The Common Approach to Calculating the Characteristics of the Signal Parameters Joint Estimates under the Violation of the Decision Statistics Regularity Conditions

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Abstract—**We study the problem of the measurement of the quasi-deterministic or stochastic signal with the unknown discontinuous parameters against Gaussian interferences. Our goal is to determine the characteristics of the joint maximum likelihood estimates of the unknown parameters under the violation of the decision statistics regularity conditions. For this purpose, we introduce the local additive approximation method. According to this method, the decision statistics being a multidimensional random field is represented in the form of the sum of the products of the independent Markov random processes. Such representation proves to be valid in the small neighborhood of the point of the true values of the unknown parameters. Further, by applying the Markov random processes technique, it is possible to obtain the asymptotic analytical expressions for the probability density and the conditional moments of the resulting estimates. The accuracy of the specified formulas increases with the signal-to-noise ratio. Finally, we illustrate how the local additive approximation method can be applied when analyzing the performance of the two receiving devices: the measurer of the time of appearance and the duration of the quasi-deterministic video pulse and the measurer of the time of appearance and the band center of the Gaussian radio pulse. By means of statistical computer simulation, it is established that the application of this method allows obtaining the closed formulas for the accuracy characteristics of discontinuous signal measurers which are operable in a wide range of signal-to-noise ratios.**

Index Terms—**Discontinuous signal parameter, maximum likelihood method, local Markov approximation method, local additive approximation method, accuracy characteristics of estimate**

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I. Introduction

In various fields of physics and engineering, there is a problem of measurement (estimation) of the parameters of the information signals or images observed against random interferences [1]-[3]. The signal parameters estimation problem is of particular importance in radio physics, radio astronomy, hydro acoustics, seismology and geology, as well as in many radio engineering applications, such as radio communication, radio control, telemetry, radiolocation and radio navigation. Technical diagnostics and process control are also the fields where this problem is a challenge [4]-[8]. One of the most common methods of synthesizing the algorithm for estimating the parameters of the signals against interferences is the maximum likelihood (ML) method [7]-[10]. Application of ML method allows us to obtain both simple and sufficiently effective algorithms for the estimation of the information signal parameters. A special advantage of such algorithms is that they require minimum amount of prior information. However, the final conclusion about the appropriateness of the maximum likelihood estimates (MLEs) application for the solution of the certain practical tasks should be made only on the basis of the analysis of the estimate characteristics.

To a great extent, the very possibility of the practical application of the common methods for calculating the characteristics of the joint estimates of the signal parameters depends upon the analytical properties of the decision statistics of the examined algorithm. In particular, when analyzing the accuracy of MLEs, what we consider to be a fundamental characteristic is the regularity of the logarithm of the functional of the likelihood ratio (FLR) as the function of the estimated signal parameters, which is the decision statistics in this case [9]-[15]. That means that, if, at least, the second derivatives of the first two moments of the logarithm of FLR by the specified parameter exist, then the logarithm of FLR is regular by this parameter [9]-[14]. The signal parameters that satisfy these conditions of regularity are called the regular ones [10], [11]. By applying the small parameter method [7]-[9] or the Ibragimov-Has'minskii method [12]-[15] we can find the asymptotically exact (with increasing the signal-to-noise ratio (SNR)) expressions for the characteristics of the joint MLEs of the regular parameters of the signal observed against Gaussian noise.

At the same time, there is a wide class of the signals commonly known as discontinuous, or nonanalytical [10]-[14]. By certain discontinuous signal parameters the conditions of regularity for the logarithm of FLR are not satisfied. Following [11]-[16] we will call such parameters the discontinuous ones. The simplest examples of the discontinuous parameters are the time of arrival and the duration of rectangular video and radio pulses, the band center of the signal with the uniform band spectrum, the time delay of some discrete complex signals, etc. [11], [14], [17]-[21]. The small parameter method [7]-[9] cannot be applied to find the MLEs characteristics of the discontinuous parameters, because it presupposes the regularity of the logarithm of FLR. Otherwise, we may get, for example, the zero value of the variance of the discontinuous parameter estimate and a number of other incorrect results.

In order to calculate the asymptotically exact (with SNR increasing) expressions for the MLEs characteristics of the discontinuous signal parameter, the local Markov approximation (LMA) method can be used [10], [11], [19]. The idea of the LMA method is to approximate the logarithm of FLR, or its increment, by Markov or local Markov random process. Then, by applying the mathematical apparatus [24] of the Markov processes theory, we can obtain the asymptotically exact expressions for the MLEs characteristics of the discontinuous signal parameters, including the anomalous effects. However, the LMA method is only applicable for calculating the characteristics of the separate MLEs of the discontinuous signal parameters. Here the separate estimate refers to the estimate found in case when other discontinuous signal parameters are a priori known. The LMA method does not apply, if we calculate the characteristics of the joint estimates of several discontinuous parameters.

In practice, signal processing is implemented, as a rule, in the conditions of prior uncertainty when we are unaware of both the informative signal parameters to be estimated and some other (spurious) parameters. Then we have to carry out a joint estimation of several unknown signal parameters which may be discontinuous. In [16], [22], the procedure is proposed that makes it possible to calculate the asymptotic (with SNR increasing) characteristics of the joint MLEs of one discontinuous and several regular parameters. In [13]-[15], based on the Ibragimov-Has'minskii method generalized by Y.A. Kutoyants, the procedure is presented that helps to describe how the joint Bayesian and ML estimates of the discontinuous parameters converge in the distribution by the specified random variables. The said approach also provides determining the rate of this convergence under SNR tending to infinity. However, the universal methods for obtaining the analytical expressions for the characteristics of the joint MLEs of the several discontinuous signal parameters are still unknown. At present, the approximate formulas for the characteristics of the joint estimates of the discontinuous parameters can be obtained for certain special tasks only [17], [20], [23]. That is why it is difficult to analyze and compare the performance of measuring systems when using the discontinuous signal models.

The specified difficulties faced while calculating the characteristics of the joint MLEs of the discontinuous signal parameters can be overcome, if the moments of the logarithm of FLR allow the additive-multiplicative representation. In this case, the mathematical expectation, the correlation function and some other moments of the logarithm of FLR are expressed as the sums of the finite number of summands, each of which is the product of the functions of one parameter only. Then, in order to find the asymptotically exact (with SNR increasing) expressions for the characteristics of the joint MLEs of the discontinuous signal parameters, we can apply the local additive approximation (LAA) method that is considered below. The LAA method allows us to reduce the problem of calculating the characteristics of the joint estimates of the discontinuous signal parameters to the simpler problem of finding the characteristics of the separate estimates of the corresponding parameters. At that, to get the characteristics of the separate MLEs of the discontinuous signal parameters, we apply the LMA method taking into account necessary generalizations.

The additive-multiplicative representation of the moments of the logarithm of FLR is possible for a wide class of various signals parameters. For example, when estimating the parameters of the stochastic Gaussian pulse [4], [18]-[21] occurring in radio and hydrolocation, communications, radio astronomy, etc., such representation of the moments of the logarithm of FLR is carried out by time and frequency parameters of the pulse. These parameters include time of arrival, duration, moments of appearance and disappearance of the pulse as well as band center and bandwidth of the spectral density of its random substructure.

Below, on the basis of the LAA method, the asymptotically exact (with SNR increasing) expressions are obtained for the characteristics of the joint MLEs of the discontinuous signal parameters, while the additive-multiplicative representation of the moments of the logarithm of FLR by the estimated parameters is valid. To determine the MLEs characteristics, we find the probability distribution of the position of the absolute (greatest) maximum of the logarithm of FLR being the random field and calculate the statistical moments of this distribution [9]-[12].

II. Maximum Likelihood Estimates and Local Representations of the Moments of Decision Statistics

Let the mix $x(t)$ of the information signal $s(t,\mathbf{l}_0)$ and the noise $n(t)$ is passed to the input of the processing unit. Here $\mathbf{l}_0 = \|l_{01}, l_{02}, \ldots, l_{0p}\|$ is the p-dimensional vector of the unknown signal parameters which is a priori unknown and possesses the values from the specified domain of the definition $\Re$. Based on the observable realization $x(t)$ and the available prior information, it is necessary to measure (estimate) the parameters l_{0i}, $i = 1,2,\ldots,p$ of the received signal $s(t,\mathbf{l}_0)$.

