Research on Fuzzy Multi-objective Multi-period Portfolio by Hybrid Genetic Algorithm with Wavelet Neural Network

Yechun Yu, Xue Deng, Chuangjie Chen and Kai Cheng

Abstract—**This paper deals with fuzzy multi-objective multi-period portfolio selection problems. The portfolio selection is proposed by taking into account three criteria of final return, cumulative risk and entropy. In the model, the return level is quantified by the possibilistic mean value of return, and the risk is quantified by the possibilistic variance of return while fuzzy entropy is adopted to increase the risk dispersion degree to some extent. Then a fuzzy multi-objective multi-period portfolio model is presented in a more complex market environment. To solve the complex model, the multi-objective functions are transformed into a single objective and the risk preference parameter is introduced to balance the return and risk to meet with investors' preferences. To ensure the investor can obtain the optimal portfolio strategy, a hybrid intelligent algorithm is designed by combining both genetic algorithm and wavelet neural network algorithm, which not only utilizes the good localization property of wavelet transform but also utilizes the effective self-learning function of neural network. Finally, a numerical example is presented to illustrate this approach and the designed algorithm. The results show that the proposed model and the designed algorithm are practical and flexible, while they are meaningful for the study on portfolio selection and multi-objective programming.**

Index Terms—**portfolio selection, multi-period, multi-objective, entropy, hybrid intelligent algorithm**

I. Introduction

Modern portfolio theory was originally proposed by Markowitz [1] in 1952. He put forward a well-known mean-variance model in which portfolio return is quantified by the mean and portfolio risk is quantified by the variance. This model has been proved to be effective and useful, so it is widely used in both portfolio selection and asset allocation. Although many scholars have accepted the idea of using mean and variance to measure portfolio return and risk[2]–[3], with the development of the technology and the increasing complexity of security market, many scholars have done deeper research on other kind of measures of uncertain return and risk.

On the one hand, the security uncertainty is a hot topic in the portfolio research. In papers [1]–[3], security returns are regarded as random variables. However, security returns are not always about the random uncertainty in many situations. In fact, some scholars have found that the portfolio selection problem also includes fuzzy uncertainty and have used fuzzy set theory in their researches such as Watada [4], Deng et. al [5]–[8].

On the other hand, the risk measurement is another hot topic in the portfolio research. For example, Markowitz [9] considered the semi-variance of return rate as risk, Konno and Yamazaki [10] defined the absolute-deviation of return rate as risk, and Shannon [11] defined the concept of information entropy and then used this concept to measure risk.

It is noted that the entropy can also present the uncertainty of security risk. For this reason, some scholars have done some researches about entropy. For example, Philippatos and Wilson [12] regarded random entropy as risk and then studied the relationship between entropy, market risk and the selection of efficient portfolios. Qin [13] et. al discussed the Kapur cross-entropy minimization model for portfolio selection problem under fuzzy environment, which can minimize the divergence of the fuzzy investment return from a priori one.

Multi-period portfolio selection model has a wider application than single-period model because investors usually want to invest in a long term. Sadjadi [14] et. al constructed a fuzzy multi-period portfolio selection model with different rates for borrowing and lending, and Liu [15] et. al proposed a robust multi-period portfolio model based on prospect theory. Nevertheless, there are still few researches on multi-objective multi-period portfolio selection. It is necessary to do some research work about multi-objective and multi-period portfolio problem. In this paper, a multi-objective multi-period model was formulated originally based on the above researches. In this model, not only the variance but also the entropy of the asset is considered as portfolio risk. Considering that the objective functions are non-smooth in some points, thus, a hybrid

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intelligent algorithm is designed. The algorithm combines genetic algorithm (GA) with wavelet neural network algorithm (WNN) and is used to solve the proposed model. Finally, a numerical example is presented to validate the model and the designed algorithm. The research is practical and meaning for economic investment.

The rest of this paper is organized as follows. In Section 2, some basic knowledge is introduced with fuzzy variables, random entropy and fuzzy entropy. In Section 3, a mean-fuzzy entropy-variance model is proposed and then simplified. In Section 4, a hybrid intelligent algorithm is designed for solving the proposed model. In Section 5, a numerical example is given to illustrate this approach. Section 6 summarizes the research work.

II. Preliminaries

A. Possibilistic Mean and Variance and Covariance of Fuzzy Numbers

Let us first review some basic concepts about fuzzy number, which are necessary for the following sections. Assume that $\tilde{A}$ is a fuzzy set of the real line with a normal, fuzzy convex and continuous membership function of bounded support, the following definitions is given by Carlsson and Fuller [16]:

Definition 1. Let $\tilde{A}$ be a fuzzy number, then the γ-level set of $\tilde{A}$ is denoted by $\tilde{A}^{\gamma} = [a_1(\gamma), a_2(\gamma)], \forall \gamma \in [0,1]$.

