A Numerical Method for Solving Fractional Variational Problems by the Operational Matrix Based on Chelyshkov Polynomials

Linna Li, Zhirou Wei and Qiongdan Huang

Abstract—**In this study, an accurate and effective method is proposed to solve fractional variational problems (FVPs). Fractional derivative is described in the sense of Caputo. In the method, we simplify the fractional variational problems by the operational matrices. The operational matrices are based on the Chelyshkov polynomials. Using this method, the fractional variational problem can be transformed to a set of algebraic equations. And using the Lagranger multiplier technique, the unkown coefficients can be solved. The numerical solution of FVPs is calculated by maple. Through different graphical and tables, the numerical results of illustrative evidence show that the method is reliable and powerful in solving fractional variational problems.**

Index Terms—**fractional variational problem, caputo fractional derivative, chelyshkov polynomials, operational matrices**

I. Introduction

The fractional calculation has been a popular topic in recent years [1]. The academics who first proposed fractional differential equations were Leibniz and L'Hopital. Fractional differential equations are applied in many different fields, such as control science and engineering [2], mathematics [3], etc. Many academics have studied different theories in fractional differential equations [4]. Many different numerical methods are introduced to develop approximate solutions to the fractional differential equations and systems, such as the Legendre wavelet method [5], the finite element method [6], homotopy perturbation method [7] and so on.

Recently, one of the numerical techniques is presented, which expands the approximate solution into the elements of the Chelyshkov polynomials for calculation. A numerical matrix method [8], based on Chelyshkov polynomials, was presented to solve the linear functional integro-differential equations. Talaei and Asgari [9] proposed the Chelyshkov-collocation spectral method to solve multi-order fractional differential equations under the supplementary conditions. In [10], a technique to solve two-dimensional Fredholm-Volterra integral equations was presented by the Chelyshkov poly-nomials. Sezer [11] put forward an approach, based on Chelyshkov polynomi-als, to solve the linear functional integro-differential equations with variable coefficients under the initial-boundary conditions. Talaei [12] worked out the weakly singular Volterra integral equations by the Chelyshkov polynomials. Jaradat [13] proposed an iterative technique based on the generalized Taylor series residual power series to obtain an approximate solution of a generalized time-fractional Drinfeld-Sokolov-Wilson system.

Many problems in life can be regarded as optimal control problems, which have been spotted in all aspects of life. Many scholars [14, 15] were devoted to the research of optimal control problems. The variational problem is a special kind of optimal control problem. The variational problem is to determine the maximum or minimum values of the functions. Because of the importanceof the variational problems in many areas, such as clinical medicine [16], physics [17] and so on, researchers are committed to the study of these problems. Amini, Weymouth and Jain [18] proposed a functional minimization algorithm and its implementation, which an iterative constraint satisfies the procedures of local surface smoothness properties. Horng and Chou [19] solved variational problems by Shifted Chebyshev polynomials. The fractional calculus of variational was a new field. Riewe [20, 21] was the first who discovered derivatives of fractional order using a Lagrangian to solve variational problems. Chen [22] proposed a symmetric dual problem for a class of multiobjective fractional variational problems. The simplest fractional variational problem and the fractional variational problem of La-grange were considered by Agrawal [23]. And many other numerical methods solved fractional variational problems, such as M¨untz-Legendre polynomials [24], shifted Legendre orthonormal polynomials [25], Jacobi polynomials [26] and so on.

In this paper, the operational matrices based on the Chelyshkov polynomials are used to solve fractional variational problems. These operational matrices are used to calculate the approximate solutions of the FVPs. This approach reduces the fractional variational problem to a set of algebraic equations by operational matrices. With the properties of the Chelyshkov polynomials, we can simplify the fractional variational problem. Moreover, a new numerical technique based on the Chelyshkov polynomials is

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proposed for the following fractional variational problem,

$$MinimunJ=\int_0^{t_f} F(t,x(t),D^{\alpha}x(t),D^{\beta}x(t))dt, \qquad (1)$$
$$\lceil\alpha\rceil-1<\alpha\le\lceil\alpha\rceil, \quad 0\le\alpha\le\beta$$

With the boundary conditions

$$x(0)=a, x(t_f)=b,$$

Where $\lceil\alpha\rceil$ is the ceiling function of α.

