On the Global Convergence of Improved Halley's Method

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Abstract—**In this work, we will present an important variant of Halley's method for solving non-linear equations: Improved Halley's method (IHM). Analysis of convergence shows that the proposed method is cubically convergent for a simple root. A fairly detailed study of its global convergence will performed. We will prove that the proposed method is very interesting since, under certain conditions, it converges faster than Halley's method, Super-Halley's method and Chebyshev's method which are three of the best iterative techniques. On comparison with the sixth and eighth order methods, they behave either similarly or better for the examples considered.**

Index Terms—**Global convergence, Halley's method, Improved Halley's iterative methods, Nonlinear equations, Rate of convergence, Third order method.**

I. Introduction

ONE of the most interesting and most encountered problem in mathematics, engineering and economy [1]-[3], is that of solving nonlinear equations:

$$f(x) = 0. \tag{1}$$

Where f is a real analytic function. One manner to approach this zero α, supposed simple, is to use a fixed-point iteration method in which, we find another function F, called an Iteration Function (I.F.) for f, and from an initial value x_0 [4]-[7], [40]-[44], we define a sequence:

$$x_{n+1} = F(x_n) \quad for \quad n = 0, 1, 2 \ldots \tag{2}$$

The best known example of these types of methods is two order Newton's method [1] given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad for \quad n=0,\ 1,\ 2\ldots \tag{3}$$

In order to improve the order of convergence of Newton's method, the new algorithms with third order [1]-[31] and higher order [32]-[39] have been developed. We cite in particular Halley's method [1], [7]-[13], [19], [27]-[29], [31], given by:

$$x_{n+1}^0 = x_n^0 - \frac{f(x_n^0)}{f'(x_n^0)} W_0(L_n) \qquad n=0,\ 1,\ 2,\ 3,\ldots \tag{4}$$

Where $W_0(L_n) = \frac{2}{2-L_n}$ and $L_n = L_f(x_n) = \frac{f(x_n^0)f''(x_n^0)}{f'(x_n^0)^2}$

A special case of (2) with iteration function:

$$H(x) = x - \frac{f(x)}{f'(x)}\left(\frac{2}{2-L_f(x)}\right). \tag{5}$$

In this paper, based on the Halley's method, we will propose a new method for finding simple roots of nonlinear equations with cubical convergence. In order to show its power, we will make a comparative analytic study between the proposed method and other much known methods. We will also test the efficiency of this method on a number of numerical examples. We will make a comparison with many third and higher order methods. The simplicity and power of the proposed formula pushed us to do a first study of its global convergence.

II. New Method and Analysis of Its Convergence

A. Development of the method

Using second-order Taylor polynomial of f at x_n, we get:

$$y(x) = f(x_n) + f'(x_n)(x - x_n) + \frac{f''(x_n)}{2}(x - x_n)^2. \tag{6}$$

Where x_n is again an approximate value of α. The graph of y will intersect the $x - axis$ at some point $(x_{n+1}, 0)$, which is the solution of the following equation for x_{n+1}:

$$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{f''(x_n)}{2}(x_{n+1} - x_n)^2. \tag{7}$$

Factoring $(x_{n+1} - x_n)$ from last two terms [7], we deduce:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) + \frac{f''(x_n)}{2}(x_{n+1} - x_n)}. \tag{8}$$

By replacing $(x_{n+1} - x_n)$ remaining in the denominator of right-hand side of (8) by Halley's correction given in (4), we get the following famous method of Super Halley:

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$$x_{n+1} = x_n - \frac{f(x_n)}{2f'(x_n)}\left(\frac{2-L_n}{1-L_n}\right). \qquad (9)$$

A special case of (2) with the following I.F.:

$$SH(x) = x - \frac{f(x)}{2f'(x)}\left(\frac{2-L_f(x)}{1-L_f(x)}\right), \qquad (10)$$

