (2.24) leads to

$$\lim_{t\to+\infty} x(t) = 0. \qquad (2.26)$$

Similarly, by using (2.15) and the comparison principle, from the second equation we could draw the conclusion

$$\lim_{t\to+\infty} y(t) = 0. \qquad (2.27)$$

(2) From (2.15) and the second equation of (1.1), by using the comparison principle, we have

$$\lim_{t\to+\infty} y(t) = 0. \qquad (2.28)$$

For $\varepsilon > 0$ enough small, it follows from (2.28) that there exists a $T_1 > 0$ such that

$$y(t) < \varepsilon \ \text{ for all } \ t \ge T_1. \qquad (2.29)$$

From (2.29) and the first equation of system (1.1), for $t \ge T_1$, we have

$$\frac{dx}{dt} \le x\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x - \frac{a_{12}x}{\varepsilon+k_1}\Big). \qquad (2.30)$$

Now let us consider the equation

$$\frac{du}{dt} = u\Big(\frac{b_{11}}{b_{12}+b_{13}u} - b_{14} - a_{11}u - \frac{a_{12}u}{\varepsilon+k_1}\Big). \qquad (2.31)$$

It follows from Lemma 2.2 that system (2.31) admits a unique positive equilibrium $u^*(\varepsilon)$, which is globally asymptotically stable. Hence, from (2.30), (2.31) and the comparison principle, one has

$$\limsup_{t\to+\infty} x(t) \le u^*(\varepsilon). \qquad (2.32)$$

From the continuity theorem of solution on parameters, one could see that

$$u^*(\varepsilon) \to x^* \ \text{ as } \ \varepsilon \to 0.$$

On the other hand, from the first equation of system (1.1), we also have

$$\frac{dx}{dt} \ge x\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x - \frac{a_{12}x}{k_1}\Big). \qquad (2.33)$$

Now let us consider the equation

$$\frac{du}{dt} = u\Big(\frac{b_{11}}{b_{12}+b_{13}u} - b_{14} - a_{11}u - \frac{a_{12}u}{k_1}\Big). \qquad (2.34)$$

It follows from Lemma 2.2 that system (2.34) admits a unique positive equilibrium x^*, which is globally asymptotically stable. Hence, from (2.33), (2.34) and the comparison principle, one has

$$\liminf_{t\to+\infty} x(t) \ge x^*. \qquad (2.35)$$

(2.32) together with (2.35) leads to

$$x^* \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le u^*(\varepsilon). \qquad (2.36)$$

Setting $\varepsilon \to 0$ in (2.36) leads to

$$\lim_{t\to+\infty} x(t) = x^*. \qquad (2.37)$$

(3) From (2.19) and the first equation of (1.1), by using the comparison principle, similarly to the analysis of (2.23)-(2.26), we have

$$\lim_{t\to+\infty} x(t) = 0. \qquad (2.38)$$

For $\varepsilon > 0$ enough small, it follows from (2.38) that there exists a $T_2 > 0$ such that

$$x(t) < \varepsilon \ \text{ for all } \ t \ge T_2. \qquad (2.39)$$

From (2.39) and the first equation of system (1.1), for $t \ge T_2$, we have

$$\frac{dy}{dt} \le y\Big(\frac{b_{21}}{b_{22}+b_{23}y} - b_{24} - a_{22}y - \frac{a_{21}y}{\varepsilon+k_2}\Big), \qquad (2.40)$$

Now let us consider the equation

$$\frac{dv}{dt} = v\Big(\frac{b_{21}}{b_{22}+b_{23}v} - b_{24} - a_{22}v - \frac{a_{21}v}{\varepsilon+k_2}\Big). \qquad (2.41)$$

It follows from Lemma 2.2 that system (2.41) admits a unique positive equilibrium $v^*(\varepsilon)$, which is globally asymptotically stable. Hence, from (2.40), (2.41) and the comparison principle, one has

$$\limsup_{t\to+\infty} y(t) \le v^*(\varepsilon). \qquad (2.42)$$

From the continuity theorem of solution on parameters, one could see that

$$v^*(\varepsilon) \to y^* \ \text{ as } \ \varepsilon \to 0.$$

On the other hand, from the second equation of system (1.1), we also have

$$\frac{dy}{dt} \ge y\Big(\frac{b_{21}}{b_{22}+b_{23}y} - b_{24} - a_{22}y - \frac{a_{21}y}{k_2}\Big). \qquad (2.43)$$

Now let us consider the equation

$$\frac{dv}{dt} = v\Big(\frac{b_{21}}{b_{22}+b_{23}v} - b_{24} - a_{22}v - \frac{a_{21}v}{k_2}\Big). \qquad (2.44)$$

