Oscillation for a Class of Conformable Fractional Dynamic Equations on Time Scales

Qinghua Feng*

Abstract—**In this paper, some new oscillation criteria for a class of fractional dynamic equations with damping term on time scales are established by use of the properties of fractional calculus and generalized Riccati transformation technique, where the fractional derivative is defined in the sense of the conformable fractional derivative. The established oscillation criteria unify continuous and discrete analysis, and are new results so far in the literature. Oscillation criteria for corresponding dynamic equations on time scales involving integer order derivative are special cases of the present results. For illustrating the established results, some examples are also presented.**

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Index Terms—**oscillation; fractional dynamic equations; time scales; conformable fractional derivative; damping**

I. Introduction

It is well known that research on solutions of various differential equations, fractional differential equations, dynamic equations is very important in the literature, such as the qualitative properties involving stability, existence and so on [1-3], the numerical methods [4-8], the analytical method for finding exact solutions [9,10]. Oscillation belongs to the range of qualitative properties analysis. In the last few decades, research for oscillation of various equations including differential equations, difference equations has been a hot topic in the literature, and much effort has been done to establish new oscillatory criteria for these equations so far (for example, see [11-22], and the references therein). In [23], Hilger initiated the theory of time scale trying to treat continuous and discrete analysis in a consistent way. Based on the theory of time scale, Many authors have taken research in oscillation of various dynamic equations on time scales (see [24-40] for example). In these investigations for oscillation of dynamic equations on time scales, we notice that most of the results are concerned of dynamic equations involving derivatives of integer order, while none attention has been paid to the research of oscillation of fractional dynamic equations on time scales so far in the literature.

A time scale is an arbitrary nonempty closed subset of the real numbers. In this paper, $\mathbf{T}$ denotes an arbitrary time scale. On $\mathbf{T}$ we define the forward and backward jump operators $\sigma \in (\mathbf{T},\mathbf{T})$ and $\rho \in (\mathbf{T},\mathbf{T})$ such that $\sigma(t) = \inf\{s \in \mathbf{T}, s > t\}$, $\rho(t) = \sup\{s \in \mathbf{T}, s < t\}$.

Definition 1.1. A point $t \in \mathbf{T}$ is said to be left-dense if $\rho(t) = t$ and $t \neq \inf \mathbf{T}$, right-dense if $\sigma(t) = t$ and $t \neq \sup \mathbf{T}$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The set $\mathbf{T}^\kappa$ is defined to be $\mathbf{T}$ if $\mathbf{T}$ does not have a left-scattered maximum, otherwise it is $\mathbf{T}$ without the left-scattered maximum.

Definition 1.2. A function $f \in (\mathbf{T},\mathbf{R})$ is called rd-continuous if it is continuous at right-dense points and if the left-sided limits exist at left-dense points, while f is called regressive if $1 + \mu(t)f(t) \neq 0$, where $\mu(t) = \sigma(t) - t$. C_{rd} denotes the set of rd-continuous functions, while $\mathcal{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathcal{R}^+ = \{f|f \in \mathcal{R},\ 1 + \mu(t)f(t) > 0,\ \forall t \in \mathbf{T}\}$.

Definition 1.3: For some $t \in \mathbf{T}^\kappa$, and a function $f \in (\mathbf{T},\mathbf{R})$, the $delta \ \ derivative$ of f at t is denoted by $f^\Delta(t)$ (provided it exists) with the property such that for every $\varepsilon > 0$ there exists a neighborhood $\mathcal{U}$ of t satisfying

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in \mathcal{U}$.

Note that if $\mathbf{T} = \mathbf{R}$, then $f^\Delta(t)$ becomes the usual derivative $f'(t)$, while $f^\Delta(t) = f(t+1) - f(t)$ if $\mathbf{T} = \mathbf{Z}$, which represents the forward difference.

Definition 1.4: For $p \in \mathcal{R}$, the $exponential \ function$ is defined by

$$e_p(t,s) = \exp(\textstyle\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau) \text{ for } s,t \in \mathbf{T}.$$

If $\mathbf{T} = \mathbf{R}$, then

$$e_p(t,s) = \exp(\textstyle\int_s^t p(\tau)d\tau), \quad for \ \ s,\ t \in \mathbf{R},$$

If $\mathbf{T} = \mathbf{Z}$, then

$$e_p(t,s) = \prod_{\tau=s}^{t-1}[1 + p(\tau)], \quad for \ \ s,\ t \in \mathbf{Z} \ \ and \ \ s < t.$$

The following two theorems include some known properties on the $exponential \ function$.