Definition 2. Let $\tilde{A}$ be a fuzzy number, then the possibilistic mean value of $\tilde{A}$ can be defined as:

$$E(\tilde{A}) = \int_0^1 \gamma [a_1(\gamma) + a_2(\gamma)] d\gamma. \tag{1}$$

In portfolio selection, possibilistic mean value is used to denote the expected returns of a fuzzy variable.

Definition 3. The possibilistic variance of $\tilde{A}$ can be defined as:

$$Var(\tilde{A}) = \int_0^1 \gamma \left(\left[E(\tilde{A}) - a_1(\gamma)\right]^2 + \left[E(\tilde{A}) - a_2(\gamma)\right]^2 \right) \mathrm{d}\gamma. \tag{2}$$

Definition 4. Let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers, then the covariance between $\tilde{A}$ and $\tilde{B}$ is defined as:

$$\begin{aligned} Cov(\tilde{A},\tilde{B}) = &\int_0^1 \gamma \left(E(\tilde{A}) - a_1(\gamma)\right)\left(E(\tilde{B}) - b_1(\gamma)\right) \mathrm{d}\gamma \\ &+ \int_0^1 \gamma \left(E(\tilde{A}) - a_2(\gamma)\right)\left(E(\tilde{B}) - b_2(\gamma)\right) \mathrm{d}\gamma, \end{aligned} \tag{3}$$

where $\tilde{A}^{\gamma} = [a_1(\gamma), a_2(\gamma)]$, $\tilde{B}^{\gamma} = [b_1(\gamma), b_2(\gamma)]$, $\forall \gamma \in [0,1]$.

In portfolio selection, possibilistic variance and possibilistic covariance are often used to denote the risk of a portfolio that composed of fuzzy variables.

Based on the Zedeh Extension Principle, when $\tilde{A}_i$ $(i = 1,2,\cdots,n)$ are all fuzzy variables, we get:

$$E\left[\lambda_1\tilde{A}_1 + \lambda_2\tilde{A}_2 + \cdots + \lambda_n\tilde{A}_n\right] = \sum_{i=1}^{n} \lambda_i \tilde{A}_i, \tag{4}$$

$$Var\left[\lambda_1\tilde{A}_1 + \lambda_2\tilde{A}_2 + \cdots + \lambda_n\tilde{A}_n\right] = \sum_{j=1}^{n}\sum_{i=1}^{n} \lambda_i\lambda_j Cov(\tilde{A}_i, \tilde{A}_j). \tag{5}$$

If $\tilde{A} = (a,b,\alpha,\beta)$ is a trapezoid fuzzy variable with the following membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \dfrac{a-x}{\alpha}, & \text{if } a - \alpha \le x \le a, \\ 1, & \text{if } a \le x \le b, \\ 1 - \dfrac{x-b}{\beta}, & \text{if } b \le x \le b + \beta, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

It is not difficult to find that the γ-level set of $\tilde{A}$ is $\tilde{A}^{\gamma} = [a - (1-\gamma)\alpha, b + (1-\gamma)\beta], \forall \gamma \in [0,1]$. Then, according to (1) and (2), the expected value and variance of the trapezoid fuzzy variable are given by:

$$E(\tilde{A}) = \frac{a+b}{2} + \frac{\beta - \alpha}{6}, \tag{7}$$

$$Var(\tilde{A}) = \left(\frac{b-a}{2} + \frac{\alpha+\beta}{6}\right)^2 + \frac{(\alpha+\beta)^2}{72} + \frac{(\alpha-\beta)^2}{72}. \tag{8}$$

Let $\tilde{A}$ and $\tilde{B}$ be trapezoid fuzzy variables, then the covariance between $\tilde{A}$ and $\tilde{B}$ is:

$$\begin{aligned} Cov(\tilde{A},\tilde{B}) = &\left(\frac{b_1 - a_1}{2} + \frac{\alpha_1 + \beta_1}{6}\right)\left(\frac{b_2 - a_2}{2} + \frac{\alpha_2 + \beta_2}{6}\right) \\ &+ \frac{1}{36}(\alpha_1\alpha_2 + \beta_1\beta_2). \end{aligned} \tag{9}$$

B. Entropy

The concept of entropy was originally derived from thermo-dynamics, and later developed to the statistical mechanics, information theory and other disciplines. As its description in thermo-dynamics, entropy can measure the disorder of a specific system. In other words, it can measure the internal uncertainty of something. Here, we introduced the original definition of entropy and its extension to fuzzy variable.

Definition 5. Consider a probabilistic test with n results and a discrete probability p_i $(i = 1,2,\cdots,n)$. Entropy is defined as:

$$S_n = -\sum_{i=1}^{n} p_i \ln p_i. \tag{10}$$

where $p_i \ge 0$ $(i = 1,2,\cdots,n)$ and $\sum_{i=1}^{n} p_i = 1$.