It follows from Lemma 2.2 that system (2.44) admits a unique positive equilibrium y^*, which is globally asymptotically stable. Hence, from (2.43), (2.44) and the comparison principle, one has

$$\liminf_{t\to+\infty} y(t) \geq y^*. \tag{2.45}$$

(2.42) together with (2.45) leads to

$$y^* \leq \liminf_{t\to+\infty} y(t) \leq \limsup_{t\to+\infty} y(t) \leq v^*(\varepsilon). \tag{2.46}$$

Setting $\varepsilon \to 0$ in (2.46) leads to

$$\lim_{t\to+\infty} y(t) = y^*. \tag{2.47}$$

This ends the proof of Theorem 2.1.

Remark 2.1. As was pointed out in the introduction section, for the traditional May cooperative system (1.3), the system admits a unique positive equilibrium which is globally asymptotically stable. That is, for system (1.3), two species could be coexist in a stable state. None of the species will be driven to extinction. Our result (Theorem 2.1) shows that by introducing the density dependent birth rate, the system may become extinct in the sense that both species x and y will be driven to extinction or the system maybe partial survival, in the sense that one of the species will be driven to extinction while the other species will be survival in the long run. Therefore, the dynamic behaviors of the system become complicated.

III. Permanence

Previously, we had discussed the extinction property of the system (1.1). This section we will discuss the persistent property of the system.

Definition 3.1. *Let $(x(t), y(t))$ be any positive solution of system (1.1), if there exist positive constants $m_i, M_i, i = 1, 2$ such that*

$$m_1 \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq M_1,$$
$$m_2 \leq \liminf_{t\to+\infty} y(t) \leq \limsup_{t\to+\infty} y(t) \leq M_2.$$

Then system (1.1) is permanent.

By means of permanence, two species could be coexist in the long run.

Theorem 3.1 *Assume that (2.11) and (2.16) hold, then system (1.1) is permanent.*

Proof. It follows from the first equation of system (1.1) that

$$\begin{aligned} \frac{dx}{dt} &= x\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x - \frac{a_{12}x}{y+k_1}\Big) \\ &\leq x\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x\Big). \end{aligned} \tag{3.1}$$

Now let us consider the equation

$$\frac{du}{dt} = u\Big(\frac{b_{11}}{b_{12}+b_{13}u} - b_{14} - a_{11}u\Big) \tag{3.2}$$

It follows from (2.11) and Lemma 2.1 that $u = u^*$ of system (3.2) is globally attractive, i.e.,

$$\lim_{t\to+\infty} u(t) = u^*. \tag{3.3}$$

Applying the caparison principle of differential equation to (3.1) and (3.2) leads to

$$\limsup_{t\to+\infty} x(t) \leq u^*. \tag{3.4}$$

Similarly, by using (2.16) and the comparison principle, from the second equation we could draw the conclusion

$$\limsup_{t\to+\infty} y(t) \leq v^*, \tag{3.5}$$

where v^* is the positive solution of the equation

$$\frac{b_{21}}{b_{22}+b_{23}v} - b_{24} - a_{22}v = 0. \tag{3.6}$$

On the other hand, from the first equation of system (1.1), we also have

$$\frac{dx}{dt} \geq x\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x - \frac{a_{12}x}{k_1}\Big). \tag{3.7}$$

Similarly to the analysis of (2.33)-(2.35), we have

$$\liminf_{t\to+\infty} x(t) \geq x^*. \tag{3.8}$$

From the second equation of system (1.1), we also have

$$\frac{dy}{dt} \geq y\Big(\frac{b_{21}}{b_{22}+b_{23}y} - b_{24} - a_{22}y - \frac{a_{21}y}{k_2}\Big). \tag{3.9}$$

similarly to the analysis of (2.43)-(2.35), we have

$$\liminf_{t\to+\infty} y(t) \geq y^*. \tag{3.10}$$

Noting that x^*, y^*, u^*, v^* are all can be expressed by the coefficients of the system (1.1), and independent of the solutions of system (1.1). Hence, (3.4), (3.5), (3.8) and (3.10) show that under the assumption of Theorem 3.1, system (1.1) is permanent.

IV. Global stability of the positive equilibrium

Concerned with the existence of the positive equilibrium of the system (1.1), we have the following result.