Now, by replacing $(x_{n+1} - x_n)$located on the right-hand side of (8) by Super Halley's correction given in (9), we get:

$$x_{n+1}^2 = x_n^2 - \frac{f(x_n^2)}{f'(x_n^2)}.W_2(L_n) \qquad (11)$$

Where $\qquad W_2(L_n) = \frac{4(1-L_n)}{L_n^2-6L_n+4}$

That we call "*Improved Halley's method*"(IHM), a special case of (2), with the following I.F.:

$$IH(x) = x - \frac{4f(x)}{f'(x)}\left(\frac{1-L_f(x)}{4-6L_f(x)+L_f^2(x)}\right). \qquad (12)$$

B. Order of convergence

We shall present the mathematical proof that the method (IHM) given by sequence (11) is cubically convergent.

Theorem 1. *Suppose that the function f has at least two continuous derivatives in the neighborhood of a zero, α, say. Further, assume that $f'(\alpha) \neq 0$ and x_0 is sufficiently close to α. Then Improved Halley's method, given by (11), converges cubically and satisfies the error equation:*

$$e_{n+1} = -\frac{f^{(3)}(\alpha)}{3!\,f'(\alpha)}e_n^3 + O(e_n^4). \qquad (13)$$

Where $e_n = x_n - \alpha$ is the error at nth iteration.

Proof: Let α be a simple root of f and e_n be the error in approximating α by the nth iterate x_n. We use the following Taylor expansions about α:

$$\begin{cases} f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5)] \\ f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + O(e_n^4)] \\ f''(x_n) = f'(\alpha)[2c_2 + 6c_3e_n + 12c_4e_n^2 + O(e_n^3)]. \end{cases} \qquad (14)$$

Where $\quad c_i = \frac{f^{(i)}(\alpha)}{i!\,f'(\alpha)}, \qquad i = 2, 3 \ldots$

Using (14) we get:

$$[f'(x_n)]^2 = [f'(\alpha)]^2[1 + 4c_2e_n + 2(2c_2^2 + 3c_3)e_n^2 + 4(3c_2c_3 + 2c_4)e_n^3 + O(e_n^4)],$$

and $\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4), \qquad (15)$

$$L_n = 2c_2e_n - 6(c_2^2 - c_3)e_n^2 + 4(4c_2^3 - 7c_2c_3 + 3c_4)e_n^3 + O(e_n^4). \qquad (16)$$

Using the Taylor's series expansion of $W(L_n)$ [22] about $L(\alpha)$, we obtain:

$$W(L_n) = W(L(\alpha)) + (L_n - L(\alpha))W'(L(\alpha)) + \frac{(L_n - L(\alpha))^2}{2}W''(L(\alpha)) + O\left((L_n - L(\alpha))^3\right)$$

Taking into account that $L(\alpha) = 0$, we obtain:

$$W(L_n) = W(0) + L_nW'(0) + \tfrac{1}{2}L_n^2W''(0) + O(L_n^3). \qquad (17)$$

The formula (11) show that function W is defined by:

$$W(t) = \frac{4(1-t)}{4-6t+t^2}$$

By a simple calculation, we obtain its first two derivatives:

$$W'(t) = \frac{4(t^2-2t+2)}{(4-6t+t^2)^2} \quad \text{and} \quad W''(t) = \frac{8\,(-t^3+3t^2-6t+8)}{(4-6t+t^2)^3}$$

We see then that function W check the following conditions:

$$W(0) = 1; \quad W'(0) = \tfrac{1}{2} \quad and \quad W''(0) = 1. \qquad (18)$$

Thus, (17) becomes:

$$W(L_n) = 1 + \tfrac{1}{2}L_n + \tfrac{1}{2}L_n^2 + O(L_n^3) \qquad (19)$$

Using (16), we get:

$$W(L_n) = 1 + c_2e_n + [-c_2^2 + 3c_3]e_n^2 + O(e_n^3) \qquad (20)$$

Substituting (15) and (20) in (11), we obtain error equation:

$$e_{n+1} = -c_3e_n^3 + O(e_n^4).$$

Thus, the three-order convergence of the method *(IHM)* is established. This completes the proof.