We designate $L(\mathbf{l}) \equiv L(l_1, l_2, \ldots, l_p)$ the logarithm of FLR being the functional from observed data $x(t)$ as the function of the current values $\mathbf{l} = \|l_1, l_2, \ldots, l_p\|$ of the unknown parameters $\mathbf{l}_0$ [7]-[10]. Then, according to the definition [7]-[10], [12]-[14], the joint MLEs $l_{1m}, l_{2m}, \ldots, l_{pm}$ of the parameters $l_{01}, l_{02}, \ldots, l_{0p}$ are the coordinates of the position of the absolute maximum of the logarithm of FLR $L(l_1, l_2, \ldots, l_p)$ within the prior domain $\Re$. As a result, the vector of the signal parameters joint MLEs $\mathbf{l}_m = \|l_{1m}, l_{2m}, \ldots, l_{pm}\|$ can be written in the form of

$$\mathbf{l}_m = \arg\sup_{\mathbf{l}\in\Re} L(\mathbf{l}). \tag{1}$$

The probability characteristics of the MLEs (1) are uniquely defined by the statistical properties of the decision statistics of the estimation algorithm, i.e. the logarithm of FLR $L(\mathbf{l})$. Therefore, we will now consider the characteristics of the logarithm of FLR.

Following [9]-[12], [16]-[19], we presuppose that the logarithm of FLR $L(\mathbf{l})$ is Gaussian random field. We present the functional $L(\mathbf{l})$ as the sum $L(\mathbf{l}) = S(\mathbf{l}) + N(\mathbf{l})$ of the signal (deterministic) component $S(\mathbf{l}) = \langle L(\mathbf{l})\rangle$ and the noise (fluctuating) component $N(\mathbf{l}) = L(\mathbf{l}) - \langle L(\mathbf{l})\rangle$, where $\langle\ \rangle$ means the averaging over all the possible realizations of the logarithm of FLR under fixed true values of $\mathbf{l}_0 \in \Re$ of the estimated signal parameters [9], [10], [18]. Then, while calculating the characteristics of the joint MLEs (1), we see that, as the logarithm of FLR has Gaussian character, we may confine its study to analyzing its first two moments – signal component $S(\mathbf{l})$ and the correlation function $K(\mathbf{l}_1, \mathbf{l}_2) = \langle N(\mathbf{l}_1) N(\mathbf{l}_2)\rangle$ of the noise component $N(\mathbf{l})$. Here the designations are: $\mathbf{l}_j = \|l_{j1}, l_{j2}, \ldots, l_{jp}\|$, $j = 1,2$.

If any random variable is added to the logarithm of FLR, then the values of the MLEs do not change. Therefore, the estimates (1) can be always presented in the form of $\mathbf{l}_m = \arg\sup_{\mathbf{l}\in\Re} \Delta(\mathbf{l})$, where $\Delta(\mathbf{l}) \equiv \Delta(l_1, l_2, \ldots, l_p) = L(\mathbf{l}) - L(\mathbf{l}^*)$ is Gaussian functional of the increments of the logarithm of FLR and $\mathbf{l}^* = \|l_1^*, l_2^*, \ldots, l_p^*\|$ is some fixed value of the vector of the parameters $\mathbf{l}$. Therefore, when we calculate the characteristics of the MLEs (1) instead of the correlation function $K(\mathbf{l}_1, \mathbf{l}_2)$ of the logarithm of FLR $L(\mathbf{l})$, we turn to the correlation function $K_\Delta(\mathbf{l}_1, \mathbf{l}_2) = \langle[\Delta(\mathbf{l}_1) - \langle\Delta(\mathbf{l}_1)\rangle][\Delta(\mathbf{l}_2) - \langle\Delta(\mathbf{l}_2)\rangle]\rangle$ of its increments $\Delta(\mathbf{l})$.

Let the signal component $S(\mathbf{l})$ has a unique maximum in the point $\mathbf{l} = \mathbf{l}_0$ of the true values of the estimated signal parameters, while $A_S = S(\mathbf{l}_0) > 0$ and the realizations of the noise component $N(\mathbf{l})$ are continuous with the probability equal to 1. In practice, these conditions are usually satisfied [9]-[12], [17]. Then the output SNR for the estimation algorithm (1) can be written in the form of

$$z = S(\mathbf{l}_0) \Big/ \sqrt{\langle N^2(\mathbf{l}_0)\rangle} = A_S / \sigma_N , \tag{2}$$

where $\sigma_N^2 = \langle N^2(\mathbf{l}_0)\rangle$ is the dispersion of the noise component under $\mathbf{l} = \mathbf{l}_0$. We presuppose that SNR (2) is so big that the high posterior accuracy of the estimates can be achieved [9]-[11]. In this case, the MLEs $\mathbf{l}_m$ (1) are located in the small neighborhood of the point $\mathbf{l} = \mathbf{l}_0$ of the maximum of the signal component, and the estimate $\mathbf{l}_m$ converges to $\mathbf{l}_0$ in mean square [9], [10], [12]. Thus, to determine the characteristics of the MLEs (1), it is sufficient to study the behavior of the signal component $S(\mathbf{l})$ and the correlation function $K_\Delta(\mathbf{l}_1, \mathbf{l}_2)$ of the increments of the logarithm of FLR in the small neighborhood of the point $\mathbf{l} = \mathbf{l}_0$. The size of this neighborhood decreases with SNR z increasing.

It is well known that the analytical properties of the logarithm of FLR $L(\mathbf{l})$ in the neighborhood of the point $\mathbf{l} = \mathbf{l}_0$ depend on the fulfillment of the conditions of regularity of this functional by each of the estimated parameters l_i, $i = 1,2,\ldots,p$ [9]-[14]. Therefore, we specify the local (in the small neighborhood of the point $\mathbf{l} = \mathbf{l}_0$) representations of the first two moments of the logarithm of FLR while estimating the discontinuous signal parameters.

Now we pass to the common class of the discontinuous parameters. For it the sections $S_i(l_i) = S(\mathbf{l})\big|_{l_k = l_{0k},\, k=1,2,\ldots,p,\, k\neq i}$, $i = 1,2,\ldots,p$ of the signal function $S(\mathbf{l})$ by each of the parameters in the small neighborhood of the point $\mathbf{l} = \mathbf{l}_0$ of the maximum of the signal component allow the asymptotic representations

$$S_i(l_i)=A_S\begin{cases}1-d_{1i}|l_i-l_{0i}|+o(\delta_i),\ l_i<l_{0i}\,,\\ 1-d_{2i}|l_i-l_{0i}|+o(\delta_i),\ l_i\ge l_{0i}\end{cases} \tag{3}$$

under $\delta_i=|l_i-l_{0i}|\to 0$. The corresponding sections $K_{\Delta i}(l_{1i},l_{2i})=K_\Delta(\mathbf{l}_1,\mathbf{l}_2)|_{l_{jk}=l_{0k},\,k=1,2,\ldots,p,\,k\neq i,\,j=1,2}$ of the correlation function $K_\Delta(\mathbf{l}_1,\mathbf{l}_2)$ of the increments of the logarithm of FLR under $\delta_{ni}^*=\max(|l_{1i}-l_{0i}|,|l_{2i}-l_{0i}|,|l_i^*-l_{0i}|)\to 0$ take the form of

$$K_{\Delta i}(l_{1i},l_{2i})=\begin{cases}B_{1i}\min(|l_{1i}-l_i^*|,|l_{2i}-l_i^*|)+C_{1i}+o(\delta_{ni}^*),\\ \qquad\qquad l_{1i},l_{2i}<l_{0i}\,,\\ B_{2i}\min(|l_{1i}-l_i^*|,|l_{2i}-l_i^*|)+C_{2i}+o(\delta_{ni}^*),\\ \qquad\qquad l_{1i},l_{2i}\ge l_{0i}\,,\end{cases} \tag{4}$$

if $(l_{1i}-l_i^*)(l_{2i}-l_i^*)\ge 0$, while $K_{\Delta i}(l_{1i},l_{2i})=0$, $(l_{1i}-l_i^*)(l_{2i}-l_i^*)<0$. Here $d_{ki}>0$, $B_{ki}>0$, $C_{ki}\ge 0$, $k=1,2$, $i=1,2,\ldots,p$ are some constants and $o(\delta)$ denotes the higher-order infinitesimal terms compared with δ. From (3), (4), we can see that the moments of the decision statistics $\Delta(\mathbf{l})$ are nondifferentiable by the discontinuous parameters at the point $\mathbf{l}=\mathbf{l}_0$ as the derivatives of the functions (3), (4) at this point have discontinuities of the first kind. This condition does not allow us to apply the small parameter method [7]-[9] to calculate the characteristics of the MLEs of the discontinuous signal parameters.