Lemma 4.1. *Assume that (2.11) and (2.16) holds, then system (1.1) admits the unique positive equilibrium $A_4(x_1, y_1)$.*

Proof. The positive equilibrium is the solution of the equations

$$\begin{aligned} \frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x - \frac{a_{12}x}{y+k_1} &= 0, \\ \frac{b_{21}}{b_{22}+b_{23}y} - b_{24} - a_{22}y - \frac{a_{21}y}{x+k_2} &= 0. \end{aligned}$$

Set

$$F_1(x,y) = \frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x - \frac{a_{12}x}{y+k_1}, \tag{4.1}$$

$$F_2(x,y) = \frac{b_{21}}{b_{22}+b_{23}y} - b_{24} - a_{22}y - \frac{a_{21}y}{x+k_2}. \tag{4.2}$$

Obviously, $F_1(x,y) = 0$ and $F_2(x,y) = 0$ define two lines, we denote them by $l_1 : y = y_1(x)$ and $l_2 : x = x_2(x)$, respectively.

Noting that

$$\frac{dF_1}{dx} = -\frac{b_{11}\, b_{13}}{(b_{13}\, x + b_{12})^2} - a_{11} - \frac{a_{12}}{k_1+y} < 0, \tag{4.3}$$

$$\frac{dF_1}{dy} = \frac{a_{12}\,x}{(k_1+y)^2} > 0. \tag{4.4}$$

Hence,

$$\frac{dy_1}{dx} = -\frac{F_{1x}}{F_{1y}} > 0. \tag{4.5}$$

That is, l_1 is the strictly increasing function, also, one could easily see that l_1 pass through $A_2(x^*, 0)$, and $y_1(x) \to +\infty$ as $x \to x_1^*$, where x_1^* is the unique positive solution of the equation

$$\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x = 0. \tag{4.6}$$

On the other hand, by computing, for l_2, we also have

$$\frac{dF_2}{dx} = \frac{a_{21}\,y}{(k_2+x)^2} > 0, \tag{4.7}$$

$$\frac{dF_2}{dy} = -\frac{b_{21}\,b_{23}}{(b_{23}\,y+b_{22})^2} - a_{22} - \frac{a_{21}}{k_2+x} < 0. \tag{4.8}$$

Hence,

$$\frac{dy_2}{dx} = -\frac{F_{2x}}{F_{2y}} > 0. \tag{4.9}$$

That is, l_2 is the strictly increasing function, also, one could easily see that l_2 pass through $A_3(0, y^*)$, and $x_2(y) \to +\infty$ as $y \to y_1^*$, where y_1^* is the unique positive solution of the equation

$$\frac{b_{21}}{b_{22}+b_{23}y} - b_{24} - a_{22}y = 0. \tag{4.10}$$

Above analysis shows that l_1 and l_2 must intersect at least one point, and the two lines could intersect at most one point. Denote this point by $A_4(x_1, y_1)$. Then, system (1.1) admits the unique positive equilibrium A_4. This ends the proof of Lemma 4.1.

Lemma 4.2. *Assume that (2.11) and (2.16) hold, then the system (1.1) admits a unique positive equilibrium $A_4(x_1, y_1)$, which is locally asymptotically stable.*

Proof. The existence and uniqueness of the positive equilibrium is ensured by Lemma 4.1. To end the proof of Lemma 4.2, it is enough to show that $A_4(x_1, y_1)$ is locally asymptotically stable.

The variational matrix of the continuous-time system (1.1) at an equilibrium solution (x, y) is

$$J(x,y) = \begin{pmatrix} G_{11}(x,y) & G_{12}(x,y) \\ G_{21}(x,y) & G_{22}(x,y) \end{pmatrix}, \tag{4.11}$$

where

$$\begin{aligned}
G_{11}(x,y) &= \frac{b_{11}}{b_{13}\,x+b_{12}} - b_{14} - a_{11}\,x - \frac{a_{12}\,x}{k_1+y} \\
&\quad + x\left(-\frac{b_{11}\,b_{13}}{(b_{13}\,x+b_{12})^2} - a_{11} - \frac{a_{12}}{k_1+y}\right), \\
G_{12}(x,y) &= \frac{x^2 a_{12}}{(k_1+y)^2}, \\
G_{21}(x,y) &= \frac{y^2 a_{21}}{(k_2+x)^2}, \\
G_{22}(x,y) &= \frac{b_{21}}{b_{23}\,y+b_{22}} - b_{24} - a_{22}\,y - \frac{a_{21}\,y}{k_2+x} \\
&\quad + y\left(-\frac{b_{21}\,b_{23}}{(b_{23}\,y+b_{22})^2} - a_{22} - \frac{a_{21}}{k_2+x}\right).
\end{aligned}$$