The rest of the structure is as follows. In section 2, some definitions about fractional calculus and the properties of Chelyshkov polynomials are introduced respectively. How to solve the fractional variational problem by Chelyshkov polynomials is proposed in section 3. In section 4, illustrative examples are presented. At last, the conclusion is drawn in section 5.

II. Preliminaries

In this section, the definitions of fractional calculus are introduced systematically. And this section also presents the most details of the Chelyshkov polynomials.

A. Fractional calculus

Definition 1 [27, 28, 29]. According to Riemann-Liouville, the fractional integral operator of order $\alpha\geqslant 0$ is defined as

$$I^{\alpha}f(t)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t\frac{f(s)}{(t-s)^{1-\alpha}}ds=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}*f(t) & \alpha>0,t>0\\ f(t) & \alpha=0\end{cases} \qquad (2)$$

Where $t^{\alpha-1}*f(t)$ is the convolution product of $t^{\alpha-1}$ and $f(t)$.

For the Riemann-Liouville fractional integral we have

$1.I^{\alpha}t^{\beta}=\dfrac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}t^{\alpha+\beta},\beta>-1,$

$2.I^{\alpha}\left(\lambda f(t)+\mu g(t)\right)=\lambda I^{\alpha}f(t)+\mu I^{\alpha}g(t),$

where λ and μ are real constants.

Definition 1 [30, 31]. Let $f(t):[0,+\infty)\to R$ be a function, and $\lceil\alpha\rceil$ be the upper positive integer of α ($\alpha>0$). The Caputo fractional derivative is defined by

$$D^{\alpha}f(t)=\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_0^t\frac{f^{\lceil\alpha\rceil}(s)}{(t-s)^{\alpha+1-\lceil\alpha\rceil}}ds, \qquad (3)$$
$$\lceil\alpha\rceil-1<\alpha\le\lceil\alpha\rceil, n\in N$$

For the Caputo derivative we have

$1.D^{\alpha}I^{\alpha}f(t)=f(t),$

$2.I^{\alpha}D^{\alpha}f(t)=f(t)-\sum_{i=0}^{n-1}y^{(i)}(0)\dfrac{t^i}{i!},$

$3.D^{\alpha}t^{\beta}=\begin{cases}\dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}t^{\beta-\alpha} & \beta\ge\alpha\\ 0 & \beta<\alpha\end{cases},$

$4.D^{\alpha}c=0,$

$5.D^{\alpha}\left(\lambda f(t)+\mu g(t)\right)=\lambda D^{\alpha}f(t)+\mu D^{\alpha}g(t),$

where λ_1, λ_2 and c are real constants.

B. Chelyshkov polynomials

Chelyshkov [32] has introduced sequences of polynomials which are orthogonal in the interval [0, 1] with the weight function 1. These polynomials are defined by

$$C_{N,n}(t)=\sum_{j=0}^{N-n}(-1)^j\binom{N-n}{j}\binom{N+n+j+1}{N-n}t^{n+j}, n=0,1,...,N \qquad (4)$$

Then, we can get the following approximate solutions of the FVPs by the Chelyshkov polynomials

$$x(t)\simeq x_N(t)=\sum_{n=0}^{N}a_nC_t=A.C.T_t \qquad (5)$$

where

$$A=(a_0,a_1,...,a_N), C_t=\begin{pmatrix}C_{N,0}(t)\\ C_{N,1}(t)\\ \vdots\\ C_{N,N}(t)\end{pmatrix}, T_t=\begin{pmatrix}1\\ t\\ \vdots\\ t^N\end{pmatrix} \qquad (6)$$

if N is odd,

$$C=\begin{pmatrix}\binom{N}{0}\binom{N+1}{N} & -\binom{N}{1}\binom{N+2}{N} & \cdots & \binom{N}{N-1}\binom{2N}{N} & -\binom{N}{N}\binom{2N+1}{N}\\ 0 & \binom{N-1}{0}\binom{N+2}{N-1} & \cdots & -\binom{N-1}{N-2}\binom{2N}{N-1} & \binom{N-1}{N-1}\binom{2N+1}{N-1}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \binom{1}{0}\binom{2N}{1} & -\binom{1}{1}\binom{2N+1}{1}\\ 0 & 0 & \cdots & 0 & 1\end{pmatrix} \qquad (7)$$