We will notice that the correlation function $K(\mathbf{l}_1,\mathbf{l}_2)$ of the logarithm of FLR satisfies the expression (4), if, in particular, its sections $K_i(l_{1i},l_{2i})=K(\mathbf{l}_1,\mathbf{l}_2)|_{l_{jk}=l_{0k},\,k=1,2,\ldots,p,\,k\neq i,\,j=1,2}$ under $\delta_{ni}=\max(|l_{1i}-l_{0i}|,|l_{2i}-l_{0i}|)\to 0$ allow the following asymptotic representations:

$$\begin{aligned}K_i(l_{1i},l_{2i})&=D_{1i}\min(l_{1i},l_{2i})+D_{2i}\min(l_{1i},l_{2i},l_{0i})+\\&+f_{1i}(l_{1i},l_{0i})+f_{2i}(l_{2i},l_{0i})+B_{0i}+o(\delta_{ni}).\end{aligned} \tag{5}$$

Here $D_{1i}>0$, $D_{2i}\ge 0$, B_{0i} are some constants and the functions $f_{ki}(l_{ki},l_{0i})$, $k=1,2$ usually satisfy the condition $f_{ki}(l_{0i},l_{0i})=0$. In this case, in (4), we have $B_{1i}=D_{1i}+D_{2i}$, $B_{2i}=D_{1i}$. The representation (5) of the correlation function of the logarithm of FLR is frequently found when estimating the discontinuous power signal parameter such as, for example, duration, bandwidth, et. al. [10], [11], [17], [18].

Another example of the correlation function satisfying the expression (4) is the function with the sections allowing the asymptotic representation like this:

$$K_i(l_{1i},l_{2i})=\sigma_N^2\begin{cases}1-\rho_i|l_{1i}-l_{2i}|-\\ -g_i\min(|l_{1i}-l_{0i}|,|l_{2i}-l_{0i}|)+o(\delta_i),\\ \qquad\qquad (l_{1i}-l_{0i})(l_{2i}-l_{0i})\ge 0\,,\\ 1-\rho_i|l_{1i}-l_{2i}|+o(\delta_i),\\ \qquad\qquad (l_{1i}-l_{0i})(l_{2i}-l_{0i})<0\,,\end{cases} \tag{6}$$

where $\sigma_N>0$, $\rho_i>0$, $g_i\ge 0$. Then, in (4), we should set $C_{1i}=C_{2i}=0$, $B_{1i}=B_{2i}=\sigma_N^2(2\rho_i-g_i)$. The representation (6) is frequently found when estimating the discontinuous non-power signal parameter such as, for example, time of arrival, band center [10]-[12], [18]-[22].

We find the expressions (3), (4) general enough and including a wide class of the discontinuous signal parameters. Assuming in (3), (4) that $A_S=z$, $d_{ki}=d$, $B_{ki}=2d$, $C_{ki}=0$, $k=1,2$ we then obtain, as special case, the asymptotic representations of the moments of the normalized logarithm of FLR, while estimating the discontinuous parameters of quasi-deterministic signals, [10], [11], [16], [17]. We also see that the general expressions (3), (4) for the moments of the logarithm of FLR are valid when estimating the discontinuous parameters of the Gaussian pulse signals [18]-[21]. Such parameters may include time of arrival, duration, moments of appearance and disappearance of a pulse as well as band center and bandwidth of the spectral density of its random substructure.

As a condition for applicability of the LAA method, we presuppose that the signal component $S(\mathbf{l})$ and the correlation function $K(\mathbf{l}_1,\mathbf{l}_2)$ of the logarithm of FLR $L(\mathbf{l})$ in the small neighborhood of the point $\mathbf{l}=\mathbf{l}_0$ allow the additive-multiplicative representations

$$S(\mathbf{l})=\sum_{k=1}^{u}\sum_{j=1}^{a_k}\prod_{i=t_{jk}+1}^{t_{(j+1)k}}V_{ki}(l_i),\ K(\mathbf{l}_1,\mathbf{l}_2)=\sum_{k=1}^{r}\sum_{j=1}^{v_k}\prod_{i=\theta_{jk}+1}^{\theta_{(j+1)k}}U_{ki}(l_{1i},l_{2i}), \tag{7}$$

where $0=t_{1k}<t_{2k}<\ldots<t_{(a_k+1)k}=p$, $0=\theta_{1k}<\theta_{2k}<\ldots<\theta_{(v_k+1)k}=p$. Here the derivatives of the functions $V_{ki}(l_i)$, $U_{ki}(l_{1i},l_{2i})$, $i=1,2,\ldots,p$ are continuous on the left and on the right from the point l_{0i}, but they can have discontinuities of the first kind at this point.

In special case, when $a_k=v_k=1$, from (7) we get

$$S(\mathbf{l})=\sum_{k=1}^{u}\prod_{i=1}^{p}V_{ki}(l_i),\qquad K(\mathbf{l}_1,\mathbf{l}_2)=\sum_{k=1}^{r}\prod_{i=1}^{p}U_{ki}(l_{1i},l_{2i}). \tag{8}$$

If $a_k=v_k=1$ and $u=r=1$, then the moments (7) factorize by the estimated signal parameters and allow the multiplicative representation

$$S(\mathbf{l})=\prod_{i=1}^{p}V_{1i}(l_i)=\prod_{i=1}^{p}S_i(l_i),$$
$$K(\mathbf{l}_1,\mathbf{l}_2)=\prod_{i=1}^{p}U_{1i}(l_{1i},l_{2i})=\prod_{i=1}^{p}K_i(l_{1i},l_{2i}),$$

where $S_i(l_i)=V_{1i}(l_i)$ and $K_i(l_{1i},l_{2i})=U_{1i}(l_{1i},l_{2i})$ are the sections of the signal component and the correlation function of the noise component of the logarithm of FLR. Finally, under $a_k=v_k=p$, the additive-multiplicative representation (7) of the moments of the logarithm of FLR transforms into the additive representation

$$S(\mathbf{l})=\sum_{k=1}^{u}\sum_{i=1}^{p}V_{ki}(l_i)=\sum_{i=1}^{p}S_i(l_i)-A_S(p-1),$$
$$K(\mathbf{l}_1,\mathbf{l}_2)=\sum_{k=1}^{r}\sum_{i=1}^{p}U_{ki}(l_{1i},l_{2i})=\sum_{i=1}^{p}K_i(l_{1i},l_{2i})-\sigma_N^2(p-1). \quad (9)$$

III. The Characteristics of the Maximum Likelihood Estimates of the Discontinuous Signal Parameters

Let us find the asymptotically exact (with SNR (2) increasing) expressions for the characteristics of the joint MLEs (1) of the discontinuous parameters l_i, $i=1\ldots p$ when the additive-multiplicative representation (7) of the moments of the logarithm of FLR holds. We presuppose that SNR (2) is so big that the high posterior accuracy of estimates is achieved [9], [10], [19]. For the calculation of the characteristics of the MLEs (1), it is sufficient to consider the local behavior of the moments of the logarithm of FLR $L(\mathbf{l})$ in the small neighborhood of the point $\mathbf{l}=\mathbf{l}_0$ [10], [18]. We expand the functions $V_{ki}(l_i)$ and $U_{ki}(l_{1i},l_{2i})$ into the Taylor series on the left and on the right from the points l_{0i}. By substituting these expansions in (7) and taking into account the summands of the first infinitesimal order by δ_i (or by δ_{ni}) only, we then obtain

$$S(\mathbf{l})=\sum_{i=1}^{p}S_i(l_i)-A_S(p-1)+o(\delta)$$
$$\text{under } \delta=\max_{i=1,2,\ldots,p}\delta_i\to 0,$$
$$K(\mathbf{l}_1,\mathbf{l}_2)=\sum_{i=1}^{p}K_i(l_{1i},l_{2i})-\sigma_N^2(p-1)+o(\delta_n) \quad (10)$$
$$\text{under } \delta_n=\max_{i=1,2,\ldots,p}\delta_{ni}\to 0.$$

Thus, the moments of the logarithm of FLR $L(\mathbf{l})$ allow the locally additive representation (10) in the small neighborhood of the point $\mathbf{l}_0$.