The characteristic equation of the variational matrix is

$$\lambda^2 - tr(J)\lambda + det(J) = 0. \tag{4.12}$$

For a continuous-time system, asymptotic stability of an equilibrium solution is satisfied as long as $tr(J) < 0$ and $det(J) > 0$. Now, at the positive equilibrium $A_4(x_1, y_1)$, one has

$$\begin{aligned}
& tr(J) \\
= \;& x_1\left(-\frac{b_{11}\,b_{13}}{(b_{13}\,x_1+b_{12})^2} - a_{11} - \frac{a_{12}}{k_1+y_1}\right) \\
& + y_1\left(-\frac{b_{21}\,b_{23}}{(b_{23}\,y_1+b_{22})^2} - a_{22} - \frac{a_{21}}{k_2+x_1}\right) \\
< \;& 0,
\end{aligned}$$

$$\begin{aligned}
& det(J) \\
= \;& x_1 y_1\left(\frac{b_{11}\,b_{13}}{(b_{13}\,x_1+b_{12})^2} + a_{11} + \frac{a_{12}}{k_1+y_1}\right) \\
& \times\left(\frac{b_{21}\,b_{23}}{(b_{23}\,y_1+b_{22})^2} + a_{22} + \frac{a_{21}}{k_2+x_1}\right) \\
& - \frac{{x_1}^2 a_{12}}{(k_1+y_1)^2}\frac{{y_1}^2 a_{21}}{(k_2+x_1)^2} \\
> \;& x_1 y_1\left(\frac{b_{11}\,b_{13}}{(b_{13}\,x_1+b_{12})^2} + a_{11}\right) \\
& \times\left(\frac{b_{21}\,b_{23}}{(b_{23}\,y_1+b_{22})^2} + a_{22}\right) \\
> \;& 0.
\end{aligned} \tag{4.13}$$

Hence the positive steady-state solution is locally asymptotically stable. This ends the proof of Lemma 4.2.

Concerned with the stability property of the positive equilibrium, we have the following result.

Theorem 4.1. *Assume that (2.11) and (2.16) hold, then the system (1.1) admits a unique positive equilibrium $A_4(x_1, y_1)$, which is globally asymptotically stable.*

Proof. Under the assumption of Theorem 4.1, it follows from Theorem 3.1 that system (1.1) is uniformly bounded. Let

$$D = \Big\{(x,y) \in R_+^2 : x < \frac{3}{2}u^*,\;\; y < \frac{3}{2}v^*\Big\}.$$

Then every solution of system (1.1) starts in R_+^2 is uniformly bounded on D. Also, from the permanence of the system, one could see that the boundary equilibria A_0, A_1, A_2 are all unstable. Lemma 4.2 shows that there is a unique local stable positive equilibrium $A_4(x_1, y_1)$. To show that $A_4(x_1, y_1)$ is globally stable, it's enough to show that the system admits no limit cycle in the area D. To this end, let's consider the Dulac function $B(x, y) = x^{-1}y^{-1}$, then

$$\begin{aligned}
& \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} \\
= \;& -\frac{1}{y}\Big(\frac{b_{11}\,b_{13}}{(b_{13}\,x+b_{12})^2} + a_{11} + \frac{a_{12}}{k_1+y}\Big) \\
& -\frac{1}{x}\Big(\frac{b_{21}\,b_{23}}{(b_{23}\,y+b_{22})^2} + a_{22} + \frac{a_{21}}{k_2+x}\Big),
\end{aligned} \tag{4.14}$$

where

$$P(x,y) = x\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x - \frac{a_{12}x}{y+k_1}\Big),$$
$$Q(x,y) = y\Big(\frac{b_{21}}{b_{22}+b_{23}y} - b_{24} - a_{22}y - \frac{a_{21}y}{x+k_2}\Big).$$

By Dulac Theorem[33], there is no closed orbit in area D. Consequently, $A_4(x_1, y_1)$ is globally asymptotically stable. This completes the proof of Theorem 4.1.

Remark 4.1 Noting that inequalities (2.11) and (2.16) are independent of the parameters b_{13}, b_{23}, hence, from the first sight, it seems that nonlinear density dependent birth rate has no influence to the persistent and stability of the positive equilibrium, consequently, it has no influence to the dynamic behaviors of the system, however, Example 5.4 in the next section shows that with the increasing of b_{13}, the final density of the first species is decreasing, and this may increasing the extinct chance of the first species, it's in this sense that the nonlinear birth rate has negative effect on the persistent property of the species.

V. Numeric simulations

Now let's consider the following three examples.

Example 5.1

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{1}{2+x} - 1 - 2x - \frac{x}{1+y}\Big),\\ \frac{dy}{dt} &= y\Big(\frac{1}{4+y} - 1 - y - \frac{y}{2+x}\Big).\end{aligned} \tag{5.1}$$

In this system, corresponding to system (1.1), we take $b_{11} = b_{13} = b_{14} = a_{12} = b_{21} = b_{23} = b_{24} = a_{22} = a_{21} = 1, a_{11} = 2, b_{12} = 2, b_{22} = 4, k_1 = 1, k_2 = 2$. Since $b_{11} < b_{12}b_{14}, b_{21} < b_{22}b_{24}$, it follows from Theorem 2.1(1) that the boundary equilibrium $A_1(0,0)$ is globally asymptotically stable. Fig. 1 supports this assertion.