III. Study of the Global Convergence of Improved Halley's Method

We will give a first study on global convergence of proposed method in its three forms: monotonic convergence toward the root, convergence towards the root by oscillating around it (between two successive iterations, we have always oscillation) and convergence (between two successive iterations, one can have monotonic or oscillating convergence) [15], [27], [28], [30].
We will use the following lemma to study the convergence of method (IHM).

Lemma 1. *Let us write the I.F. of method* (IHM)

$$IH(x) = x - \frac{4f(x)}{f'(x)}\left[\frac{1-L_f(x)}{L_f^2(x)-6L_f(x)+4}\right]. \qquad (21)$$

Then, the derivative of function IH is given by:

$$IH'(x) = \frac{[L_f(x)]^2\left[5L_f^2(x) - 4L_{f'}(x)\left(L_f^2(x) - 2L_f(x) + 2\right)\right]}{\left[L_f^2(x) - 6L_f(x) + 4\right]^2}. \qquad (22)$$

A. Monotonic convergence of Improved Halley's method

Let us consider function g of a real variable defined on interval $(-\infty,\ 3-\sqrt{5})$ by:

$$g(x)=\frac{5x^2}{4(x^2-2x+2)}. \tag{23}$$

Theorem 2. *Let f''' be continuous, $f'\neq 0$, $f''\neq 0$, $L_f<3-\sqrt{5}$ and $L_{f'}(x)\leq g\left(L_f(x)\right)$ on an interval $[a,b]$ containing root α of f. Then Improved Halley's method, given by (11), is decreasing (resp. increasing) and converges to α from any point $x_0\in[a,b]$ such that $f(x_0)f'(x_0)>0$ (resp. $f(x_0)f'(x_0)<0$).*

Proof: It is clear that if $f(x_0)f'(x_0)>0$, then $x_0>\alpha$. By applying the Mean Value Theorem, we obtain:

$$x_1-\alpha=IH(x_0)-IH(\alpha)=IH'(c)(x_0-\alpha),$$

For some $c\in(\alpha,x_0)$. Knowing that $L_{f'}(x)\leq g\left(L_f(x)\right)$, we deduce that $IH'\geq 0$ on $[\alpha,\mathrm{b}]$. So $x_1\geq\alpha$. By induction, we obtain that $x_n\geq\alpha$ for all $n\in\mathbb{N}$.
On the other hand, according (11), we have, for $n=0$:

$$x_1-x_0=-\frac{4f(x_0)}{f'(x_0)}\left(\frac{1-L_0}{L_0^2-6L_0+4}\right),$$

Since $L_0=L_f(x_0)<3-\sqrt{5}$, so $\frac{1-L_0}{L_0^2-6L_0+4}>0$, and as $\frac{f(x_0)}{f'(x_0)}>0$, we deduce that $x_1<x_0$. Now it is easy to prove by induction that $x_{n+1}\leq x_n$ for all $n\in\mathbb{N}$.
Thereby, the sequence (x_n), given by (11), is decreasing and converges to a limit r, where $r\in[a,b]$ and $r\geq\alpha$. So, taking limits in (11) we get:

$$r=r-\frac{4f(r)}{f'(r)}\left(\frac{1-L_f(r)}{L_f^2(r)-6L_f(r)+4}\right).$$

Consequently $f(r)=0$, because we know that $L_f(r)\neq 1$. As α is unique root of f on $[a,b]$, so $r=\alpha$. This completes the proof of theorem.

Corollary 1. *Let f''' be continuous, $f'\neq 0$, $f''\neq 0$, $L_f<3-\sqrt{5}$ and $L_{f'}\leq 0$ on an interval $[a,b]$ containing root α of f. Then Improved Halley's method, given by (11), is decreasing (resp. increasing) and converges to α, from any point $x_0\in[a,b]$ such that $f(x_0)f'(x_0)>0$ (resp. $f(x_0)f'(x_0)<0$).*

Proof: Taking into account that $g\geq 0$ on $(-\infty,3-\sqrt{5})$, it follows that condition $L_{f'}\leq g(\,L_f)$ is well satisfied when $L_f<3-\sqrt{5}$ and $L_{f'}\leq 0$. By applying theorem 2, we complete proof.