Now we introduce $M_i(l_i)$, $i=1,2,\ldots,p$ that are the statistically independent Gaussian random processes for which the mathematical expectations $S_{Mi}(l_i)$ and the correlation functions $K_{Mi}(l_{1i},l_{2i})$ in the neighborhoods of the points $l_i=l_{0i}$ are expressed in the form of

$$S_{Mi}(l_i)=S_i(l_i)-A_S(p-1)/p,$$
$$K_{Mi}(l_{1i},l_{2i})=K_i(l_{1i},l_{2i})-\sigma_N^2(p-1)/p. \quad (11)$$

Here the functions $S_i(l_i)$ and $K_i(l_{1i},l_{2i})$ satisfy the conditions (3), (4). Then, according to (10), we get the random field $L(\mathbf{l})$ that converges in distribution to the sum $M(\mathbf{l})=\sum_{i=1}^{p}M_i(l_i)$ of the statistically independent random processes $M_i(l_i)$ under $\delta\to 0$. As it is noted above, the characteristics of the MLEs (1) under the big SNR z are defined by the behavior of the logarithm of FLR $L(\mathbf{l})$ in the small neighborhood of the point $\mathbf{l}=\mathbf{l}_0$. The size of this neighborhood vanishes, if $z\to\infty$. We assume that SNR z is so big and the sizes δ_i of the specified neighborhoods of the points $l_i=l_{0i}$ are so small that, within the intervals $l_i\in[l_{0i}-\delta_i,l_{0i}+\delta_i]$, the representations (11) of the moments of the random processes $M_i(l_i)$ are valid. Then the joint probability density $W(l_1,l_2,\ldots,l_p)$ of the estimates $l_{1m},l_{2m},\ldots,l_{pm}$ (1) can be approximated by the product

$$W(l_1,l_2,\ldots,l_p)=\prod_{i=1}^{p}W_i(l_i) \quad (12)$$

of the probability densities $W_i(l_i)$ of the separate estimates

$$l_{ir}=\underset{l_i\in[l_{0i}-\delta_i,l_{0i}+\delta_i]}{\arg\sup}M_i(l_i),\qquad i=1,2,\ldots,p. \quad (13)$$

Here δ_i is the size of the neighborhood of the points $l_i=l_{0i}$ and $\delta_i\to 0$ while $z\to\infty$. The accuracy of the representations (12), (13) under the fixed δ_i increases with SNR z.

Thus, the characteristics of the joint MLEs l_{im} (1) of the discontinuous parameters l_{0i} coincide asymptotically (with SNR increasing) with the corresponding characteristics of the separate estimates l_{ir} (13) of the same parameters.

In order to find the characteristics of the separate estimates l_{ir} (13) of the discontinuous signal parameters, we apply the LMA method [10], [11], [19]. In [10], [11], [19], by means of the LMA method, the asymptotically exact (with SNR increasing) expressions are obtained for the distribution function of the single MLE of the discontinuous parameter of quasi-deterministic or stochastic signal as well as for the conditional bias and variance of the estimate. In this, it is considered that the mathematical expectation and the correlation function of the logarithm of FLR allow the representations (3), (4), where $A_S=z$, $d_{ki}=d$, $B_{ki}=2d$, $C_{ki}=0$, $k=1,2$. Further we will obtain the asymptotically exact expressions for the characteristics of the estimates l_{ir} (13) in a general case when the coefficients $d_{ki}>0$, $B_{ki}>0$, $C_{ki}\geq 0$ in (3), (4) are arbitrary.

Let us introduce Gaussian random processes $\Delta_i(l_i)=M_i(l_i)-M_i(l_i^*)$, $l_i^* \in [l_{0i}-\delta_i, l_{0i}+\delta_i]$ with mathematical expectations $S_i(l_i)-S_i(l_i^*)$ and correlation functions $K_{\Delta i}(l_{1i},l_{2i})$. In this case, the distribution function of the estimate l_{ir} (13) can be presented in the form of [10], [11], [19]

$$F_i(l_i^*)=P[l_{ir}<l_i^*]=P[\sup_{l_i\in[l_{0i}-\delta_i,l_i^*)}\Delta_i(l_i)>\sup_{l_i\in[l_i^*,l_{0i}+\delta_i]}\Delta_i(l_i)],$$

where $P[A]$ is the probability of the event *A*. From (4), it follows that the segments of the realizations of the random processes $\Delta_i(l_i)$ within the intervals $[l_{0i}-\delta_i,l_i^*)$, $[l_i^*,l_{0i}+\delta_i]$ are not correlated, and therefore they are statistically independent, as being Gaussian. Then [10], [11]

$$F_i(l_i^*)=\int_0^\infty P_{2i}(u)\,\mathrm{d}P_{1i}(u)=1-\int_0^\infty P_{1i}(u)\,\mathrm{d}P_{2i}(u), \tag{14}$$

where

$$\begin{aligned}P_{1i}(u)&=P[\sup_{l_i\in[l_{0i}-\delta_i,l_i^*)}\Delta_i(l_i)<u\,],\\ P_{2i}(u)&=P[\sup_{l_i\in[l_i^*,l_{0i}+\delta_i]}\Delta_i(l_i)<u\,]\end{aligned} \tag{15}$$

are the distribution functions of the absolute maxima of the random process $\Delta_i(l_i)$ within the intervals $[l_{0i}-\delta_i,l_i^*)$ and $[l_i^*,l_{0i}+\delta_i]$, respectively. In (14), we take into account that $\sup\Delta_i(l_i)\ge 0$, therefore, $P_{1i}(u)=0$ and $P_{2i}(u)=0$, if $u<0$.

Now we need to obtain the expression for the function $P_{2i}(u)$, $u\ge 0$ (15). For this purpose, we introduce the random processes $r_i(l_i)=u-\Delta_i(l_i)$. By applying the Doob's theorem [25] in the Kailath's wording [26] and taking into account the representations (3), (4) of the moments of the logarithm of FLR, we can show that the random processes $\Delta_i(l_i)$ and $r_i(l_i)$ are the Gaussian Markov random processes of the diffusion type [24], [25] within the interval $[l_i^*,l_{0i}+\delta_i]$. According to (3), (4), under $l_i>l_i^*$, the drift Γ_{1i} and the diffusion Γ_{2i} coefficients of the processes $r_i(l_i)$ are equal to

$$\Gamma_{1i}=\begin{cases}A_S d_{2i}, & l_i<l_{0i},\\ -A_S d_{1i}, & l_i\ge l_{0i},\end{cases}\qquad \Gamma_{2i}=\begin{cases}B_{2i}, & l_i<l_{0i},\\ B_{1i}, & l_i\ge l_{0i}.\end{cases} \tag{16}$$

Through applying Markov properties of the process $r_i(l_i)$ as described in [24], we express the probability $P_{2i}(u)$ (15) in the form of

$$P_{2i}(u)=P[\,\underset{l_i\in[l_i^*,l_{0i}+\delta_i]}{r_i(l_i)}>0\,]=\int_0^\infty W_{ri}(x,l_{0i}+\delta_i)\,\mathrm{d}x, \tag{17}$$

where $W_{ri}(x,l)$ is the solution of the direct Fokker-Planck-Kolmogorov equation [25]

$$\frac{\partial W_{ri}(x,l)}{\partial l}-\frac{\partial}{\partial x}[\Gamma_{1i}(l)W_{ri}(x,l)]-\frac{1}{2}\frac{\partial^2}{\partial x^2}[\Gamma_{2i}(l)W_{ri}(x,l)]=0 \tag{18}$$

with the coefficients (16) under the starting condition $W_{ri}(x,l_i^*)=\delta(x-u)$ and the boundary conditions $W_{ri}(0,l)=0$, $W_{ri}(\infty,l)=0$. Here $\delta(x)$ is the delta-function. After solving the equation (18) similarly [10], [11], [18], [19] and substituting the found solution into the formula (17), we obtain

$$\begin{aligned}P_{2i}(u)=&\frac{1}{\sqrt{2\pi B_{1i}(l_{0i}-l_i^*)}}\int_0^\infty\left\{\exp\left[-\frac{\left(u-A_S d_{1i}(l_{0i}-l_i^*)-\zeta\right)^2}{2B_{1i}(l_{0i}-l_i^*)}\right]-\right.\\ &\left.-\exp\left(\frac{2A_S d_{1i}}{B_{1i}}u\right)\exp\left[-\frac{\left(u+A_S d_{1i}(l_{0i}-l_i^*)+\zeta\right)^2}{2B_{1i}(l_{0i}-l_i^*)}\right]\right\}\times\\ &\times\left\{\Phi\left(\frac{A_S d_{2i}\delta_i+\zeta}{\sqrt{B_{2i}\delta_i}}\right)-\exp\left(-\frac{2A_S d_{2i}}{B_{2i}}\zeta\right)\Phi\left(\frac{A_S d_{2i}\delta_i-\zeta}{\sqrt{B_{2i}\delta_i}}\right)\right\}\mathrm{d}\zeta\end{aligned} \tag{19}$$

under $l_{0i}-\delta_i\le l_i^*<l_{0i}$ and

$$\begin{aligned}P_{2i}(u)=&\Phi\left[\frac{A_S d_{2i}(l_{0i}+\delta_i-l_i^*)+u}{\sqrt{B_{2i}(l_{0i}+\delta_i-l_i^*)}}\right]-\\ &-\exp\left(-\frac{2A_S d_{2i}}{B_{2i}}u\right)\Phi\left[\frac{A_S d_{2i}(l_{0i}+\delta_i-l_i^*)-u}{\sqrt{B_{2i}(l_{0i}+\delta_i-l_i^*)}}\right]\end{aligned} \tag{20}$$

under $l_{0i}\le l_i^*\le l_{0i}+\delta_i$. Here $\Phi(x)=\int_{-\infty}^x\exp(-t^2/2)\mathrm{d}t/\sqrt{2\pi}$ is the probability integral.