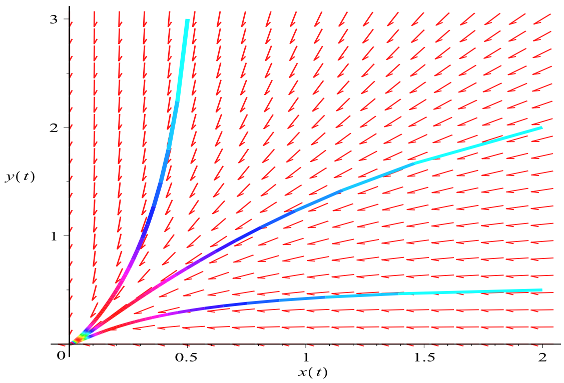

Fig. 1. Dynamic behaviors of the system (5.1) the initial condition $(x(0), y(0)) = (2, 0.5), (2, 2)$ and $(0.5, 3)$, respectively.

Example 5.2

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{2}{1+x} - 1 - x - \frac{x}{1+y}\Big),\\ \frac{dy}{dt} &= y\Big(\frac{2}{1+y} - 1 - y - \frac{y}{1+x}\Big).\end{aligned} \tag{5.2}$$

In this system, corresponding to system (1.1), we take $b_{12} = b_{13} = b_{14} = a_{11} = a_{12} = b_{22} = b_{23} = b_{24} = a_{22} = a_{12} = a_{21} = 1, b_{11} = b_{21} = 2, k_1 = k_2 = 1$. Since $b_{11} > b_{12}b_{14}, b_{21} > b_{22}b_{24}$, it follows from Theorem 3.1 and 4.1 that the system is permanent, and the unique positive equilibrium $A_4(0.3050995891, 0.3419788843)$ is globally asymptotically stable. Fig. 2 supports this assertion.

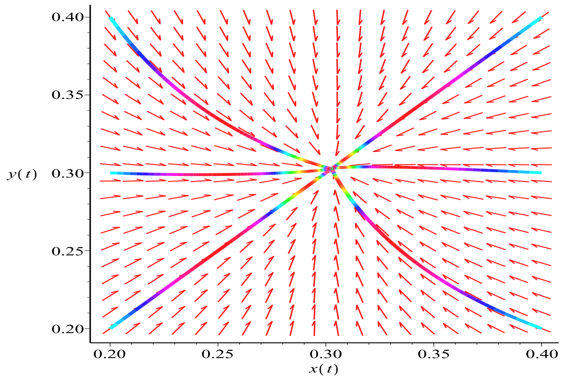

Fig. 2. Dynamic behaviors of the system (5.2) the initial condition $(x(0), y(0)) = (0.2, 0.4), (0.2, 0.2), (0.4, 0.3)$ $(0.4, 0.2)$, $(0.2, 0.3)$ and $(0.4, 0.4)$, respectively.

Example 5.3

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{1}{2+x} - 1 - 2x - \frac{x}{1+y}\Big),\\ \frac{dy}{dt} &= y\Big(\frac{4}{1+y} - 1 - y - \frac{y}{2+x}\Big).\end{aligned} \tag{5.3}$$

In this system, corresponding to system (1.1), we take $b_{11} = b_{13} = b_{14} = a_{12} = b_{22} = b_{23} = b_{24} = a_{22} = a_{21} = 1, a_{11} = 2, b_{12} = 2, b_{21} = 4, k_1 = 1, k_2 = 2$. Since $b_{11} < b_{12}b_{14}, b_{21} > b_{22}b_{24}$, it follows from Theorem 2.1(3) that the boundary equilibrium $A_2(0, 0.8081429670)$ is globally asymptotically stable. Fig. 3 supports this assertion.

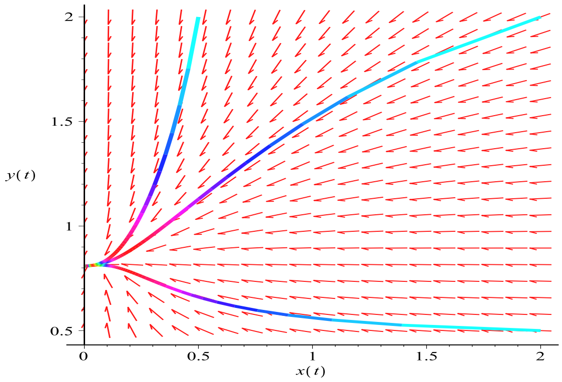

Fig. 3. Dynamic behaviors of the system (5.3) the initial condition $(x(0), y(0)) = (2, 0.5), (2, 2)$ and $(0.5, 2)$, respectively.