Definition 6. Let ξ be a continuous fuzzy variable with membership function $\mu(x)$, and then its entropy is defined as:

$$H[\xi] = -\int_{-\infty}^{\infty} \left(\frac{\mu(x)}{2}\ln\frac{\mu(x)}{2} + (1 - \frac{\mu(x)}{2})\ln(1 - \frac{\mu(x)}{2})\right)dx. \tag{11}$$

So the entropy of a trapezoid fuzzy variable $\tilde{A} = (a,b,\alpha,\beta)$ is:

$$H[\tilde{A}] = \frac{\alpha+\beta}{2} + (b-a)\ln 2. \tag{12}$$

$$H(b_1\xi_1 + b_2\xi_2 + \cdots + b_n\xi_n) = \sum_{i=1}^{n} b_i H(\xi_i), \tag{13}$$

where b_i $(i = 1,2,\cdots,n)$ is a real number.

III. Model

To make it easier to understand, we put together all the notations that will be used hereafter.

Notations

x_k : the investment proportion of riskless asset at period k ;

y_k^i : the investment proportion of risky asset i at period k ;

θ_k^i : the investment strategy of risky asset i at the beginning of period k , when $\theta_k^i < 0$, θ_k^i is the amount of asset i which is sold at the beginning of period k , when $\theta_k^i > 0$, θ_k^i is the amount of asset i which is purchased at the beginning of period k ;

η_k^i : the return rate of risky asset i at period k ;

r_k : the return rate of riskless asset at period k ;

α_k^i : the unit transaction cost of risky asset i at period k ($k = 0,1,\cdots,T-1, i = 1,2,\cdots,n$).

In this paper, assume that the whole investment process is self-financing, that is, the investor does not invest the additional capital during the investment period. According to the previous section, the mean value of the return rate of $(y_k^1, y_k^2, \cdots, y_k^n)$ at period k is determined by

$$E_{k1} = \sum_{i=1}^{n} E[\eta_k^i](y_k^i + \theta_k^i). \tag{14}$$

The return rate of the riskless asset at period k is determined by

$$E_{k2} = r_k \cdot [x_k - \sum_{i=1}^{n} (\alpha_k^i \left|\theta_k^i\right| + \theta_k^i)]. \tag{15}$$

Then the total mean value at period k is

$$\begin{aligned} E_k = E_{k1} + E_{k2} = &\sum_{i=1}^{n} E[\eta_k^i](y_k^i + \theta_k^i) \\ &+ r_k \cdot [x_k - \sum_{i=1}^{n} (\alpha_k^i \left|\theta_k^i\right| + \theta_k^i)]. \end{aligned} \tag{16}$$

So the investor's final return is determined by

$$\sum_{k=0}^{T-1} \left\{ \sum_{i=1}^{n} E[\eta_k^i](y_k^i + \theta_k^i) + r_k \cdot [x_k - \sum_{i=1}^{n} (\alpha_k^i \left|\theta_k^i\right| + \theta_k^i)] \right\}. \tag{17}$$

According to Section 2, the variance of the portfolio at period k is determined by

$$R_k = \sum_{i,j=1}^{n} (y_k^i + \theta_k^i) \cdot (y_k^j + \theta_k^j) Cov(\eta_k^i, \eta_k^j). \tag{18}$$

The total risk of the portfolio is determined by

$$\sum_{k=0}^{T-1} \left\{ \sum_{i,j=1}^{n} (y_k^i + \theta_k^i) \cdot (y_k^j + \theta_k^j) Cov(\eta_k^i, \eta_k^j) \right\}. \tag{19}$$

The entropy of the portfolio at period k is determined by

$$H_k = \sum_{i=1}^{n} (y_k^i + \theta_k^i) H[\eta_k^i]. \tag{20}$$

The final entropy of the portfolio is determined by

$$\sum_{k=0}^{T-1} \left\{ \sum_{i=1}^{n} (y_k^i + \theta_k^i) H[\eta_k^i] \right\}. \tag{21}$$

In each period, we have

$$\begin{cases} x_{k+1} = (1 + r_k)[x_k - \sum_{i=1}^{n} (\theta_k^i + \alpha_k^i \left|\theta_k^i\right|)], \\ y_k^{i+1} = (1 + E[\eta_k^i]) \cdot (Y_k^i + \theta_k^i), \end{cases} \tag{22}$$

and $y_k^i \geq 0$, $i = 1,2,\cdots,n;$ $k = 0,1,\cdots,T-1$.

According to the concept of Markowitz's mean-variance model, the goal of investors is to maximize the return of portfolio while minimizing the whole risk. Considering that multi-period investment is a special form of investment many times, it is rational to pursue the maximal of the final return and the minimal of the cumulative risk. In addition, to reduce the uncertainty of the portfolio and then bring investors a better experience, the entropy of the portfolio should be minimized. To state simply, entropy can be regarded as other kind of risk measure in portfolio.

Thus, we can summary the investor's goal and requirement in (23).

This is a multi-objective optimization model whose solution depends strongly on the investors' preference for each objective. In this paper, we require that the entropy value of each period cannot be too much or too small. That is, we change the goal of minimizing the entropy value to a special constrain of the model and then place more emphasis on the return and risk. It should be noticed that the entropy value cannot be too small, or it would conflict with the other two goals.