B. Oscillating Convergence of Improved Halley's method

Let us consider function h of a real variable defined on interval $J=(-\infty,0)\cup(0,\frac{1}{2}]$ by:

$$h(x)=\frac{3}{2}+\frac{4(2x^2-3x+1)}{x^2(x^2-2x+2)}. \tag{24}$$

Theorem 3. *Let f''' be continuous, $f'\neq 0, f''\neq 0$, and $L_f\leq\frac{1}{2}$ on an interval $[a,b]$ containing root α of f.
If function $L_{f'}$ satisfies condition:*

$$g\left(L_f(x)\right)\leq L_{f'}(x)<h\left(L_f(x)\right), \tag{25}$$

On $[a,\alpha)\cup(\alpha,b]$, then sequence (x_n), given by (11), converges towards root α by oscillating around it from any point $x_0\in[a,\alpha)\cup(\alpha,b]$, such that $a\leq IH(x_0)\leq b$.

Proof: Suppose that the starting value satisfies $\alpha<x_0\leq b$. By Mean Value Theorem we have:

$$x_1-\alpha=IH(x_0)-IH(\alpha)=\ IH'(c)(x_0-\alpha)\,,$$

For some $c\in(\alpha,x_0)$.We have $IH'(\alpha)=0$. If $L_f\leq\frac{1}{2}$ on $[a,b]$ and $L_{f'}(x)$ satisfies the condition (25) on $[a,\alpha)\cup(\alpha,b]$, it follows that $-1<IH'(x)\leq 0$ on $[a,b]$. Thus, there exists $M\in(0,1)$ such that $|IH'(x)|\leq M$ on $[a,b]$. On the other hand, since $IH'(x)\leq 0$ on $[a,b]$, $\alpha<x_0\leq b$ and $a\leq x_1=IH(x_0)\leq b$, it is immediate that $x_1\in[a,\alpha]$ and $x_2\geq\alpha$. Besides, from the inequality $|x_2-\alpha|\leq M^2|x_0-\alpha|<|x_0-\alpha|$, it follows that $x_2\in[\alpha,b]$. By mathematical induction, we obtain that $x_{2n}\in[\alpha,b]$ and $x_{2n+1}\in[a,\alpha]$ for all $n\in\mathbb{N}$. Moreover $|x_n-\alpha|\leq M^n|x_0-\alpha|$ and therefore, the sequence (11) converges to root α in an oscillating fashion.
The case $f(x_0)f'(x_0)<0$ is similar.
It should be noted that in the present case, between two successive iterations, there is always an oscillation around α.

Corollary 2. *Let f''' be continuous, $f'\neq 0$, $f''\neq 0$, If $-1-\sqrt{3}\leq L_f\leq\frac{1}{2}$ and $\frac{5}{8}\leq L_{f'}<\frac{3}{2}$ on an interval $[a,b]$ containing root α of f, then sequence (x_n), given by (11), converges towards root α by oscillating around it, starting from any point $x_0\in[a,b]$ where $a\leq IH(x_0)\leq b$.*

Proof: Taking into account that $g(t)\leq\frac{5}{8}$ and $h(t)\geq\frac{3}{2}$ for all $t\in\left[-1-\sqrt{3},\ 0\right)\cup\left(0,\frac{1}{2}\right]$, it follows that the condition (25) is satisfied whether $-1-\sqrt{3}\leq L_f\leq\frac{1}{2}$ and $\frac{5}{8}\leq L_{f'}<\frac{3}{2}$ on $[a,b]$. By applying theorem 3, we complete proof.