It is not difficult to show that the probability $P_{1i}(u)$ (15) is also determined from (19), (20) where the coefficients d_{ji}, $j=1,2$ should be changed for $d_{(3-j)i}$, the coefficients B_{ji}, $j=1,2$ – for $B_{(3-j)i}$ and the difference $l_{0i}-l_i^*$ – for $l_i^*-l_{0i}$.

Then we substitute the expressions (19), (20) that have been produced for the probabilities $P_{1i}(u)$ and $P_{2i}(u)$ in the formula (14) and find the conditional (under the fixed l_{0i}) distribution functions $F_i(l_i)$ of the estimates l_{ir} (13):

$$F_i(l_i)=\begin{cases}\Psi(|l_i-l_{0i}|,z_{1i},z_{2i},\chi_i), & l_{0i}-\delta_i\le l_i<l_{0i},\\ 1-\Psi(|l_i-l_{0i}|,z_{1i},z_{2i},1/\chi_i), & l_{0i}\le l_i\le l_{0i}+\delta_i,\end{cases} \tag{21}$$

where

$$\Psi(l_i, z_{1i}, z_{2i}, \chi_i) = \frac{1}{\sqrt{2\pi|l_i|}} \int_0^\infty \int_0^\infty \left\{ \exp\left[-\frac{(u - z_{1i}|l_i| - \zeta)^2}{2|l_i|} \right] - \right.$$
$$\left. - \exp(2z_{1i}u) \exp\left[-\frac{(u - z_{1i}|l_i| - \zeta)^2}{2|l_i|} \right] \right\} \left\{ 2z_{1i} \exp(-2z_{1i}u) \times \right.$$
$$\left. \times \Phi\left[\frac{z_{1i}(\delta_i - |l_i|) - u}{\sqrt{\delta_i - |l_i|}} \right] + \sqrt{\frac{2}{\pi(\delta_i - |l_i|)}} \exp\left[-\frac{(z_{1i}(\delta_i - |l_i|) + u)^2}{2(\delta_i - |l_i|)} \right] \right\} \times$$
$$\times \left[\Phi\left(\frac{z_{2i}\delta_i + \zeta\chi_i}{\sqrt{\delta_i}} \right) - \exp(-2z_{2i}\chi_i\zeta) \Phi\left(\frac{z_{2i}\delta_i - \zeta\chi_i}{\sqrt{\delta_i}} \right) \right] du d\zeta,$$

and $z_{1i} = A_S d_{1i} / \sqrt{B_{1i}}$, $z_{2i} = A_S d_{2i} / \sqrt{B_{2i}}$, $\chi_i = \sqrt{B_{1i}/B_{2i}}$.

In practical calculations, the expression (21) appears to be rather complicated and inconvenient. In this connection, we take into account that the ratios z_{1i} and z_{2i} in (21) are the values of z order. By assuming that SNR z is very big, similarly to [10], [19], [24] we find the asymptotic expressions for the conditional (under the fixed l_{0i}) probability densities $W_i(l_i)$ of the estimates (13):

$$W_i(l_i) = \begin{cases} 2z_{1i}^2 W_0\left[2z_{1i}^2(l_{0i} - l_i), 1/R_i\right], & l_i < l_{0i}, \\ 2z_{2i}^2 W_0\left[2z_{2i}^2(l_i - l_{0i}), R_i\right], & l_i \ge l_{0i}, \end{cases} \quad i = 1, 2, \ldots, p, \quad (22)$$

$$W_0(x, u) = \Phi\left(\sqrt{\frac{|x|}{2}}\right) - 1 + \frac{2+u}{u} \exp\left(|x| \frac{1+u}{u^2}\right) \left[1 - \Phi\left(\sqrt{\frac{|x|}{2}} \frac{2+u}{u}\right)\right],$$

where $R_i = B_{1i} d_{2i} / B_{2i} d_{1i} = \chi_i z_{2i} / z_{1i}$. The accuracy of the expression (22) increases with SNR z (with the ratios z_{1i} and z_{2i}).

Thus, the joint probability density of the MLEs $l_{1m}, l_{2m}, \ldots, l_{pm}$ (1) is presented in the form of the product (12) of the probability densities (22) of the estimates (13). We apply the distributions (22) and find the conditional biases $b_i = \langle l_{im} - l_{0i} \rangle$ and variances $V_i = \langle (l_{im} - l_{0i})^2 \rangle$ of the estimates (1) as well as the third-order and the four-order moments $Y_i = \langle (l_{im} - l_{0i})^3 \rangle$ and $Q_i = \langle (l_{im} - l_{0i})^4 \rangle$ of the errors of these estimates:

$$b_i = \frac{2R_i + 1}{2z_{2i}^2 (R_i + 1)^2} - \frac{R_i (R_i + 2)}{2z_{1i}^2 (R_i + 1)^2},$$
$$V_i = \frac{5R_i^2 + 6R_i + 2}{2z_{2i}^4 (R_i + 1)^3} + \frac{R_i (2R_i^2 + 6R_i + 5)}{2z_{1i}^4 (R_i + 1)^3}, \quad (23)$$

$$Y_i = \frac{3}{4(R_i + 1)^4} \left[\frac{14R_i^3 + 28R_i^2 + 20R_i + 5}{z_{2i}^6} - \frac{R_i (5R_i^3 + 20R_i^2 + 28R_i + 14)}{z_{1i}^6} \right],$$
$$Q_i = \frac{3}{2(R_i + 1)^5} \left[\frac{42R_i^4 + 120R_i^3 + 135R_i^2 + 70R_i + 14}{z_{2i}^8} + \frac{R_i (14R_i^4 + 70R_i^3 + 135R_i^2 + 120R_i + 42)}{z_{1i}^8} \right]. \quad (24)$$

The accuracy of the formulas (23), (24) increases with SNR z (with z_{1i}, z_{2i}).

The formulas (22)-(24) become significantly simpler, if $d_{1i} = d_{2i} = d_i$, $B_{1i} = B_{2i} = B_i$. In this case, in (22) $z_{1i} = z_{2i} = z_i = A_S d_i / \sqrt{B_i}$, $R_i = 1$ and the probability densities $W_i(l_i)$ of the estimates (13) take the form of

$$W_i(l_i) = 2z_i^2 W_{01}\left(2z_i^2 |l_i - l_{0i}|\right), \qquad i = 1, 2, \ldots, p, \quad (25)$$
$$W_{01}(x) = \Phi\left(\sqrt{|x|/2}\right) - 1 + 3\exp(2|x|)\left[1 - \Phi\left(3\sqrt{|x|/2}\right)\right],$$

while the moments (23), (24) are being written as

$$b_i = 0, \quad V_i = 13/8z_i^4, \quad Y_i = 0, \quad Q_i = 1143/32z_i^8. \quad (26)$$

During the estimation of the discontinuous parameters of the quasi-deterministic signal we still have $z_i = z$ [10], [11], [19] and the expressions (26) get the following form: $b_i = 0$, $V_i = 13/8z^4$, $Y_i = 0$, $Q_i = 1143/32z^8$.