Then, for other two goals, let: $X = \left\{ (x_k, y_k^1, y_k^2, \cdots, y_k^n) \;\middle|\; x_k + \sum_{i=1}^{n} y_k^i, x_k \geq 0, y_k^i \geq 0, i = 1,2,\cdots,n \right\}$, $E_k^+ = \max_{x \in X} E_k(x)$, $E_k^- = \min_{x \in X} E_k(x)$, $R_k^+ = \max_{x \in X} R_k(x)$, $R_k^+ = \min_{x \in X} R_k(x)$. Obviously, E_k^+ and E_k^- respectively represent the maximal and the minimal expected returns at period k while R_k^+ and R_k^- respectively represent the maximal and

$$\begin{cases} \max \ \sum_{k=0}^{T-1} \left\{ \sum_{i=1}^{n} E[\eta_k^i](y_k^i + \theta_k^i) + r_k \cdot [x_k - \sum_{i=1}^{n} (\alpha_k^i \left|\theta_k^i\right| + \theta_k^i)] \right\} \\ \min \ \sum_{k=0}^{T-1} \left\{ \sum_{i,j=1}^{n} (y_k^i + \theta_k^i) \cdot (y_k^j + \theta_k^j) Cov(\eta_k^i, \eta_k^j) \right\} \\ \min \ \sum_{k=0}^{T-1} \left\{ \sum_{i=1}^{n} (y_k^i + \theta_k^i) H[\eta_k^i] \right\} \\ \text{s.t.} \ \ x_{k+1} = (1 + r_k)[x_k - \sum_{i=1}^{n} (\theta_k^i + \alpha_k^i \left|\theta_k^i\right|)], \\ \qquad y_k^{i+1} = (1 + E[\eta_k^i]) \cdot (y_k^i + \theta_k^i), \\ \qquad \theta_k^i \geq -y_k^i, x_k \geq 0, \theta_k^i \geq 0, k = 0,1,\cdots,T-1, \ i = 1,2,\cdots,n. \end{cases} \tag{23}$$

$$\begin{cases} \min & \sum_{k=0}^{T-1}\left\{\lambda\dfrac{E_k^+-E_k}{E_k^+-E_k^-}+(1-\lambda)\dfrac{R_k-R_k^-}{R_k^+-R_k^-}\right\} \\ \text{s.t.} & H_{k0}^-\le\sum_{i=1}^{n}(y_k^i+\theta_k^i)H[\eta_k^i]\le H_{k0}^+, \\ & x_{k+1}=(1+r_k)[x_k-\sum_{i=1}^{n}(\theta_k^i+\alpha_k^i\left|\theta_k^i\right|)], \\ & y_k^{i+1}=(1+E[\eta_k^i])\cdot(y_k^i+\theta_k^i), \\ & \theta_k^i\ge -y_k^i, x_k\ge 0, \theta_k^i\ge 0, k=0,1,\cdots,T-1, i=1,2,\cdots,n. \end{cases} \tag{25}$$

the minimal risks at period k, so that we have $\dfrac{E_k^+-E_k}{E_k^+-E_k^-}\in[0,1]$ and $\dfrac{R_k-R_k^+}{R_k^+-R_k^-}\in[0,1]$.

As mentioned above, the goals of (23) are to maximize E_k while minimizing R_k, which are equivalent to minimize both $\dfrac{E_k^+-E_k}{E_k^+-E_k^-}$ and $\dfrac{R_k-R_k^+}{R_k^+-R_k^-}$. Thus, we can combine the last two goals. To make the final goal more flexible, the risk preference parameter $\lambda\in[0,1]$ is introduced. Then we could build a new objective function as follow:

$$\min \quad \lambda\frac{E_k^+-E_k}{E_k^+-E_k^-}+(1-\lambda)\frac{R_k-R_k^+}{R_k^+-R_k^-}. \tag{24}$$

Also, it can be pointed out that the new objective function still has a range of $[0,1]$. When the parameter is higher than 0.5, the final return is paid more attention; conversely, the cumulative risk is paid more attention. Thus, it is convenient for investors to choose the value of the parameter according to their risk preferences. The new model is obtained as (25).

IV. Hybrid Intelligent Algorithm

A. The Basic Idea of the Algorithm

Since it is difficult to find the optimal solution of (25) in traditional ways, a hybrid intelligent algorithm combining genetic algorithm (GA) with wavelet neural network algorithm (WNN) is designed to help investors find the optimal solution. The GA was initialized by Holland [17] in 1975, and has been well developed. WNN is structured by combining wavelet transform with neural network, which not only utilizes the good localization property of wavelet transform but also utilizes the self-learning function of neural network. So WNN can solve the prediction problem of decision variables with nonlinear relation. In addition, the experiments show that the parameters of wavelet neural network have good theoretical basis, the nonlinear function approximation method has high accuracy, and the local extreme can be jumped out to find the global optimum. The method of reverse dynamic programming is used to solve (25).