Example 5.4

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{2}{1+b_{13}x} - 1 - x - \frac{x}{1+y}\Big),\\ \frac{dy}{dt} &= y\Big(\frac{2}{1+y} - 1 - y - \frac{y}{1+x}\Big).\end{aligned} \tag{5.4}$$

In this system, corresponding to system (1.1), we take $b_{12} = b_{14} = a_{11} = a_{12} = b_{22} = b_{23} = b_{24} = a_{22} = a_{12} = a_{21} = 1, b_{11} = b_{21} = 2, k_1 = k_2 = 1$, since $b_{11} > b_{12}b_{14}, b_{21} > b_{22}b_{24}$, it follows from Theorem 3.1 and 4.1 that for any positive constant b_{13}, the system is permanent, and the unique positive equilibrium which is globally asymptotically stable. For $b_{13} = 1, 2$ and 3, respectively, Fig. 4-6 shows the dynamic behaviors of $x(t)$. From Fig.4-6, one could see that with the increasing of b_{13}, the final density of the first species is decreasing.

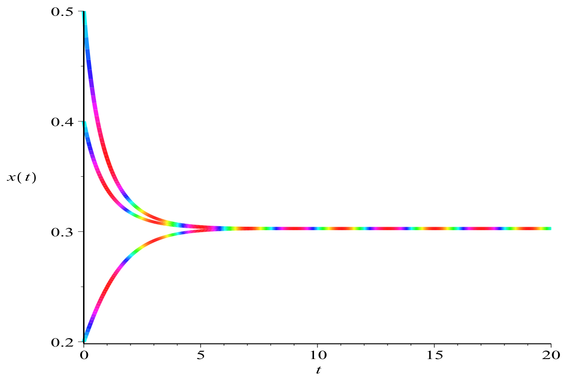

Fig. 4. Dynamic behaviors of the species $x(t)$ in system (5.4) the initial condition $(x(0), y(0)) = (0.2, 0.2), (0.4, 0.4)$ and $(0.5, 0.5)$, respectively. $b_{13} = 1$.

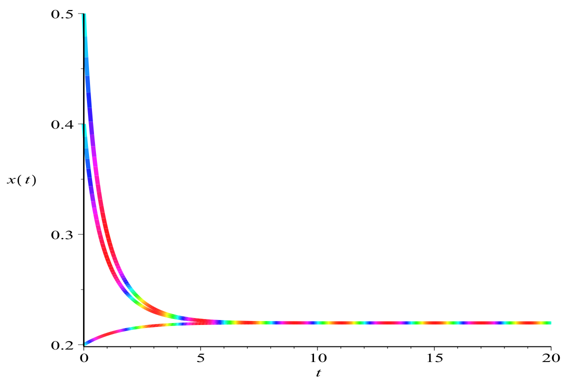

Fig. 5. Dynamic behaviors of the species $x(t)$ in system (5.4) the initial condition $(x(0), y(0)) = (0.2, 0.2), (0.4, 0.4)$ and $(0.5, 0.5)$, respectively. $b_{13} = 2$.

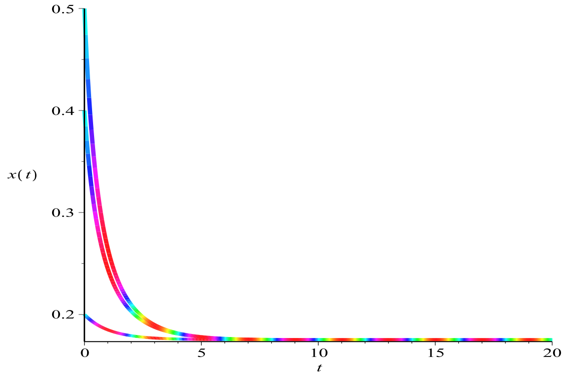

Fig. 6. Dynamic behaviors of the species $x(t)$ in system (5.4) the initial condition $(x(0), y(0)) = (0.2, 0.2), (0.4, 0.4)$ and $(0.5, 0.5)$, respectively. $b_{13} = 4$.