and if N is even,

$$C=\begin{pmatrix}\binom{N}{0}\binom{N+1}{N} & -\binom{N}{1}\binom{N+2}{N} & \cdots & -\binom{N}{N-1}\binom{2N}{N} & \binom{N}{N}\binom{2N+1}{N}\\ 0 & \binom{N-1}{0}\binom{N+2}{N-1} & \cdots & \binom{N-1}{N-2}\binom{2N}{N-1} & -\binom{N-1}{N-1}\binom{2N+1}{N-1}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \binom{1}{0}\binom{2N}{1} & -\binom{1}{1}\binom{2N+1}{1}\\ 0 & 0 & \cdots & 0 & 1\end{pmatrix} \qquad (8)$$

These polynomials satisfy the orthogonality condition

$$\int_0^1 C_{N,p}(t)C_{N,q}(t)dt=\begin{cases}0 & p\ne q\\ \dfrac{1}{p+q+1} & p=q\end{cases} \qquad (9)$$

Theorem 1 [7]. Assume that Ct be Chelyshkov vector defined in (6) and $\alpha\in R_+$. Then

$$D^{\alpha}C_t\simeq D^{(\alpha)}C_t \qquad (10)$$

where $D^{(\alpha)}$ is the $(N+1)\times(N+1)$ operational matrix of fractional derivative as

$$D^{(\alpha)}=\begin{pmatrix}w_{0,0}^{\alpha} & w_{0,1}^{\alpha} & \cdots & w_{0,N}^{\alpha}\\ w_{1,0}^{\alpha} & w_{1,1}^{\alpha} & \cdots & w_{1,N}^{\alpha}\\ \vdots & \vdots & \vdots & \vdots\\ w_{n,0}^{\alpha} & w_{n,1}^{\alpha} & \cdots & w_{n,N}^{\alpha}\\ \vdots & \vdots & \vdots & \vdots\\ w_{N,0}^{\alpha} & w_{N,1}^{\alpha} & \cdots & w_{N,N}^{\alpha}\end{pmatrix} \qquad (11)$$

where

$$w_{n,l}^{\alpha}=\sum_{j=\lceil\alpha\rceil}^{N}(-1)^{j-n}\binom{N-n}{j-n}\binom{N+j+1}{N-n}\times\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}d_{l,j}, \qquad (12)$$
$$n,l=0,1,...,N$$

and

$$\begin{aligned} d_{l,j} &= (2l+1)\int_0^1 t^{j-\alpha} C_{N,l}(t)dt \\ &= (2l+1)\sum_{s=0}^{N-l}(-1)^s \binom{N-l}{s}\binom{N-l+s+1}{N-l}\int_0^1 t^{i-\alpha+l+s}dt \\ &= \sum_{s=0}^{N-l}(-1)^s \binom{N-l}{s}\binom{N-l+s+1}{N-l}\left(\frac{2l+1}{j-\alpha+l+s+1}\right) \end{aligned} \tag{13}$$

III. SOLVING THE FRACTIONAL VARIATIONAL PROBLEMS BY CHELYSHKOV POLYNOMIALS

In this section, a numerical technique is presented to solve the fractional variational problems. Some results about the Chelyshkov polynomials are proposed in section 2. The general form of this problem is

$$MinimumJ = \int_0^{t_f} F\left(t, x(t), D^{\alpha}x(t), D^{\beta}x(t)\right)dt, \tag{14}$$

$$\lceil \alpha \rceil - 1 < \alpha \le \lceil \alpha \rceil, 0 \le \alpha \le \beta$$

with the boundary conditions

$$x(0)=a,\ x(t_f)=b.$$

Here F is a linear or nonlinear function. To solve the fractional variational problem, we give the following steps，

Step 1. Expand the function x with the Chelyshkov polynomials in (5), as

$$x(t) \simeq x_N(t) = \sum_{n=0}^{N} a_n C_t = A.C.T_t \tag{15}$$

and the fractional derivatives of $D^{\alpha}x(t)$, $D^{\beta}x(t)$ can be written as

$$D^{\alpha}x(t) \simeq D^{\alpha}x_N(t) = A.D^{(\alpha)}.C_t = A.D^{(\alpha)}.C.T_t \tag{16}$$

$$D^{\beta}x(t) \simeq D^{\beta}x_N(t) = A.D^{(\beta)}.C_t = A.D^{(\beta)}.C.T_t \tag{17}$$