Combined with genetic algorithm and wavelet neural network, the basic idea of using the reverse dynamic programming method which the initial state has given to solve model is as follows:

1) The investment process is divided into some stages, and in this paper, each period is regarded as a stage. Considering the period k_0, select the investment portfolio $\{x_{k_0}, y_{k_0}^i\}$ at the beginning of period k_0 being the state variable, the optimal investment decision $\{\theta_{k_0}^i\}$ being the decision variable, and define the objective function as $\sum_{k=k_0}^{T-1}\left\{\lambda\dfrac{E_k^+-E_k}{E_k^+-E_k^-}+(1-\lambda)\dfrac{R_k-R_k^-}{R_k^+-R_k^-}\right\}$. Thus, the original model is decomposed into a series of sub-models of the same type, and then the sub-models are solved by one by one. The optimal investment decision is solved by genetic algorithm.
2) Solve the model gradually from the period $T-1$ using the method of reverse dynamic programming. After solving the sub model of period $T-1$, the state variables are used as the input values of the wavelet network, and the optimal investment decision and the optimal index are used as the output values of the wavelet network. The wavelet network is trained by using the function relation between input and output values. In order to solve the period $T-2$ sub model, we not only need to calculate the objective function of the period $T-2$, but also need the objective function value of the period $T-1$ sub model. Adding the two objective-function values, and let this sum be the objective function value of the period $T-2$ sub model. Then every sub-model can be solved like this.

By using the above optimization theory, the multi-period portfolio problem solving process can be expressed as a continuous recursive process, which is calculated by the backward forward step by step.

B. Specific Steps

Let $\theta_k=\{\theta_k^1,\theta_k^2,\cdots,\theta_k^n\}(k=0,1,\cdots,T-1)$.

Step 1: when $k=T-1$, according to (25), we can decompose the whole goals into multiple single goals for each period. Then, the objective function of $T-1$ period is:

$$\min \quad \Gamma_{T-1}(\theta_{T-1})=\lambda\frac{E_{T-1}^+-E_{T-1}}{E_{T-1}^+-E_{T-1}^-}+(1-\lambda)\frac{R_{T-1}-R_{T-1}^-}{R_{T-1}^+-R_{T-1}^-}, \tag{26}$$

where E_{T-1}^+, E_{T-1}^-, R_{T-1}^+ and R_{T-1}^- can be got according to (14), (15), (16) and (18).

1) Randomly generate a set of initial investment portfolio $\{x_{T-1}, y_{T-1}^1,\cdots,y_{T-1}^n\}$ which satisfies the constraint conditions.
2) Use genetic algorithm to solve the optimal investment strategy $\theta_{T-1}=\{\theta_{T-1}^1,\theta_{T-1}^2,\cdots,\theta_{T-1}^n\}$ and the objective

function value Γ_{T-1}.

3) Repeat 1) and 2) for N times to get N sets of initial investment portfolio, the optimal investment strategy and the objective function value, and record them. Let N sets of $\{x_{T-1}, y_{T-1}^1, \cdots, y_{T-1}^n\}$ be the input values of the wavelet network, N sets of $\{\theta_{T-1}^1, \theta_{T-1}^2, \cdots, \theta_{T-1}^n, \Gamma_{T-1}\}$ be the output values of the wavelet network. So, we can get the following approximation function:

$$\begin{cases} \theta_{T-1}^1 = \Phi_{T-1}^1(x_{T-1}, y_{T-1}^1, y_{T-1}^2, \cdots, y_{T-1}^n), \\ \theta_{T-1}^2 = \Phi_{T-1}^2(x_{T-1}, y_{T-1}^1, y_{T-1}^2, \cdots, y_{T-1}^n), \\ \qquad \vdots \\ \theta_{T-1}^n = \Phi_{T-1}^n(x_{T-1}, y_{T-1}^1, y_{T-1}^2, \cdots, y_{T-1}^n), \\ \Gamma_{T-1} = \Psi_{T-1}^n(x_{T-1}, y_{T-1}^1, y_{T-1}^2, \cdots, y_{T-1}^n) \end{cases} \tag{27}$$

Step 2: when $k = T-2$, the objective function is written as:

$$\min \ \Gamma_{T-2}(\theta_{T-2}) = \left[\lambda \frac{E_{T-2}^+ - E_{T-2}}{E_{T-2}^+ - E_{T-2}^-} + (1-\lambda)\frac{R_{T-2} - R_{T-2}^-}{R_{T-2}^+ - R_{T-2}^-}\right] + \Gamma_{T-1}. \tag{28}$$

1) Randomly generate a set of initial investment portfolio $\{x_{T-2}, y_{T-2}^1, \cdots, y_{T-2}^n\}$ which satisfies the constraint conditions.

2) Use genetic algorithm to obtain the optimal investment strategy $\theta_{T-2} = \{\theta_{T-2}^1, \theta_{T-2}^2, \cdots, \theta_{T-2}^n\}$, and get Γ_{T-1} according to Step1. Then calculate the objective function value Γ_{T-2}.