C. Convergence of Improved Halley's method

We consider the function k of a real variable defined on interval $I=(-\infty,0)\cup(0,\frac{1}{2}]$ by:

$$k(x)=\frac{x^4+3x^3-11x^2+12x-4}{x^2(x^2-2x+2)}. \tag{26}$$

Theorem 4. *Let f''' be continuous, $f'\neq 0, f''\neq 0$ and $L_f\leq\frac{1}{2}$ on an interval [a, b] containing root α of f. If:*

$$k\left(L_f(x)\right)<L_{f'}(x)<h\left(L_f(x)\right), \tag{27}$$

On $[\mathrm{a},\alpha)\cup(\alpha,\mathrm{b}]$, then Improved Halley's method, given by (11), converges to α from any point $\mathrm{x}_0\in[\mathrm{a},\mathrm{b}]$ such that $\mathrm{a}\leq\mathrm{IH}(\mathrm{x}_0)\leq\mathrm{b}$.

Proof: Suppose that the initial value satisfies $\alpha < x_0 \le b$. By Mean Value Theorem we have:

$$x_1 - \alpha = IH(x_0) - IH(\alpha) = IH'(\lambda)(x_0 - \alpha),$$

For some $\lambda \in (\alpha, x_0)$. Taking into account of (22), (27) and that $IH'(\alpha)=0$, it follows that $-1 < IH' < 1$ on $[a,b]$. Thus, there exists $M \in (0,1)$ such that $|IH'| \le M$ on $[a,b]$ and consequently $|x_1 - \alpha| \le M|x_0 - \alpha|$. By induction, we obtain that $|x_n - \alpha| \le M^n|x_0 - \alpha|$ for all $n \in \mathbb{N}$. Besides, since $a \le IH(x_0) \le b$, it is immediate that $a \le x_n \le b$ for all $n \in \mathbb{N}$ and therefore sequence (11) converges to α.
If $a \le x_0 < \alpha$, all the previous results turn out to be valid by modifying slightly the reasoning used.
It should be noted that, in present case, the convergence can take one of three forms: monotonic convergence, oscillating convergence in a regular way (between two successive iterations, there is always oscillation) or non-regular oscillation (between two successive iterations, it sometimes there is oscillation, sometimes no).

Corollary 3. *Let f''' be continuous, $f' \neq 0$ and $f'' \neq 0$ on an interval $[a,b]$ containing root α of f. If $-5 \le L_f \le \frac{1}{2}$ and $0 \le L_{f'} < \frac{3}{2}$ on $[a,b]$, then sequence (x_n), given by (11), converges to α from any point $x_0 \in [a,b]$ such that $a \le IH(x_0) \le b$.*

Proof: We have $k < 0$ and $h \ge \frac{3}{2}$ on $\left[-5, \frac{1}{2}\right]$. It follows that condition (27) is well verified when $0 \le L_{f'} < \frac{3}{2}$ and $-5 \le L_f \le \frac{1}{2}$. By applying theorem 4, we complete proof.

IV. Advantages of improved Halley's method

To illustrate the efficiency of Improved Halley's method, we will make an analytical comparison of its speed of convergence with those of Halley, Super-Halley and Chebyshev's methods [30].
The methods of Halley, Super Halley, Improved Halley and their Iterative functions are respectively given by (4), (9), (11) and (5), (10), (12).
The Chebyshev's method and its Iterative Function are given respectively by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + \frac{L_n}{2}\right). \tag{28}$$

$$C(x) = x - \frac{f(x)}{f'(x)}\left(1 + \frac{L_f(x)}{2}\right). \tag{29}$$

Theorem 5. *Let f''' be continuous, $f' \neq 0$, $f'' \neq 0$, $L_{f'} \le 0$ and $0 \le L_f < 3 - \sqrt{5}$ on an interval $[a,b]$ containing root α of f. Starting from the same initial point $x_0 \in [a,b]$, the rate of convergence the method (IHM) is higher than those of the methods of Halley, Super-Halley and Chebychev.*

Proof: Suppose that starting value satisfies $f(x_0)f'(x_0) > 0$ so $x_0 > \alpha$. According to several authors in ([2], [6], [7], [9], [10], [13], [15], [27], [28]), and from corollary 1 given above, we now know that if:

- $L_f < 2$ and $L_{f'} \le \frac{3}{2}$,
- $L_f < 1$ and $L_{f'} \le L_f$,
- $L_f > -2$ and $L_{f'} \le 3$ and
- $L_f < 3 - \sqrt{5}$ and $L_{f'} \le 0$,

On an interval $[a,b]$ containing root α of f, the sequences defined by (4), (9), (28) and (11) are respectively decreasing and converges to α from any point $x_0 \in [a,b]$. To get simultaneous convergence for all them, we consider the same following assumptions $L_{f'} \le 0$ and $0 \le L_f < 3 - \sqrt{5}$ on $[a,b]$.
Let us consider the sequences (x_n), (y_n), (z_n) and (t_n) defined respectively by (11), (4), (28) and (9). Since $x_0 = y_0 = z_0 = t_0$ and the four sequences are decreasing, we expect that $x_n \le y_n$, $x_n \le z_n$ and $x_n \le t_n$ for all $n \in \mathbb{N}$. This can be proved by induction. Let $n = 1$, then:

$$x_1 - y_1 = 2\frac{f(x_0)}{f'(x_0)}\left(\frac{L_0^2}{(L_0 - 2)(L_0^2 - 6L_0 + 4)}\right) \le 0,$$

$$x_1 - z_1 = \frac{f(x_0)}{2f'(x_0)}\left(\frac{L_0^2(L_0 - 4)}{(L_0^2 - 6L_0 + 4)}\right) \le 0,$$

$$x_1 - t_1 = -\frac{f(x_0)}{f'(x_0)}\left(\frac{L_0^3}{2(1 - L_0)(L_0^2 - 6L_0 + 4)}\right) \le 0.$$

Now we assume that $x_n \le y_n$, $x_n \le z_n$ and $x_n \le t_n$. Since, under the above hypotheses, IH is increasing function on $[a,b]$, we obtain $IH(x_n) \le IH(y_n)$, $IH(x_n) \le IH(z_n)$ and $IH(x_n) \le IH(t_n)$.
On the other hand, we have:

$$\begin{aligned} &IH(y_n) - H(y_n) \\ &= 2\frac{f(y_n)}{f'(y_n)}\left(\frac{L_f^2(y_n)}{(L_f(y_n) - 2)(L_f^2(y_n) - 6L_f(y_n) + 4)}\right) \le 0, \end{aligned}$$

$$IH(z_n) - C(z_n) = \frac{f(z_n)}{2f'(z_n)}\left(\frac{L_f^2(z_n)(L_f(z_n) - 4)}{(L_f^2(z_n) - 6L_f(z_n) + 4)}\right) \le 0,$$

$$\begin{aligned} &IH(t_n) - SH(t_n) \\ &= -\frac{f(t_n)}{f'(t_n)}\left(\frac{L_f^3(t_n)}{2(1 - L_f(t_n))(L_f^2(t_n) - 6L_f(t_n) + 4)}\right) \le 0. \end{aligned}$$

We deduce that $IH(x_n) \le H(y_n)$, $IH(x_n) \le C(z_n)$ and $IH(x_n) \le SH(t_n)$. So $x_{n+1} \le y_{n+1}$, $x_{n+1} \le z_{n+1}$ and $x_{n+1} \le t_{n+1}$. This completes induction.
The case $f(x_0)f'(x_0) < 0$ is analogous to the previous one.

V. Numerical results

Numerical computations reported here have been carried out in MATLAB $R2015b$ and the stopping criterion has been taken as $|x_{n+1} - x_n| \le 10^{-15}$ and $|f(x_n)| \le 10^{-15}$.

A. Monotonic convergence of new method

We Consider function $f(x) = 2\sin x - 1$ defined on interval $I = \left[\frac{\Pi}{10}, \frac{9\Pi}{20}\right]$. We have $f' \neq 0$, $f'' \neq 0$, f''' continuous on I, $L_{f'}(x) = \frac{-\cos^2(x)}{\sin^2(x)}$. It is easy to check that

$L_f < 3-\sqrt{5}$ and that $L_{f'} \le 0$ on *I*.