Step 2. Substitute Eqs.(15)-(17), the general function of Eq.(14) can be written in the following approximate form as

$$MinimumJ = \int_0^{t_f} F\left(t, x_N(t), D^{\alpha}x_N(t), D^{\beta}x_N(t)\right)dt \tag{18}$$

Step 3. Approximate the boundary condition, as

$$x(0) - a \cong x_N(0) - a = 0 \tag{19}$$

$$x(t_f) - b \cong x_N(t_f) - b = 0 \tag{20}$$

Step 4. Then let

$$G^T = [x_N(0)-a, x_N(t_f)-b]$$

where G is a 2×1 vectors. Consider

$$J^*[c_0, c_1, c_2, \ldots, c_N, \mu_1, \mu_2] = J^*[c_0, c_1, c_2, \ldots, c_N] + G^T\mu \tag{21}$$

where

$$\mu = [\mu_1, \mu_2]^T$$

is the Lagrange multiplier vector.

Step 5. The fractional variational problem of Eq.(14) can be transformed to the unconstrained problems, as

$$\frac{\partial J^*}{\partial c_n} = 0, n = 1, 2, \ldots, N, \frac{\partial J^*}{\partial \mu_i} = 0, i = 1, 2 \tag{22}$$

Step 6. Solving the Eq.(14), we can obtain the coefficients of c_n and μ_i. Then the approximate solution of Eq.(7) is

$$x(t) \simeq A.C.T_t.$$

IV. ILLUSTRATIVE EXAMPLES

In this section, some fractional variational problems are analysed by Maple in Windows (64bit). Using the operational matrices based on the Chelyshkov polynomials and the Lagrange multiplier method, FVPs can be transformed to the unconstrained problems. The approximate solutions of the FVPs are calculated. And the results are presented by charts and graphics.

Example 1 As the first example, we consider the following fractional variational problem,

$$MinimumJ = \int_0^1 (D^{0.5}x(t) - \frac{2}{\Gamma(2.5)}t^{1.5})^2 dt$$

with the boundary conditions,

$$x(0) = 0,\ x(1) = 1.$$

The exact solution of this fractional variational problem is $x(t) = t^2$. We present the absolute errors for different values of N in TABLE I. The different solutions of $x(t)$ are plotted in Figure.1 for different N. We can find that our method can have a good approximate solution to this fractional variational problem.

TABLE I
THE ABSOLUTE ERRORS FOR DIFFERENT VALUES OF N

t	N=3	N=4	N=5
0.1	1.2189×10^{-10}	1.0423×10^{-10}	4.4557×10^{-10}
0.3	1.9168×10^{-10}	1.2548×10^{-10}	5.5492×10^{-10}
0.5	1.0947×10^{-10}	2.3313×10^{-10}	9.3147×10^{-10}
0.7	2.8739×10^{-11}	3.5998×10^{-10}	1.1240×10^{-10}
0.9	1.2695×10^{-10}	9.8768×10^{-11}	1.0654×10^{-10}