3) Repeat 1) and 2) for N times to get N sets of initial investment portfolio, the optimal investment strategy and the objective function value, and record them. Let N sets of $\{x_{T-2}, y_{T-2}^1, \cdots, y_{T-2}^n\}$ be the input values of the wavelet network, $\{\theta_{T-2}^1, \theta_{T-2}^2, \cdots, \theta_{T-2}^n, \Gamma_{T-2}\}$ be the output values of the wavelet network. The following approximation function can be obtained:

$$\begin{cases} \theta_{T-2}^1 = \Phi_{T-2}^1(x_{T-2}, y_{T-2}^1, y_{T-2}^2, \cdots, y_{T-2}^n), \\ \theta_{T-2}^2 = \Phi_{T-2}^2(x_{T-2}, y_{T-2}^1, y_{T-2}^2, \cdots, y_{T-2}^n), \\ \qquad \cdots \\ \theta_{T-2}^n = \Phi_{T-2}^n(x_{T-2}, y_{T-2}^1, y_{T-2}^2, \cdots, y_{T-2}^n), \\ \Gamma_{T-2} = \Psi_{T-2}^n(x_{T-2}, y_{T-2}^1, y_{T-2}^2, \cdots, y_{T-2}^n). \end{cases} \tag{29}$$

Step 3: When $k = 0$, the objective function is:

$$\min \quad \Gamma_0(\theta_0) = \left[\lambda \frac{E_0^+ - E_0}{E_0^+ - E_0^-} + (1-\lambda)\frac{R_0 - R_0^-}{R_0^+ - R_0^-}\right] + \Gamma_1. \tag{30}$$

In the same way, the following approximation function can be obtained:

$$\begin{cases} \theta_0^1 = \Phi_0^1(x_0, y_0^1, y_0^2, \cdots, y_0^n), \\ \qquad \cdots \\ \theta_0^n = \Phi_0^n(x_0, y_0^1, y_0^2, \cdots, y_0^n), \\ \Gamma_0 = \Psi_0^n(x_0, y_0^1, y_0^2, \cdots, y_0^n). \end{cases} \tag{31}$$

Step 4: Since the initial investment portfolio $\{x_0, y_0^1, y_0^2, \cdots, y_0^n\}$ is determined, the obtained T wavelet neural networks are used to get the optimal solution for each period. As shown in (32).

1) According to (31) and $\{x_0, y_0^1, y_0^2, \cdots, y_0^n\}$, the optimal investment strategy $\hat{\theta}_0$ and objective function value $\hat{\Gamma}_0$ can be obtained.

2) Then we can get $\{x_1, y_1^1, y_1^2, \cdots, y_1^n\}$, so we can obtain the optimal investment strategy $\hat{\theta}_1$ and objective function value $\hat{\Gamma}_1$.

3) With the iterative process, finally we can get $\{x_T, y_T^1, y_T^2, \cdots, y_T^n\}$. So the final asset of an investor is $x_T + \sum y_T^i$, the optimal investment strategy are $\hat{\theta}_0, \hat{\theta}_1, \cdots, \hat{\theta}_{T-1}$, and the final objective function value is $\hat{\Gamma}_0$.

V. Numerical Example

In order to illustrate the idea of our model, an example is given to simulate the real transaction. Suppose that the initial wealth of the investor is $w_0 = 1$, and the investor invested 0.15, 0.2, 0.1 and 0.2 for four risky assets and the rest for a riskless asset. Set $T = 3$, the return rate of these four risky assets is denoted by trapezoidal fuzzy number and the probability distributions of these fuzzy numbers in the three periods are listed in Table I.

TABLE I
The Probability Distributions of Four Assets in the Three Periods

Period	Asset i	a_i	b_i	α_i	β_i
$t=1$	Asset1	0.9976	1.0628	0.0958	0.1138
	Asset 2	1.0198	1.0760	0.2385	0.2027
	Asset 3	1.0050	1.1100	0.2537	0.1018
	Asset 4	0.9695	1.1075	0.1993	0.1608
$t=2$	Asset 1	0.7794	0.9834	0.0000	0.5865
	Asset 2	1.0440	1.1949	0.3183	0.3152
	Asset 3	1.0085	1.0665	0.1386	0.2631
	Asset 4	0.9954	1.0869	0.3225	0.3676
$t=3$	Asset 1	1.0073	1.1417	0.3387	0.1260
	Asset 2	0.6983	1.0443	0.3164	0.4836
	Asset 3	1.0194	1.0794	0.3575	0.1887
	Asset 4	0.9822	1.0617	0.4259	0.1883

In Table I, $(a_i, b_i, \alpha_i, \beta_i)$ represents the return rate of asset i in the form of trapezoid fuzzy number. It could be deduced that all these four assets have a stable performance in each period. But it does not mean that we can keep our investment strategy unchanged all times, which would be stated in detail below.