It is well known that the probability distributions of the joint MLEs of the regular signal parameters are the asymptotically Gaussian ones under $z \to \infty$ [9], [12]-[15]. From (22), it follows that the asymptotic distributions of the MLEs of the discontinuous signal parameters differ significantly from the Gaussian distribution. In particular, the coefficients of skewness γ_{1i} and excess γ_{2i} of the distribution (22) are equal to

$$\gamma_{1i} = \frac{\varepsilon_{3i}}{\sqrt{\varepsilon_{2i}^3}} = \frac{Y_i - 3V_i b_i + 2b_i^3}{\sqrt{(V_i - b_i^2)^3}},$$
$$\gamma_{2i} = \frac{\varepsilon_{4i}}{\varepsilon_{2i}^2} - 3 = \frac{Q_i - 4Y_i b_i + 6V_i b_i^2 - 3b_i^4}{(V_i - b_i^2)^2} - 3$$

and not equal to null, in general case. Here $\varepsilon_{ki} = \left\langle (l_{im} - \langle l_{im} \rangle)^k \right\rangle$ is the central k-th moment of the distribution (22). For example, under $R_i = 1$, $z_{1i} = z_{2i} = z$, we get $\gamma_{2i} = 1779/169 \approx 10.527$.

It should be noted that the moments of the random processes $\Delta_i(l_i)=M_i(l_i)-M_i(l_i^*)$ in the neighborhoods of the points $l_i=l_{0i}$ coincide with the corresponding moments of the random processes $\Delta_{mi}(l_i)=\Delta(\mathbf{l})|_{l_k=l_{0k},\,k=1,2,\ldots,p,\,k\neq i}$ being the sections of the functional of the increments $\Delta(\mathbf{l})=L(\mathbf{l})-L(\mathbf{l}^*)$ of the functional of FLR by the planes passing through the point $\mathbf{l}=\mathbf{l}_0$. Therefore, the statistical characteristics of both the estimates l_{ir} (13) and the joint MLEs l_{im} (1) coincide asymptotically (under SNR z increasing) with the characteristics of the separate MLEs of the signal parameters l_{0i}. It conforms to the similar conclusions drawn in [27].

IV. The Application of the Local Additive Approximation Method for Determining the Characteristics of the Estimated Signal parameters

A. The Joint Estimates of the Time of Arrival and the Duration of the Video Pulse

As the first example of the application of the LAA method, we consider the joint MLEs of the time of arrival λ_0 and the duration τ_0 of the rectangular video pulse with the amplitude a [8], [9], [17]:

$$s(t,\lambda_0,\tau_0)=a\,I\left(\frac{t-\lambda_0}{\tau_0}\right),\qquad I(x)=\begin{cases}1, & |x|\le 1/2,\\ 0, & |x|>1/2\end{cases} \tag{27}$$

observed against the Gaussian white noise $n(t)$ with the one-sided spectral density N_0.

According to [8], [9], [17] the logarithm of FLR for the realization of the observable data

$$x(t)=s(t,\lambda_0,\tau_0)+n(t) \tag{28}$$

against alternative $x(t)=n(t)$ is of the form

$$L(\lambda,\tau)=\frac{2a}{N_0}\int_{\lambda-\tau/2}^{\lambda+\tau/2}x(t)\,\mathrm{d}t-\frac{a^2\tau}{N_0}. \tag{29}$$

Here λ, τ are the current values of the unknown parameters λ_0, τ_0.

Let the time of arrival and the duration of the pulse (27) possess the values from the prior intervals $\lambda_0\in[\Lambda_1,\Lambda_2]$ and $\tau_0\in[\mathrm{T}_1,\mathrm{T}_2]$. Then the joint MLEs λ_m and τ_m of the time of arrival and the duration of the pulse are the coordinates of the position of the absolute maximum of the functional (29) within the intervals $\lambda\in[\Lambda_1,\Lambda_2]$, $\tau\in[\mathrm{T}_1,\mathrm{T}_2]$, i.e.

$$(\lambda_m,\tau_m)=\underset{\lambda\in[\Lambda_1,\Lambda_2],\,\tau\in[\mathrm{T}_1,\mathrm{T}_2]}{\arg\sup}L(\lambda,\tau). \tag{30}$$

By applying the LAA method, we find the characteristics of the joint MLEs λ_m and τ_m. For this purpose we present the functional (29) as the sum of signal $S(\lambda,\tau)=\langle L(\lambda,\tau)\rangle$ and noise $N(\lambda,\tau)=L(\lambda,\tau)-\langle L(\lambda,\tau)\rangle$ components: $L(\lambda,\tau)=S(\lambda,\tau)+N(\lambda,\tau)$. Performing averaging (29) over all the possible realizations of the observed data (28), for the signal component we find

$$\begin{aligned}S(\lambda,\tau)=\left(z_0^2/\tau_0\right)[\,&\max(0,\min(\lambda+\tau/2,\lambda_0+\tau_0/2)-\\&-\max(\lambda-\tau/2,\lambda_0-\tau_0/2)-\tau/2)\,].\end{aligned} \tag{31}$$

Here $z_0^2=2a^2\tau_0/N_0$ is the output SNR (2) for the algorithm (30) [9], [10]. The noise component $N(\lambda,\tau)$ is the Gaussian centered random field, its correlation function is equal to

$$\begin{aligned}\langle N(\lambda_1,\tau_1)N(\lambda_2,\tau_2)\rangle=(z_0/\tau_0)\max(0,\\ \min(\lambda_1+\tau_1/2,\lambda_2+\tau_2/2)-\max(\lambda_1-\tau_1/2,\lambda_2-\tau_2/2)).\end{aligned} \tag{32}$$

In (30)-(32) we pass to the new parameters

$$\theta_1=\lambda-\tau/2,\qquad \theta_2=\lambda+\tau/2. \tag{33}$$

Then the expressions (31), (32) are overwritten as follows

$$\begin{aligned}&S(\theta_1,\theta_2)=\left(z_0^2/2\right)\left[C(\tilde\theta_1-\tilde\theta_{01})+C(\tilde\theta_2-\tilde\theta_{02})-1\right],\\ &K(\theta_{11},\theta_{21},\theta_{12},\theta_{22})=\langle N(\theta_{11},\theta_{12})N(\theta_{21},\theta_{22})\rangle=\\ &=z_0\max\left(0,\min(\tilde\theta_{11},\tilde\theta_{21})+\min(\tilde\theta_{12},\tilde\theta_{22})-\tilde\theta_{11}-\tilde\theta_{21}\right),\end{aligned} \tag{34}$$

where $C(x)=\max(0,1-|x|)$, $\theta_{01}=\lambda_0-\tau_0/2$, $\theta_{02}=\lambda_0+\tau_0/2$, $\tilde\theta_k=\theta_k/\tau_0$, $\tilde\theta_{jk}=\theta_{jk}/\tau_0$, $j=0,1,2$, $k=1,2$.

From (34) we can see that the derivatives of the logarithm of FLR (29) by the current values $\tilde\theta_1$, $\tilde\theta_2$ of the normalized moments of appearance $\tilde\theta_{01}$ and disappearance $\tilde\theta_{02}$ of the pulse (27) have discontinuities of the first kind at the point $\tilde\theta_1=\tilde\theta_{01}$, $\tilde\theta_2=\tilde\theta_{02}$. Therefore, the moments of appearance θ_{01} and disappearance θ_{02} of the pulse (27) are the discontinuous parameters. In the conditions of high posterior accuracy, when $z_0^2>>1$, the sections $S_1(\tilde\theta_1)=S(\theta_1,\theta_{02})$, $S_2(\tilde\theta_2)=S(\theta_{01},\theta_2)$ and $K(\tilde\theta_{11},\tilde\theta_{21})=K(\theta_{11},\theta_{21},\theta_{02},\theta_{02})$, $K(\tilde\theta_{12},\tilde\theta_{22})=K(\theta_{01},\theta_{01},\theta_{12},\theta_{22})$ of the signal component and the correlation function (34) by the variables $\tilde\theta_1$, $\tilde\theta_2$ allow the representations (3), (5), where

$$A_S=z_0^2/2,\quad d_{1i}=d_{2i}=1,\quad D_{1i}=z_0,\quad D_{2i}=0,$$

$$f_{k1}\left(\widetilde{\theta}_{k1},\widetilde{\theta}_{01}\right)=z_0\left(\widetilde{\theta}_{01}-\widetilde{\theta}_{k1}\right),\quad f_{k2}\left(\widetilde{\theta}_{k2},\widetilde{\theta}_{02}\right)=0\,,\quad k=1,2\,,$$
$$B_{01}=z_0\left(\widetilde{\theta}_{02}-2\widetilde{\theta}_{01}\right),\qquad B_{02}=-z_0\widetilde{\theta}_{01}\,.$$

It should be also noted that the moments (34) of the decision statistics (29) allow the additive representation (9), where $p=2$, $\sigma_N^2=z_0$.