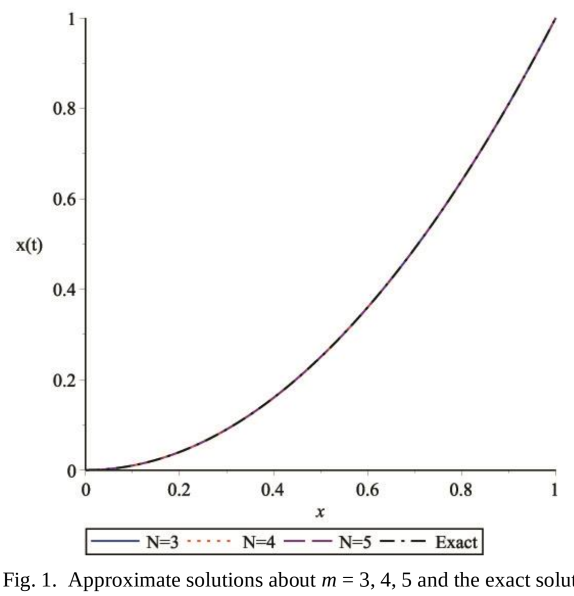


Fig. 1. Approximate solutions about m = 3, 4, 5 and the exact solution

Example 2 We consider the following fractional variational problem,

$$MinimumJ = \int_0^1 (D^{\alpha}x(t) - \frac{\Gamma(2\alpha+3)}{\Gamma(\alpha+3)}t^{\alpha+2} - \Gamma(\alpha+2)t)^2 dt,$$

with the boundary conditions,

$$x(0)=0,\ x(1)=3.$$

The exact solution about this problem is $y(t) = t^{2\alpha+2} + t^{\alpha+1} + 1$. The absolute errors are proposed for different values of α when $N = 7$ in table 2. From TABLE II, we can find that if the value of t is fixed, the larger the value of α, the smaller the absolute errors.

In Figure.2, the approximate solutions when N=3, 4, 5, 6,7 for α=1 and the exact solutions are plotted. From Figure.2, it is shown that the errors decay as N increase.

TABLE II
THE ABSOLUTE ERRORS WHEN N = 7 FOR DIFFERENT VALUES OF α

t	α=0.3	α=0.6	α=0.9	α=1
0.2	9.4066×10^{-5}	9.3011×10^{-5}	1.4114×10^{-5}	1.5343×10^{-9}
0.4	4.8693×10^{-5}	2.1125×10^{-5}	1.1682×10^{-5}	3.8318×10^{-9}
0.6	1.0568×10^{-4}	8.5799×10^{-5}	8.4150×10^{-6}	3.5262×10^{-9}
0.8	5.7636×10^{-5}	1.1878×10^{-5}	6.2915×10^{-6}	2.7671×10^{-7}
1	1.1120×10^{-6}	1.1530×10^{-6}	1.7850×10^{-6}	1.1300×10^{-6}

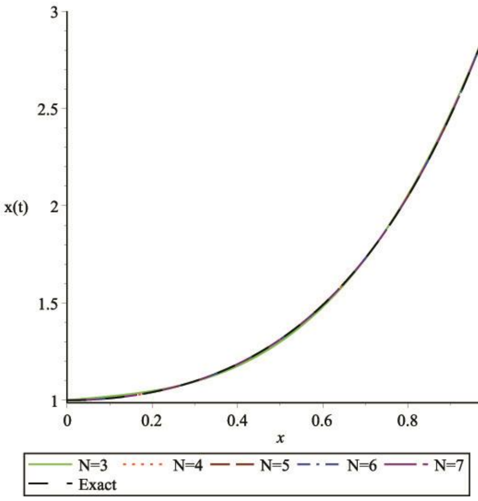


Fig. 2. Approximate solutions about N= 3, 4, 5, 6, 7 when α= 1 and the exact solution

Example 3 We consider the following fractional variational problem,

$$MinimumJ = \int_0^{\frac{\pi}{2}} (D^\alpha x(t) - \cos(t))^2 dt$$

subjected to

$$x(0)=0,\ x(\pi/2)=1,$$

with the exact solution $x(t) = \sin(t)$ for α= 1.

In TABLE III, the maximum absolute errors (MAEs) between our method and Shifted Legendre Orthonormal Polynomials (SLOP) [25] are presented at α=1 with various choices of N. We can find our method can deduce a good approximate solution.

In Figure.3, the absolute error of $x(t)$ when N=6, α= 1 are plotted. In addition, Figure.4 plots the approximate solutions of $x(t)$ when N=5 for α= 0.6,0.7,0.8, 0.9,1. From Figure.4, we can find that as approaching to 1, the solution for the integer order equation is recovered.