Theorem 1.5 [41, Theorem 5.1]. If $p \in \mathcal{R}$, and fix $t_0 \in \mathbf{T}$, then the $exponential \ function$ $e_p(t,t_0)$ is the unique solution of the following initial value problem

$$\begin{cases} y^\Delta(t) = p(t)y(t), \\ \\ y(t_0) = 1. \end{cases}$$

Theorem 1.6 [41, Theorem 5.2]. If $p \in \mathcal{R}^+$, then $e_p(t,s) > 0$ for $\forall s,\ t \in \mathbf{T}$.

Recently, Benkhettou etc. developed a conformable fractional calculus theory on arbitrary time scales [42], and

Manuscript received September 19, 2019.
Q. Feng is with the School of Mathematics and Statistics, Shandong University Of Technology, Zibo, Shandong, 255049 China *e-mail: fqhua@sina.com

established the basic tools for fractional differentiation and fractional integration on time scales.

Definition 1.7 [42, Definition 1]. For $t \in \mathbf{T}^{\kappa}$, $\alpha \in (0,1]$, and a function $f \in (\mathbf{T}, \mathbf{R})$, the fractional derivative of α order for f at t is denoted by $f^{(\alpha)}(t)$ (provided it exists) with the property such that for every $\varepsilon > 0$ there exists a neighborhood $\mathcal{U}$ of t satisfying

$$|[f(\sigma(t)) - f(s)]t^{1-\alpha} - f^{(\alpha)}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in \mathcal{U}$.

Definition 1.8 [42, Definition 28]. If $F^{(\alpha)}(t) = f(t),\ t \in \mathbf{T}^{\kappa}$, then F is called an α-order *antiderivative* of f, and the Cauchy α-fractional integral of f is defined by

$\int_a^b f(t)\Delta^{\alpha} t = \int_a^b f(t)t^{\alpha-1}\Delta t = F(b) - F(a)$, where $a,\ b \in \mathbf{T}$.

Theorem 1.9 [42, Theorem 4]. For $t \in \mathbf{T}^{\kappa}$, $\alpha \in (0,1]$, and a function $f \in (\mathbf{T}, \mathbf{R})$, the following conclusions hold:

(i). If f is conformal fractional differentiable of order α at $t > 0$, then f is continuous at t.

(ii). If f is continuous at t and t is right-scattered, then f is conformable fractional differentiable of order α at t with $f^{(\alpha)}(t) = \dfrac{f(\sigma(t)) - f(t)}{\sigma(t) - t}t^{1-\alpha} = \dfrac{f(\sigma(t)) - f(t)}{\mu(t)}t^{1-\alpha}$.

(iii). If t is right-dense, then f is conformable fractional differentiable of order α at t if, and only if, the limit $\lim\limits_{s \to t} \dfrac{f(s) - f(t)}{s - t}t^{1-\alpha}$ exists as a finite number. In this case, $f^{(\alpha)}(t) = \lim\limits_{s \to t} \dfrac{f(s) - f(t)}{s - t}t^{1-\alpha}$.

(iv). If f is fractional differentiable of order α at t, then $f(\sigma(t)) = f(t) + \mu(t)t^{1-\alpha}f^{(\alpha)}(t)$.

Corollary 1.10. According to the definition of the conformable fractional differentiable of order α, it holds that $f^{(\alpha)}(t) = t^{1-\alpha}f^{\Delta}(t)$, where $f^{\Delta}(t)$ is the usual Δ derivative in the case $\alpha = 1$. Furthermore, if $f^{(\alpha)}(t) > 0$ (< 0) for $t > 0$, then f is increasing (decreasing) for $t > 0$.

By a combination of Theorem 1.5 and Corollary 1.10 one can obtain the following theorem.

Theorem 1.11: Let $\widetilde{p}(t) = t^{\alpha-1}p(t)$, $\alpha \in (0,1]$. If $\widetilde{p} \in \mathcal{R}$, and fix $t_0 \in \mathbf{T}$, then the *exponential function* $e_{\widetilde{p}}(t, t_0)$ is the unique solution of the following initial value problem

$$\begin{cases} y^{(\alpha)}(t) = p(t)y(t), \\ y(t_0) = 1. \end{cases}$$

Theorem 1.12 [42, Theorem 15]. Assume $f,\ g \in (\mathbf{T}, \mathbf{R})$ are conformable fractional differentiable of order α. Then

(i). $(f+g)^{(\alpha)}(t) = f^{(\alpha)}(t) + g^{(\alpha)}(t)$.