$$\xrightarrow{\{x_0, y_0^1, y_0^2, \cdots, y_0^n\}} \overset{\hat{\theta}_0}{0} \xrightarrow{\{x_1, y_1^1, y_1^2, \cdots, y_1^n\}} \cdots \xrightarrow{\{x_k, y_k^1, y_k^2, \cdots, y_k^n\}} \overset{\hat{\theta}_k}{k}$$
$$\xrightarrow{\{x_{k+1}, y_{k+1}^1, y_{k+1}^2, \cdots, y_{k+1}^n\}} \cdots \xrightarrow{\{x_{T-1}, y_{T-1}^1, y_{T-1}^2, \cdots, y_{T-1}^n\}} \overset{\hat{\theta}_{T-1}}{T-1} \xrightarrow{\{x_T, y_T^1, y_T^2, \cdots, y_T^n\}} \tag{32}$$

According to (7) and (12), the means E_i and entropy values H_i of the four risky assets and the riskless asset are obtained in Table II. In addition, the covariance matrix of the risky assets is given in Table III According to (9), and the concrete steps for data processing are shown in Table IV.

TABLE II
THE MEANS AND ENTROPY VALUES OF FOUR SAMPLE ASSETS AND RISKLESS ASSET IN THE THREE PERIODS

Asset i	$t=1$ (E_i, H_i)	$t=2$ (E_i, H_i)	$t=3$ (E_i, H_i)
Asset 1	$(1.0332, 0.1500)$	$(0.9791, 0.4347)$	$(1.0391, 0.3255)$
Asset 2	$(1.0419, 0.2596)$	$(1.1189, 0.4213)$	$(0.8992, 0.6398)$
Asset 3	$(1.0572, 0.2159)$	$(1.0583, 0.2411)$	$(1.0213, 0.3147)$
Asset 4	$(1.0321, 0.2757)$	$(1.0485, 0.4090)$	$(1.9823, 0.3622)$
riskless	$(1.0270, 0.0000)$	$(1.0306, 0.0000)$	$(1.0468, 0.0000)$

From Table II, we can find that:

1) In period 1, the difference between the return rates and entropy values of each risky asset is not significant, which is consistent with Table I.
2) In period 2, it is obvious that the balance is broken. The return rate of asset 2 is the highest and the entropy value of asset 3 is the lowest.
3) In period 3, the return rate of asset 2 is the lowest, which is entirely contrary to the condition in period 2. Comparatively, the return rate of asset 4 is the highest, which is almost twice as much as that of asset 2. Besides, the entropy value of asset 2 is the highest.

In conclusion, the return rates and the uncertainty of each risky assets are changed at each period, which requires us to change the portfolio strategy to avoid serious loss and obtain more returns.

As to Table III, we will do some analysis after solving the portfolio selection model, which can make the analysis clearer.

The proposed algorithm is employed to solve the model, and the parameters setting are stated as follows:

In genetic algorithm: let the population size be 30, the length of chromosome be 5, the iteration number be 500, the crossover probability be 0.1, the mutation probability be 0.02, and the repeat number be 100. In wavelet neural network: let the number of nodes in the input layer be 5, the number of hidden layer nodes be 6, and the number of output layer nodes be 5.

In order to be closer to the actual financial market, when the program is calculated, the return on each period of each stock is multiplied by 0.1. According to the specific model and relevant data, let $\lambda = 0.2$, the results are obtained by using our algorithm under the MATLAB environment as follow (In order to avoid short selling, some amendments to the calculation result were made).

TABLE III
THE COVARIANCE MATRIX IN THE THREE PERIODS

$t=1$	$t=2$	$t=3$
$\begin{bmatrix} 0.0052 & 0.0085 & 0.0069 & 0.0099 \\ 0.0085 & 0.0131 & 0.0109 & 0.0153 \\ 0.0069 & 0.0109 & 0.0096 & 0.0128 \\ 0.0099 & 0.0153 & 0.0128 & 0.0185 \end{bmatrix}$	$\begin{bmatrix} 0.0495 & 0.0389 & 0.0202 & 0.0354 \\ 0.0389 & 0.0383 & 0.0212 & 0.0352 \\ 0.0202 & 0.0212 & 0.0117 & 0.0198 \\ 0.0354 & 0.0352 & 0.0198 & 0.0326 \end{bmatrix}$	$\begin{bmatrix} 0.0246 & 0.0497 & 0.0206 & 0.0240 \\ 0.0497 & 0.1031 & 0.0432 & 0.0500 \\ 0.0206 & 0.0432 & 0.0192 & 0.0213 \\ 0.0240 & 0.0500 & 0.0213 & 0.0262 \end{bmatrix}$

TABLE IV
THE CONCRETE STEPS FOR DATA PROCESSING

Steps	Concrete methods
Step 1：	Initialization. Generate initial values randomly of Wavelet function parameters a_j, b_j , weight w_{ij}, w_{jk} , and learning rate η ;
Step 2：	Enter training samples for Training Wavelet Neural Network;
Step 3：	Predictive output. The training samples are input into the network, the network output is calculated and the error e between the network output and the expected output is calculated.
Step 4：	Weight correction. Correcting weights and parameters according to errors e to make the predicted value of the network approach the expected value in the near future;
Step 5：	Judging whether the algorithm is over, if yes, go to Step 6; if no, return to Step 3;
Step 6：	Output final coefficients a_j, b_j and weights w_{ij}, w_{jk} , which are saved as a well-trained wavelet neural network for the next calculation.