TABLE III
COMPARISON BETWEEN OUR METHOD AND SLOP WITH THE MAES FOR α= 1

m	$MAEs_{ourmethod}$	$J_{ourmethod}$	$MAEs_{SLOP}$	J_{SLOP}
3	2.180×10^{-3}	1.218×10^{-8}	2.646×10^{-3}	6.795×10^{-6}
4	1.647×10^{-4}	1.090×10^{-6}	2.317×10^{-4}	1.022×10^{-7}
5	1.073×10^{-5}	6.724×10^{-9}	1.608×10^{-5}	2.993×10^{-10}
6	7.995×10^{-7}	-3.009×10^{-8}	9.912×10^{-7}	2.482×10^{-12}

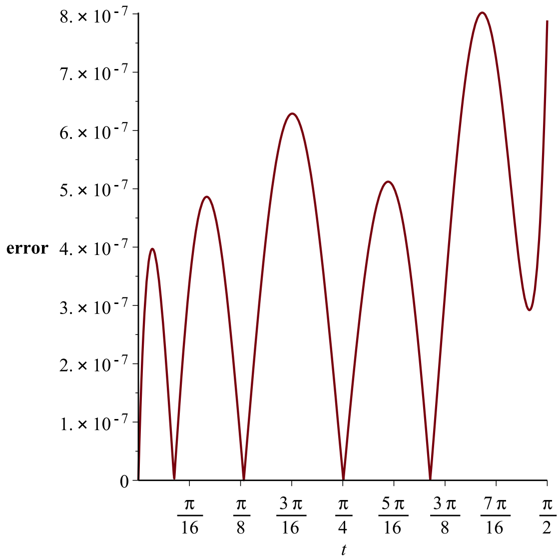


Fig. 3. The absolute error of $x(t)$ when N = 6

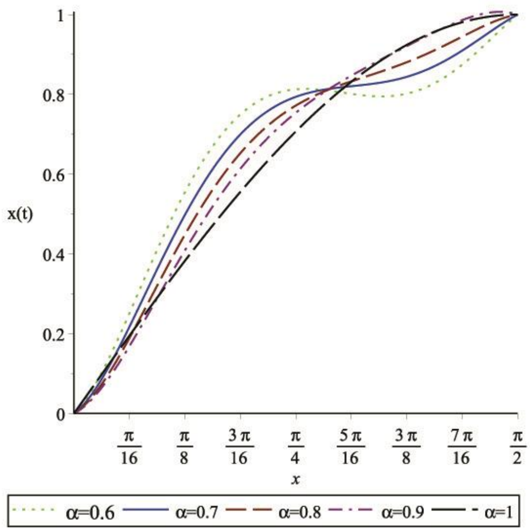


Fig. 4. The approximate solution of $x(t)$ when N= 5 for α= 0.6, 0.7, 0.8, 0.9, 1

Example 4 We consider the following fractional variational problem,

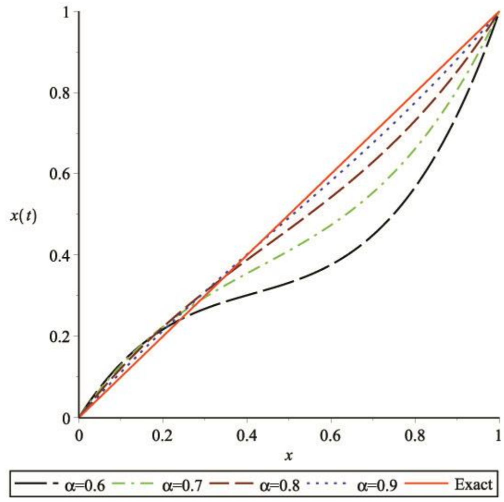
$$MinimumJ = \frac{1}{2}\int_0^1 (D^\alpha x(t))^2 dt$$

with the boundary conditions,

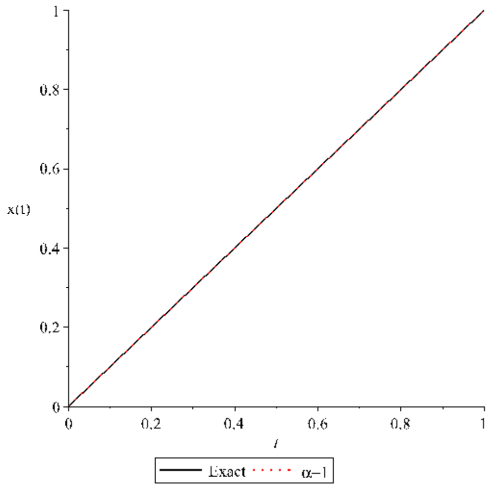
$$x(0)=0, x(1)=1.$$

The exact solution about this fractional variational problem is $x(t)=t$ for α=1.