By choosing $x_0 = \frac{2\Pi}{5}$, we have $f(x_0)f'(x_0) > 0$. Therefore, by Corollary 1, the sequence (x_n), given by (11), is decreasing and converges to the unique root $\alpha = 0.5235987755982989$ of f on I, see Table I.

TABLE I
MONOTONIC CONVERGENCE OF METHOD (IHM)

n	x_n
0	1.256637061435917
1	0.629523165216759
2	0.5237863276430287
3	0.5235987755993984
4	0.5235987755982989

B. *Oscillating convergence of Improved Halley's method*

Let us consider $f(x) = 2-2e^{1-x}$ defined on interval $T = \left[\frac{1}{3},\ 2\right]$.

We have $f' \neq 0,\ f'' \neq 0$ and f''' continuous on T.

The function $L_f(x) = 1-e^{x-1}$ satisfies the condition $-1-\sqrt{3} \le L_f(x) \le \frac{1}{2}$ on T. The function $L_{f'}(x) = 1$ and therefore it checks the condition $\frac{5}{8} \le L_{f'}(x) < \frac{3}{2}$ on T.

If $x_0 = 1.9$, then $x_1 = 0.9203035597004922 \in T$. According to corollary 3, the sequence (x_n), given by (11), must converge towards the root $\alpha = 1$ by oscillating around it, see Table II.

TABLE II
OSCILLATING CONVERGENCE OF METHOD (IHM)

n	x_n
0	1.9
1	0.9203035597004922
2	1.000089878162929
3	0.999999999999879
4	*1.000000000000000*

C. *Convergence of Improved Halley's method*

Let us consider $f(x) = x^3+4x^2-10$ defined on the interval $I = [0.47;\ 2.7]$.

We have f'''continuous, $f' \neq 0$ and $f'' \neq 0$ on I.

The function $L_f(x) = \frac{(x^3+4x^2-10)(6x+8)}{(3x^2+8x)^2}$ satisfies condition $-5 \le L_f \le \frac{1}{2}$ on I. The function $L_{f'}(x) = \frac{6(3x^2+8x)}{(6x+8)^2}$ and it checks condition $0 \le L_{f'} < \frac{3}{2}$ on I.

For $x_0 = 0.48$, we have $x_1 = 1.306504563900452 \in I$. Therefore, according to corollary 3, the sequence (x_n), given by (11), converges to the unique root $\alpha = 1.365230013414097$ of f on I, see Table III.

TABLE III
CONVERGENCE OF METHOD (IHM)

n	x_n	type of convergence between two iterations
0	0.48	
1	1.306504563900452	Monotonic
2	1.365242242550277	Oscillating
3	1.365230013414097	Monotonic

D. *Comparison with third order methods*

In this section, we present some examples to illustrate the efficiency of method(IHM). We give number of iterations (N) or/and number of function evaluations (NOFE) required to satisfy stopping criterion, FL denotes that method fails, CU denotes that method converge to undesired root and D denotes for divergence.

The tests functions that will be used in Table V and Table VI, and their roots α, are given in Table IV.

TABLE IV
TEST FUNCTIONS AND THEIR ROOTS

Test functions	Root**α**
$f_1(x)=1+(x-3)e^x$	2.947530902542285
$f_2(x)=(x-2)^4-1$	3.000000000000000
$f_3(x)=x^3-10$	2.154434690031884
$f_4(x)=(\sin x)^2-\sqrt{3}\sin x$	-3.141592653589793
$f_5(x)=(x-2)^2-\ln x$	3.057103549994738
$f_6(x)=e^x-3x^2$	0.910007572488709
$f_7(x)=x^3+4x^2-10$	1.365230013414097

We shall present the numerical results obtained by employing classical Newton methods (CN), and some third order methods: Improved Halley's method (IHM) given by (11), Chebyshev's method (CM) given by (28), Halley's method (HM) defined by (4), Super Halley's method (SHM) given by (9), Sharma's method (SM) defined by (20), with $a_n = 1$, in [17], Chun's method (CH) defined by (23), with $a_n = 1$, in [14]. The results are summarized in Table V.