VI. Discussion

Recently, many scholars studied the dynamic behaviors of the mutualism model (cooperative model), see [1]-[32] and the references cited therein. Among those works, May[18], Xie, Xue and Chen[12], Chen, Wu and Xie[15], Lei[14], Han and Chen[11], Chen, Xie and Chen[2], Zhao, Qin and Chen [7] proposed several kind of May type cooperative system, and investigated the dynamic behaviors of the system. However, all of the works of [2], [7], [11], [12], [14], [15], [18] were based on the traditional Logistic model, as was shown in the introduction section. In this paper, stimulated by recent works of [28]-[30], we argued that the birth rate of the species maybe density dependent, this leads to the system (1.1).

Theorem 2.1 shows that under some suitable assumptions, all of the three boundary equilibria maybe globally asymptotically stable. That is, by introducing the density dependent birth rate, all of the species or part of the species may be driven to extinction, such kind of dynamic behaviors is quite different to the system without density dependent (For example, May[18] showed that the unique positive equilibrium of the system (1.3) is globally asymptotically stable).

Theorem 3.1 shows that under some suitable assumption, the system could be permanent. Also, under the same assumption as that of Theorem 3.1, we could show that system (1.1) admits a unique positive equilibrium which is globally asymptotically stable. However, example 5.4 shows that with the increasing of the parameter b_{13}, the final density of the first species is decreasing, hence, under the assumption (2.11) and (2.16), though at first sight the system is permanent, the chance for the system to be extinct is increasing, the nonlinear birth rate has negative effect to the persistent property of the system.

Our study shows that the birth rate is one of the essential factors to determine the dynamic behaviors of the species. Density dependent birth rate may lead to the complicated dynamic behaviors of the system, in some case, may lead to the collapse of the system.

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References

[1] K. Yang, Z. S. Miao, Chen and Xie, "Influence of single feedback control variable on an autonomous Holling-II type cooperative system," Journal of Mathematical Analysis and Applications, vol. 435, no. 1, pp874-888, 2016.