(ii). $(fg)^{(\alpha)}(t) = f^{(\alpha)}(t)g(t) + f(\sigma(t))g^{(\alpha)}(t)$

$= f^{(\alpha)}(t)g(\sigma(t)) + f(t)g^{(\alpha)}(t)$.

(iii). $(\frac{1}{f})^{(\alpha)}(t) = -\dfrac{f^{(\alpha)}(t)}{f(t)f(\sigma(t))}$.

(iv). $(\frac{f}{g})^{(\alpha)}(t) = \dfrac{f^{(\alpha)}(t)g(t) - f(t)g^{(\alpha)}(t)}{g(t)g(\sigma(t))}$.

Theorem 1.13 [42, Theorem 31]. Let $\alpha \in (0,1]$, $a,\ b,\ c \in \mathbf{T}$, $\lambda \in \mathbf{R}$, and $f,\ g$ be two rd-continuous functions. Then the following properties hold:

(i). $\int_a^b [f(t) + g(t)]\Delta^{\alpha} t = \int_a^b f(t)\Delta^{\alpha} t + \int_a^b g(t)\Delta^{\alpha} t$.

(ii). $\int_a^b (\lambda f)(t)\Delta^{\alpha} t = \lambda \int_a^b f(t)\Delta^{\alpha} t$.

(iii). $\int_a^b f(t)\Delta^{\alpha} t = -\int_b^a f(t)\Delta^{\alpha} t$.

(iv). $\int_a^b f(t)\Delta^{\alpha} t = \int_a^c f(t)\Delta^{\alpha} t + \int_c^b f(t)\Delta^{\alpha} t$.

(v). $\int_a^a f(t)\Delta^{\alpha} t = 0$.

(vi). For $|f(t)| \leq g(t)$, it holds that $|\int_a^b f(t)\Delta^{\alpha} t| \leq \int_a^b g(t)\Delta^{\alpha} t$.

(vii). If $f(t) > 0$, then $\int_a^b f(t)\Delta^{\alpha} t \geq 0$.

Theorem 1.14. Let $\alpha \in (0,1]$, $f,\ g$ be two rd-continuous functions. Then

$$\int_a^b f^{(\alpha)}(t)g(t)\Delta^{\alpha} t = [f(t)g(t)]_a^b - \int_a^b f(\sigma(t))g^{(\alpha)}(t)\Delta^{\alpha} t.$$

The proof of Theorem 1.14 can be completed by fulfilling α-fractional integral for the first equality in Theorem 1.12 (ii).

Motivated by the analysis above, in this paper, we will consider oscillation of solutions of the following fractional dynamic equation with damping term on time scales of the following form:

$$(a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)})^{(\alpha)} + p(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}$$

$$+q(t)x(t) = 0,\ t \in \mathbf{T}_0, \tag{1.1}$$

where $\alpha \in (0,1]$, $\mathbf{T}$ is an arbitrary time scale, $\mathbf{T}_0 = [t_0, \infty) \bigcap \mathbf{T}$, $t_0 > 0$, $a,\ r,\ p,\ q \in C_{rd}(\mathbf{T}_0, \mathbf{R}_+)$.

A solution of Eq. (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Eq. (1.1) is said to be oscillatory in case all its solutions are oscillatory.

We will establish some new oscillation criteria for Eq. (1.1) by properties of conformable fractional calculus and generalized Riccati transformation technique in Section 2, and present some applications for the established results in Section 3. Some conclusions are presented in Section 4. Throughout this paper, $\mathbf{R}$ denotes the set of real numbers and $\mathbf{R}_+ = (0, \infty)$, while $\mathbf{Z}$ denotes the set of integers. $t_i \in \mathbf{T}$, $[t_i, \infty)_{\mathbf{T}} = [t_i, \infty) \bigcap \mathbf{T}$, $i = 0, 1, ..., 5$. For the sake of convenience, denote $\delta_1(t, t_i) = \int_{t_i}^t \dfrac{e_{-\frac{\widetilde{p}}{a}}(s, t_0)}{a(s)}\Delta^{\alpha} s$, where $\widetilde{p}(t) = t^{\alpha-1}p(t)$.