TABLE V
COMPARISON OF METHOD (IHM) WITH THIRD ORDER METHOD

		N: Number of iterations						
f_i	x_0	CN	CM	SM	CH	HM	SHM	**IHM**
f_1	2.55	6	6	6	6	4	4	**3**
	2.6	6	5	5	5	4	4	**3**
f_2	2.68	6	CU	5	5	4	4	**3**
f_3	1.4	6	5	5	5	4	4	**3**
	0.6	10	13	D	17	5	5	**4**
f_4	-3.6	5	5	4	4	4	3	**3**
	-2.525	5	4	4	4	4	4	**3**
f_5	5	6	4	4	4	4	4	**3**
	5.5	7	5	5	5	4	4	**3**
f_6	0.45	6	F	5	6	4	4	**4**
	1.47	5	4	4	4	4	3	**3**
f_7	0.68	6	7	7	7	4	4	**3**
	2.3	6	4	4	4	4	3	**3**

The presented results in Table V, indicate that, for the most of cases considered, the new method (IHM) are more efficient and perform better than other used third-order methods, as its converge to root much faster and take lesser number of iterations.

E. *Comparison with higher order methods*

In Table 7, we will compare the new method (IHM) with some higher order methods. (F) denotes fifth-order method of Fang and al. (formula (2) in [39]). (C) denotes sixth order method of Chun and Ham (formula (10), (11), (12) in [32]). (K) denotes sixth order method of Kou (first formula in [36]). (B) denotes eighth order method of Bi and al. (formula (36) with **α**=1 in [38]). (Z) denotes eighth order method of Zhao and al. (formula (37) with *a=3* in [33]).

The efficiency of Improved Halley's method is also confirmed by Table VI which shows that, for the examples taken, it require a similar or smaller number of function evaluations than some famous methods of Higher order.

TABLE VI
COMPARISON OF METHOD (IHM) WITH HIGHER ORDER METHODS

		N: Number of iterations						NOFE: Number of functions evaluations					
f_i	x_0	F	C	K	B	Z	$\boldsymbol{IHM}$	F	C	K	B	Z	$\boldsymbol{IHM}$
f_1	2.55	3	3	3	FL	3	**3**	12	12	12	FL	12	**9**
	2.42	6	17	3	3	4	**3**	24	68	12	12	16	**9**
f_2	2.68	D	5	2	3	3	**3**	D	20	8	12	12	**9**
f_3	1.2	6	5	3	3	FL	**3**	24	20	12	12	FL	**9**
	0.6	11	21	4	D	FL	**4**	44	84	16	D	FL	**12**
f_4	-3.67	4	4	3	FL	2	**3**	16	16	12	FL	8	**9**
f_5	2.45	4	4	3	3	3	**3**	16	16	12	12	12	**9**
	6.8	4	3	3	3	3	**4**	16	12	12	12	12	**12**
f_6	0.45	3	3	3	3	2	**3**	12	12	12	12	8	**9**
	2.422	3	3	4	3	3	**4**	12	12	16	12	12	**12**
f_7	0.5	FL	5	3	3	3	**3**	FL	20	12	12	12	**9**
	0.47	9	6	3	3	3	**3**	36	24	12	12	12	**9**

VI. CONCLUSION

In this paper, based on the Halley's method, we have constructed a simple and interesting formula of third-order for solving nonlinear equations with simple roots. A first study on the global convergence of this method was performed. We have distinguished between the case where said method converges monotonically, the one where it converges oscillating around the root, and the one where it converges randomly. On the other hand, the efficacity of this method has been analytically shown in justifying that, under certain conditions, its convergence speed is greater than Halley's method, Super Halley's method and Chebyshev's method. Finally, the numerical comparisons, with several methods of third order and much higher order, overwhelmingly have supported the theoretical results derived and have confirmed the robust and efficient nature of the proposed technique.

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