We compare the approximate solution of $x(t)$ when N = 3 for various values of those in Figure.5. From these figures, we can find the approximate solution is more accurate for the value of α is close to 1. In Figure.6, the absolute error is plotted for N=4 and α=1. From Figure.6, we obtain the absolute error is close to 0.



(a) Comparison of $x(t)$ when N=3 for α= 0.6, 0.7, 0.8,0.9



(b) Comparison of $x(t)$ when N=3 and the exact solution

Fig. 5. The comparison between approximate solutions and exact solution

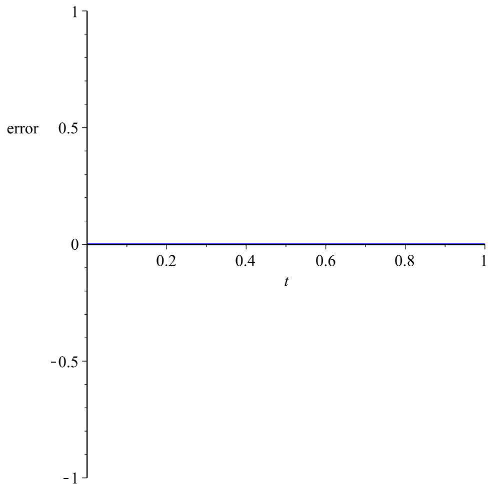


Fig. 6. The absolute error of $x(t)$ when N=4 for α= 1

Example 4 We consider the following fractional variational problem,

$$Minimum J = \int_0^1 ((D^\alpha x(t))^2 + tD^\alpha x(t) + x^2(t))dt$$

with the boundary conditions

$$x(0)=0,\ x(1)=1/4$$

In TABLE IV, the exact solution about this fractional problem is proposed. We compare the approximate solution of $x_m(t)$ for N=5 and the Legendre wavelet [33] method for k=3, M=5. The approximate solution of $x_5(t)$ is plotted for various values of α in Figure.7.

TABLE IV Comparison between our method and the Legendre wavelet method for α=1

t	*Legendre wavelet k=3,M=5*	*Our method*	*exact solution*
0	0.000000	0.000000	0.000000
0.1	0.041949	0.041951	0.041950
0.2	0.079315	0.079317	0.079316
0.3	0.112471	0.112473	0.112472
0.4	0.141749	0.141751	0.141750
0.5	0.167443	0.167443	0.167442
0.6	0.189807	0.189807	0.189806
0.7	0.209064	0.209066	0.209065
0.8	0.225411	0.225414	0.225412
0.9	0.239010	0.239013	0.239011
1.0	0.249999	0.250000	0.250000
J		0.19759399	0.19759399

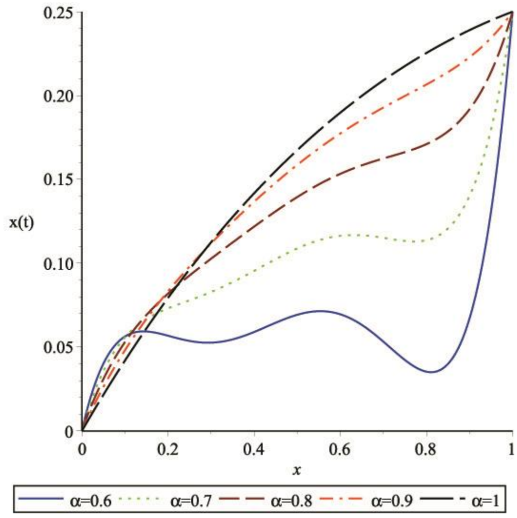


Fig. 7. The approximate solution of $x(t)$ when N=5 for α= 0.6, 0.7, 0.8, 0.9, 1

V. Conclusion

In this paper, we discuss a numerical method to solve fractional variational problems. Using the operational matrices based on the Chelyshkov polynomials, the fractional variational problems can be transformed to a set of algebraic equations. Then we can determine the unknown coefficients. The illustrative examples are presented by graphics and datum. Results show that our numerical method is an applicable and active method in finding solutions to fractional variational problems.

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