Thus, all the applicability conditions for the LAA method are satisfied and the asymptotic characteristics of the MLEs θ_{1m}, θ_{2m} of the moments of appearance θ_{01} and disappearance θ_{02} of the pulse (27) can be determined from (25), (26). Assuming in (26) that $z_i^2=z_0^2/2$, for the conditional biases $b_1=\langle\theta_{1m}-\theta_{01}\rangle$, $b_2=\langle\theta_{2m}-\theta_{02}\rangle$ and variances $V_1=\left\langle(\theta_{1m}-\theta_{01})^2\right\rangle$, $V_2=\left\langle(\theta_{2m}-\theta_{02})^2\right\rangle$ of the estimates θ_{1m}, θ_{2m} we get

$$b_1=b_2=0\,,\qquad V_1=V_2=26\tau_0^2/z_0^4\,. \tag{35}$$

The estimates θ_{1m}, θ_{2m} are the statistically independent ones and, according to (33), they are related to the estimates (30) by the linear transformations: $\lambda_m=(\theta_{1m}+\theta_{2m})/2$, $\tau_m=\theta_{2m}-\theta_{1m}$. Thus, taking into account (35), for the conditional biases and variances of MLEs λ_m, τ_m, we can write

$$\begin{gathered} b_\lambda=\langle\lambda_m-\lambda_0\rangle=0\,,\\ V_\lambda=\left\langle(\lambda_m-\lambda_0)^2\right\rangle=(V_1+V_2)/4=13\tau_0^2/z_0^4\,,\\ b_\tau=\langle\tau_m-\tau_0\rangle=0\,,\quad V_\tau=\left\langle(\tau_m-\tau_0)^2\right\rangle=V_1+V_2=52\tau_0^2/z_0^4\,. \end{gathered} \tag{36}$$

From (36) it follows that the variances of the joint estimates (30) is two times higher than the variances of the corresponding separate estimates [12], [28], i.e. there is a statistical dependence between the estimates λ_m and τ_m [17].

In order to establish the limits of applicability of the asymptotically exact formulas (36) for the characteristics of the estimates λ_m and τ_m, computer simulation of the algorithm (30) has been carried out for the case that $\Lambda_1=T_1=\tau_0/2$, $\Lambda_2=T_2=3\tau_0/2$, $\lambda_0=(\Lambda_1+\Lambda_2)/2$, $\tau_0=(T_1+T_2)/2$. During the simulation, the samples of the Gaussian random field (29) have been formed on the uniform two dimensional grid with the discretization step $10^{-3}\tau_0$, as it is described in [29]. Thus the mean-root-square error of stepwise approximations of continuous field realizations does not exceed 10 %. The number of processed realizations of the random field (29) was taken equal to 10^4. As a result, with probability of 0.9 confidence intervals boundaries deviate from experimental values no more than for 5...10 %.

In Fig. 1, curve 1 represents dependence (36) of the normalized variance $\widetilde{V}_\lambda=V_\lambda/\tau_0^2$ of the estimate λ_m from SNR z_0 (31), while curve 2 shows analogous dependence of the normalized variance $\widetilde{V}_\tau=V_\tau/\tau_0^2$ of the estimate τ_m. The experimental values of the variances $\widetilde{V}_\lambda$, $\widetilde{V}_\tau$ are designated by squares and crosses, respectively.

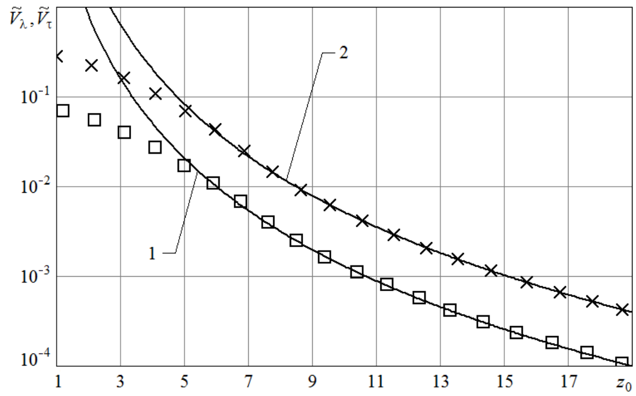

Fig. 1. The theoretical and experimental dependences of the variances of the estimates of the time of arrival and the duration of the video pulse.

As it follows from Fig. 1, theoretical dependences (36) for the variances of the estimates (30) well approximate experimental data, if SNR $z_0\ge 5$. Under $z_0<5$ the theoretical dependences for V_λ, V_τ (36) deviates from experimental data as the formulas (36) have been obtained without considering the finite lengths of the prior definition intervals $[\Lambda_1,\Lambda_2]$, $[T_1,T_2]$ of the parameters λ_0, τ_0. If necessary, a more accurate calculation of the theoretical values of the variances of the estimates can be made on the basis of the probability densities (25) using numerical integration formulas, as it is described, for example, in [30].

B. The Joint Estimates of the Time of Arrival and the Band Center of the Stochastic Radio Pulse

As the second example of the application of the LAA method, we consider the joint MLEs of the time of arrival λ_0 and the band center ν_0 of the stochastic radio pulse [18]-[21]

$$s(t,\lambda_0,\nu_0)=\xi(t)\,I[(t-\lambda_0)/\tau]. \tag{37}$$

Here $I(x)$ is the unit duration indicator (27), τ is the pulse duration, $\xi(t)$ is the high-frequency stationary centered Gaussian random process possessing the spectral density

$$G(\omega)=(\gamma/2)\{\,I[(\nu_0-\omega)/\Omega]+I[(\nu_0+\omega)/\Omega]\,\}. \tag{38}$$

In (38), the designations are: γ is the intensity, Ω is the bandwidth of the spectral density.

As before, we assume that the interferences and the registration errors are approximated by the Gaussian white noise $n(t)$ with the one-sided spectral density N_0, so that the mix

$$x(t)=s(t,\lambda_0,\tau_0)+n(t) \tag{39}$$

is available to observation. In addition, we presuppose that the correlation time of the process $\xi(t)$ is much less than the pulse (37) duration τ, i.e. the following condition is satisfied:

$$\mu=\tau\Omega/2\pi>>1\,. \tag{40}$$

As the examples of the signals described by the stochastic model (37), (38), (40) there can serve the reflected location signal, the radio pulse distorting by modulating interference, the signals in spectroscopy and astronomy [4], [8], [31], etc. The random signals (37) can be used as the noise carrier in data communication systems [32].

If the condition (40) holds, then the logarithm of FLR $L(\lambda,\nu)$ as the function of the current values λ, ν of the signal estimated parameters λ_0, ν_0 can be presented in the form of [18], [33]

$$L(\lambda,\nu)=\frac{qM(\lambda,\nu)}{N_0(1+q)}-\mu\ln(1+q)\,,\ M(\lambda,\nu)=\int_{\lambda-\tau/2}^{\lambda+\tau/2}y^2(t,\nu)\,\mathrm{d}t\,, \tag{41}$$

where $q=\gamma/N_0$, $y(t,\nu)=\int_{-\infty}^{\infty}x(t')h(t-t',\nu)\,\mathrm{d}t'$ is the response of the filter with pulse transition function $h(t,\nu)$ to the observed realization (39), while the transfer function $H(\omega,\nu)$ of this filter satisfies the condition $|H(\omega,\nu)|^2=I[(\nu-\omega)/\Omega]+I[(\nu+\omega)/\Omega]$. Then the joint MLEs λ_m and ν_m of the time of arrival and the band center of the pulse (37) are determined as follows

$$(\lambda_m,\nu_m)=\underset{\lambda\in[\Lambda_1,\Lambda_2],\,\nu\in[\Upsilon_1,\Upsilon_2]}{\arg\sup}M(\lambda,\nu). \tag{42}$$

Here $[\Lambda_1,\Lambda_2]$, $[\Upsilon_1,\Upsilon_2]$ are the prior intervals of the possible values of the estimated parameters λ_0, ν_0.

In order to find the characteristics of the joint MLEs λ_m and ν_m, we apply the LAA method. Firstly we introduce the designations: $\eta=\lambda/\tau$, $\eta_j=\lambda_j/\tau$, $\kappa=\nu/\Omega$, $\kappa_j=\nu_j/\Omega$, $j=0,1,2$. Then we present the functional $M(\lambda,\nu)$ (41) as the sum $M(\lambda,\nu)=S(\eta,\kappa)+N(\eta,\kappa)+B$, where $S(\eta,\kappa)=\langle M(\eta\tau,\kappa\Omega)\rangle-B$ is the signal component, $N(\eta,\kappa)=M(\eta\tau,\kappa\Omega)-S(\eta,\kappa)-B$ is the noise component and $B=\mu N_0$ is the inessential summand. In fulfilling (40), similarly to [18], [20], we obtain

$$S(\eta,\kappa)=S_0C(\eta-\eta_0)C(\kappa-\kappa_0), \tag{43}$$

where $S_0=\mu qN_0$ and the function $C(x)$ is determined in the same way as in (34).