TABLE V
THE OPTIMAL INVESTMENT STRATEGY WHEN $\lambda = 0.2$

Investment stage	Asset 1	Asset 2	Asset 3	Asset 4	Riskless	sum
Begin of Period 1	0.1500	0.2000	0.1000	0.2000	0.3500	1.0000
Strategy of Period 1	0.0263	-0.1003	0.0343	-0.1593	0.1990	0.0000
End of Period 1	0.1946	0.1101	0.1485	0.0449	0.6036	1.1016
Strategy of Period 2	-0.0188	-0.1101	0.0390	-0.0449	0.1348	0.0000
End of Period 2	0.1930	0.0000	0.2073	0.0000	0.8133	1.2136
Strategy of Period 3	-0.1570	0.0000	-0.1293	0.2458	-0.0405	0.0000
End of period 3	0.0397	0.0000	0.0860	0.2699	0.8256	1.2212

TABLE VI
THE OPTIMAL INVESTMENT STRATEGY WHEN $\lambda = 0.8$

Investment stage	Asset 1	Asset 2	Asset 3	Asset 4	Riskless	sum
Begin of Stage 1	0.1500	0.2000	0.1000	0.2000	0.3500	1.0000
Strategy of Stage 1	-0.0597	0.1932	0.2446	-0.0281	-0.3500	0.0000
End of Stage 1	0.0996	0.4342	0.3810	0.1896	0.0000	1.1045
Strategy of Stage 2	-0.0737	0.2061	-0.0681	-0.0643	0.0000	0.0000
End of Stage 2	0.0285	0.7119	0.3460	0.1385	0.0000	1.2249
Strategy of Stage 3	0.0888	-0.1053	0.0812	-0.1015	0.0367	0.0000
End of Stage 3	0.1295	0.6611	0.4709	0.0407	0.0373	1.3395

When $\lambda = 0.2$, investors are risk-averse, that is to say, they focus on the pursuit of low risk. From Table V, we can see that:

1) The expected return of asset 1 in period 1 is nearly equal to asset 4, but the variance value of asset 1 in period 1 is significantly less than the asset 4; asset 3 in period 1 has a higher expected return but a lower variance value. So, the strategy of the period 1 is to reduce the amount of assets 2 and 4, at the same times to increase the amount of assets 1, 3 and the riskless asset.
2) The expected return of asset 3 in period 2 is just less than asset 2, but the variance value of asset 3 in period 2 is significantly less than the other assets. So, the strategy of the period 2 is to reduce the amount of assets 1, 2 and 4, at the same times to increase the amount of asset 3 and the riskless asset.
3) The expected return of asset 2 in period 3 is the lowest in these four assets, but the variance value of asset 2 in period 3 is the highest. Although the performance of asset 4 in period 3 is not as good as asset 3, the investor still increases the amount of asset 4, and we attribute this behavior to ensure the diversification degree of portfolio selection problem.

When $\lambda = 0.8$, investors are risk-preferred and focus on the pursuit of higher return. From Table VI, we can see that:

1) The expected return of assets 2 and 3 in period 1 is higher than assets 1 and 4, and the riskless asset has a lowest expected return. Thus, the strategy of the period 1 is to reduce the amount of asset 1 and 4, and reduce the amount of the riskless asset to zero, at the same to increase the amount of assets 2 and 3.
2) The expected return of asset 2 has a highest expected return in period 2. Thus, the strategy of the period 2 is to reduce the amount of asset 1, 3 and 4, at the same to increase the amount of asset.
3) The expected return of assets 1 and 3 in period 3 is higher than assets 2 and 4.

VI. CONCLUSION

In this paper, not only variance, but also entropy was used to measure the risk of portfolios. As a tool to describe the internal uncertainty of a system, entropy can reflect the risk dispersion degree of a portfolio. The smaller the entropy value, the more concentrative the portfolio return distributes. The more concentrative the portfolio return distributes, the more likely the specific return one expects will occur, and thus, the safer the portfolio. However, as one of the objectives of the portfolio model, too small entropy value will conflict with the objective of maximizing the portfolio return. Thus, we need to control the entropy value not too much nor too small to balance the uncertainty and return.

In addition, the paper proposed a model that includes multi-period investment, and then employed a hybrid intelligent algorithm to solve the proposed multi-period model. The calculation results and analysis of the given numerical example show that the proposed model can provide investors with satisfactory strategies that meet their risk appetite, while ensuring the diversification degree of portfolio selection, so that it is practicable and effective. Moreover, the design of the intelligent hybrid algorithm is apparently superior to the traditional ways. As the result, the hybrid GA with WNN could have a better performance for the optimization models which have non-smooth functions.

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