[2] F. D. Chen, X. D. Xie, X. Chen, “Dynamic behaviors of a stage-structured cooperation model,” Communications in Mathematical Biology and Neuroscience, volume 2015, article ID 4, 2015.
[3] K. Yang, X. D. Xie, F. Chen, “Global stability of a discrete mutualism model,” Abstract and Applied Analysis, volume 2014, article ID 709124, 2014.
[4] T. T. Li, F. D. Chen, J. H. Chen and Q. X. Lin, “Stability of a stage-structured plant-pollinator mutualism model with the Beddington-DeAngelis functional response,” Journal of Nonlinear Functional Analysis, volume 2017, article ID 50, 2017.
[5] R. Y. Han, X. D. Xie, F. Chen, “Permanence and global attractivity of a discrete pollination mutualism in plant-pollinator system with feedback controls,” Advances in Difference Equations, volume 2016, article ID 199, 2016.
[6] L. J. Chen, X. D. Xie, “Feedback control variables have no influence on the permanence of a discrete N-species cooperation system,” Discrete Dynamics in Nature and Society, volume 2009, article ID 306425, 10 pages, 2009.
[7] L. Zhao, B. Qin, F. Chen, “Permanence and global stability of a May cooperative system with strong and weak cooperative partners,” Advances in Difference Equations, volume 2018, article ID 172, 2018.
[8] L. Y. Yang, X. D. Xie, et al, “Dynamic behaviors of a discrete periodic predator-prey-mutualist system,” Discrete Dynamics in Nature and Society, volume 2015, article ID 247269, 11 pages, 2015.
[9] F. D. Chen, J. H. Yang, L. Chen and X. Xie, “On a mutualism model with feedback controls,” Applied Mathematics and Computation, vol. 214, no. 3, pp581-587, 2009.
[10] L. J. Chen, L. J. Chen, Z. Li, “Permanence of a delayed discrete mutualism model with feedback controls,” Mathematical and Computer Modelling, vol. 50, no. 5, pp1083-1089, 2009.
[11] R. Y. Han, F. D. Chen, X. Xie and Z. Miao, “Global stability of May cooperative system with feedback controls,” Advances in Differences Equations, volume 2015, article ID 360, 2015.
[12] X. D. Xie, F. D. Chen, Y. Xue, “Note on the stability property of a cooperative system incorporating harvesting,” Discrete Dynamics in Nature and Society, volume 2014, article ID 327823, 5 pages, 2014.
[13] X. D. Xie, F. D. Chen, K. Yang and Y. Xue, “Global attractivity of an integrodifferential model of mutualism,” Abstract and Applied Analysis, volume 2014, article ID 928726, 2014.
[14] C. Q. Lei, “Dynamic behaviors of a non-selective harvesting May cooperative system incorporating partial closure for the populations,” Communications in Mathematical Biology and Neuroscience, volume 2018, article ID 12, 2018.
[15] F. D. Chen, H. L. Wu, X. Xie, “Global attractivity of a discrete cooperative system incorporating harvesting,” Advances in Difference Equations, volume 2016, article ID 268, 2016.
[16] W. S. Yang, X. P. Li, “Permanence of a discrete nonlinear N-species cooperation system with time delays and feedback controls,” Applied Mathematics and Computation, vol. 218, no. 7, pp3581-3586, 2011.
[17] R. X. Wu, L. Li, X. Zhou, “A commensal symbiosis model with Holling type functional response,” Journal of Mathematics and Computer Science, vol. 16, no. 2, pp364-371, 2016.
[18] R. X. Wu, L. Li, “Dynamic behaviors of a commensal symbiosis model with ratio-dependent functional response and one party can not survive independently,” Journal of Mathematics and Computer Science, vol. 16, no. 2, pp495-506, 2016.
[19] R. M. May, “Theoretical Ecology,” Principles and Applications, Sounders, Philadelphia,1976.
[20] Y. Liu, X. Xie, Q. Lin, “Permanence, partial survival, extinction, and global attractivity of a nonautonomous harvesting Lotka-Volterra commensalism model incorporating partial closure for the populations,” Advances in Difference Equations, volume 2018, article ID 211, 2018.
[21] L. Yu, X. Guan, X. Xie and Q. Lin, “On the existence and stability of positive periodic solution of a nonautonomous commensal symbiosis model with Michaelis-Menten type harvesting,” Communications in Mathematical Biology and Neuroscience, volume 2019, article ID 2, 2019.
[22] R. Y. Han, F. D. Chen, “Global stability of a commensal symbiosis model with feedback controls,” Communications in Mathematical Biology and Neuroscience, volume 2015, article ID 15, 2015.
[23] X. D. Xie, Z. S. Miao, Y. Xue, “Positive periodic solution of a discrete Lotka-Volterra commensal symbiosis model,” Communications in Mathematical Biology and Neuroscience, volume 2015, article ID 2, 2015.
[24] Y. L. Xue, X. D. Xie, F. Chen and H. Rong, “Almost periodic solution of a discrete commensalism system,” Discrete Dynamics in Nature and Society, volume 2015, article ID 295483, 11 pages, 2015.
[25] Q. F. Lin, “Allee effect increasing the final density of the species subject to the Allee effect in a Lotka-Volterra commensal symbiosis model,” Advances in Difference Equations, volume 2018, article ID 196, 2018.
[26] F. Chen, X. Xie, Z. Miao and P. Li, “Extinction in two species nonautonomous nonlinear competitive system,” Applied Mathematics and Computation, vol. 274, no. 1, pp119-124, 2016.
[27] Q. Lin, X. Xie, F. Chen and Q. Lin, “Dynamical analysis of a logistic model with impulsive Holling type-II harvesting,” Advances in Difference Equations, volume 2018, article ID 112, 2018.
[28] F. Chen, Y. Xue, Q. Lin and X. Xie, “Dynamic behaviors of a Lotka-Volterra commensal symbiosis model with density dependent birth rate,” Advances in Difference Equations, volume 2018, article ID 296, 2018.
[29] L. Zhao, B. Qin, X. Sun, “Dynamic Behavior of a commensalism model with nonmonotonic functional response and density-dependent birth rates,” Complexity, volume 2018, article ID 9862584, 2018.
[30] B. G. Chen, “The Influence of density dependent birth rate to a commensal symbiosis model with Holling type functional response,” Engineering Letters, vol. 27, no. 2, pp295-302, 2019.
[31] B. G. Chen, “Dynamic behaviors of a commensal symbiosis model involving Allee effect and one party can not survive independently,” Advances in Difference Equations, volume 2018, article ID 212, 2018.
[32] B. G. Chen, “The influence of commensalism on a Lotka-Volterra commensal symbiosis model with Michaelis-Menten type harvesting,” Advances in Difference Equations, volume 2019, article ID 43, 2019.
[33] L. S. Chen, “Mathematical Models and Methods in Ecology,” Science Press, Beijing (1988), (in Chinese).
[34] Z. W. Xiao, Z. Li, “Stability and bifurcation in a stage-structured predator-prey model with Allee effect and time delay,” IAENG International Journal of Applied Mathematics, vol. 49, no. 1, pp6-13, 2019.
[35] Q. Yue, “Permanence of a delayed biological system with stage structure and density-dependent juvenile birth rate,” Engineering Letters, vol. 27, no. 2, pp263-268, 2019.
[36] S. B. Yu, “Effect of predator mutual interference on an autonomous Leslie-Gower predator-prey model,” IAENG International Journal of Applied Mathematics, vol. 49, no. 2, pp229-233, 2019.