The noise component $N(\eta,\kappa)$ is the asymptotically (under $\mu\to\infty$) Gaussian centered random field [18]. Therefore, while the condition (40) holds, we merely consider the correlation function of the noise component:

$$\begin{gathered}K(\eta_1,\eta_2,\kappa_1,\kappa_2)=\langle N(\eta_1,\kappa_1)N(\eta_2,\kappa_2)\rangle=\\=D_1R(\eta_1,\eta_2,\eta_0)R(\kappa_1,\kappa_2,\kappa_0)+D_0C(\eta_1-\eta_2)C(\kappa_1-\kappa_2),\end{gathered} \tag{44}$$

where $D_0=\mu N_0^2$, $D_1=\mu q(2+q)N_0^2$, $R(x_1,x_2,x_0)=\max\left[0,1-\max(|x_1-x_2|,|x_1-x_0|,|x_2-x_0|)\right]$.

According to (43) the signal component $S(\eta,\kappa)$ tops under $\eta=\eta_0$, $\kappa=\kappa_0$, i.e. in the point of the true values of the time of arrival and the band center of the received pulse. Then, the output SNR (2) takes the form of

$$z^2=S_0^2/(D_1+D_0)=\mu q^2/(1+q)^2\,. \tag{45}$$

From (43), (44), it follows that the derivatives of the moments of decision statistics $M(\lambda,\nu)$ (41) by the normalized variables η and κ have discontinuities of the first kind at the point $\eta=\eta_0$, $\kappa=\kappa_0$. Therefore, the time of arrival and the band center of the stochastic pulse (37) are the discontinuous parameters. When $z^2>>1$ (45), the sections $S_1(\eta)=S(\eta,\kappa_0)$, $S_2(\kappa)=S(\eta_0,\kappa)$ and $K_1(\eta_1,\eta_2)=K(\eta_1,\eta_2,\kappa_0,\kappa_0)$, $K_2(\kappa_1,\kappa_2)=K(\eta_0,\eta_0,\kappa_1,\kappa_2)$ of the signal component (43) and the correlation function of the noise component (44) by the current values of the time of arrival and the band center allow the representations (3), (6), where

$$\begin{gathered}A_S=S_0\,,\ \ d_{1i}=d_{2i}=1,\ \ \sigma_N^2=D_1+D_0=\mu N_0^2(1+q)^2,\\ \rho_i=1,\ \ g_i=g=D_1/\sigma_N^2=q(2+q)/(1+q)^2.\end{gathered} \tag{46}$$

As it can be seen, the moments (43), (44) of the decision statistics (41) allow the additive-multiplicative representation (8), where $u=1$, $r=2$, $p=2$, $V_{11}(l_1)=S_0C(\eta-\eta_0)$, $V_{12}(l_2)=C(\kappa-\kappa_0)$, $U_{11}(l_{11},l_{21})=D_1R(\eta_1,\eta_2,\eta_0)$, $U_{21}(l_{11},l_{21})=D_0C(\eta_1-\eta_2)$, $U_{12}(l_{12},l_{22})=R(\kappa_1,\kappa_2,\kappa_0)$, $U_{22}(l_{12},l_{22})=C(\kappa_1-\kappa_2)$.

Thus, all the applicability conditions for the LAA method are satisfied and the asymptotic characteristics of the MLEs λ_m and ν_m can be determined from (25), (26). Assuming in (26) that $z_i^2=z^2/(2-g)$, we get the conditional biases and variances of the joint MLEs of the time of arrival and the band center of the stochastic pulse (37):

$$\begin{gathered}b_1=\langle\lambda_m-\lambda_0\rangle=0\,,\ V_1=\langle(\lambda_m-\lambda_0)^2\rangle=13\tau^2(2-g)^2/8z^4\,,\\ b_2=\langle\nu_m-\nu_0\rangle=0\,,\ V_2=\langle(\nu_m-\nu_0)^2\rangle=13\Omega^2(2-g)^2/8z^4\,,\end{gathered} \tag{47}$$

where SNR z and the parameter g are determined from (45), (46). The accuracy of the formulas (47) increases with SNR (45).

In order to establish the limits of applicability of the asymptotically exact formulas (47) for the characteristics of the estimates λ_m, ν_m, computer simulation of the algorithm (42) has been carried out. During the simulation, following the procedure described in [29], the samples of the Gaussian random field with mathematical expectation (43) and correlation function (44) have been formed with the discretization step $\Delta t = 0.01$ by the variables η and κ. In this case, the mean-root-square error of stepwise approximations of continuous field realizations does not exceed 15 %. Thus one has the normalized estimates determined as the coordinates of the greatest field sample within the area of variables $\eta \in [0, m_1]$, $\kappa \in [0, m_2]$, where $m_1 = (\Lambda_2 - \Lambda_1)/\tau$, $m_2 = (\Upsilon_2 - \Upsilon_1)/\tau$. Based on processing no less than 10^4 realizations of the random field, the sample characteristics of the estimates η_m, κ_m have been calculated.

In Fig. 2, for $q = 0.1$, $m_1 = m_2 = 10$, $\eta_0 = \lambda_0/\tau = m_1/2$, $\kappa_0 = \nu_0/\Omega = m_2/2$ the experimental values of the normalized variance $\widetilde{V}_1 = V_1/\tau^2$ of the estimate λ_m in the absence of anomalous errors [19] are shown by squares. Note that they practically coincide with the similar values of the normalized variance $\widetilde{V}_2 = V_2/\Omega^2$ of the estimate ν_m. By solid line there is represented the corresponding theoretical dependence $\widetilde{V} = V_1(z)/\tau^2 = V_2(z)/\Omega^2$ of the normalized variances of the estimates (42) calculated by the formulas (47).

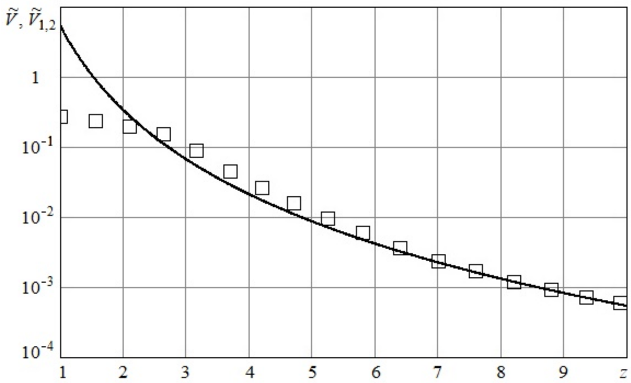

Fig. 2. The theoretical and experimental dependences of the variances of the estimates of the random pulse time-frequency parameters.

From the simulation results it follows that the asymptotically exact expressions for the characteristics of the estimates (42) satisfactorily approximate the experimental data, if $z > 1.5 \ldots 2$. Thus, since in practice, for high quality operation of the receiver it is necessary to provide an output signal-to-noise ratio of about 10 or greater, the formulas (36), (47) and their generalizations (23), (26) can be used to calculate the characteristics of the corresponding measurers without significant limitations.

It should be also noted that there is a number of other examples of the application of the LAA method for calculating the characteristics of the joint MLEs of the discontinuous time and frequency parameters of the stochastic pulse. Thus, in [34], the task is considered of determining the joint MLEs of time of arrival, duration, band center and bandwidth of the high-frequency pulse (37), while in [35] the joint MLEs are studied of time of arrival, duration and bandwidth of the low-frequency Gaussian random pulse.

V. CONCLUSION

In order to determine the performance of the optimal (maximum likelihood) measurers of the signals with the unknown discontinuous parameters, the method based on the additive-multiplicative representation of the moments of the decision statistics (the method of locally additive approximation) can be used. With the help of the specified approach, the closed analytical expressions can be found for the characteristics of the estimates of the discontinuous quasi-deterministic and Gaussian random signals observed against Gaussian interferences. These expressions adequately describe the corresponding experimental data in a wide range of the output signal-to-noise ratios. The obtained results make it possible to theoretically evaluate the practical application appropriateness of the particular algorithms for processing the discontinuous signals in each specific case.

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