II. Main Results

Lemma 2.1. Suppose $-\frac{\widetilde{p}}{a}\in\mathcal{R}_+$, and assume that

$$\int_{t_0}^{\infty}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}\Delta^\alpha s=\infty, \tag{2.1}$$

$$\int_{t_0}^{\infty}\frac{1}{r(s)}\Delta^\alpha s=\infty, \tag{2.2}$$

and Eq. (1.1) has a positive solution x on $[t_0,\infty)_{\mathbf{T}}$. Then we have the following statements:

(i). There exists a sufficiently large t_1 such that $\left(\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}\right)^{(\alpha)}<0$, $[r(t)x^{(\alpha)}(t)]^{(\alpha)}>0$ on $[t_1,\infty)_{\mathbf{T}}$.

(ii). If furthermore assume that

$$\lim_{t\to\infty}\sup\int_{t_0}^{t}\Big[\frac{1}{r(\xi)}\int_{\xi}^{\infty}\Big(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s\Big)$$
$$\Delta^\alpha\tau\Big]\Delta^\alpha\xi=\infty, \tag{2.3}$$

then either there exists a sufficiently large t_4 such that $x^{(\alpha)}(t)>0$ on $[t_4,\infty)_{\mathbf{T}}$ or $\lim_{t\to\infty}x(t)=0$.

Proof of (i). By $-\frac{\widetilde{p}}{a}\in\mathcal{R}_+$ and Theorem 1.6, we have $e_{-\frac{\widetilde{p}}{a}}(t,t_0)>0$. Since x is a positive solution of (1.1) on $[t_0,\infty)_{\mathbf{T}}$, by Theorem 1.12 (iv) and Theorem 1.11 we obtain that

$$\Big(\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}\Big)^{(\alpha)}=$$
$$\frac{1}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}\{e_{-\frac{\widetilde{p}}{a}}(t,t_0)(a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)})^{(\alpha)}$$
$$-(e_{-\frac{\widetilde{p}}{a}}(t,t_0))^{(\alpha)}a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}\}$$
$$=\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)})^{(\alpha)}+p(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}$$
$$=\frac{-q(t)x(t)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}<0. \tag{2.4}$$

According to Corollary 1.10 one can see $\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}$ is decreasing on $[t_0,\infty)_{\mathbf{T}}$. Furthermore, by Theorem 1.6 one has $e_{-\frac{\widetilde{p}}{a}}(t,t_0)>0$. So considering $a(t)>0$ one can obtain that $[r(t)x^{(\alpha)}(t)]^{(\alpha)}$ is eventually of one sign. We claim $[r(t)x^{(\alpha)}(t)]^{(\alpha)}>0$ on $[t_1,\infty)_{\mathbf{T}}$. Otherwise, assume there exists a sufficiently large $t_2>t_1$ such that $[r(t)x^{(\alpha)}(t)]^{(\alpha)}<0$ on $t\in[t_2,\infty)_{\mathbf{T}}$. Then from Corollary 1.10 one can see $r(t)x^{(\alpha)}(t)$ is decreasing on $[t_2,\infty)_{\mathbf{T}}$, and from Definition 1.8 it holds that

$$r(t)x^{(\alpha)}(t)-r(t_2)x^{(\alpha)}(t_2)$$
$$=\int_{t_2}^{t}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)a(s)[r(s)x^{(\alpha)}(s)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(s,t_0)a(s)}\Delta^\alpha s$$
$$\le\frac{a(t_2)[r(t_2)x^{(\alpha)}(t_2)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t_2,t_0)}\int_{t_2}^{t}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}\Delta^\alpha s. \tag{2.5}$$

It follows from (2.1) that $\lim_{t\to\infty}r(t)x^{(\alpha)}(t)=-\infty$, and thus there exists a sufficiently large $t_3\in[t_2,\infty)_{\mathbf{T}}$ such that $r(t)x^{(\alpha)}(t)<0$ on $[t_3,\infty)_{\mathbf{T}}$. So

$$x(t)-x(t_3)=\int_{t_3}^{t}\frac{r(s)x^{(\alpha)}(s)}{r(s)}\Delta^\alpha s$$
$$\le r(t_3)x^{(\alpha)}(t_3)\int_{t_3}^{t}\frac{1}{r(s)}\Delta^\alpha s.$$

Due to (2.2) one can deduce that $\lim_{t\to\infty}x(t)=-\infty$, which leads to a contradiction. So it holds that $[r(t)x^{(\alpha)}(t)]^{(\alpha)}>0$ on $[t_1,\infty)_{\mathbf{T}}$, and the proof is complete.

Proof of (ii). According to (i), since $[r(t)x^{(\alpha)}(t)]^{(\alpha)}>0$ on $[t_1,\infty)_{\mathbf{T}}$, from Corollary 1.10 one can see that $x^{(\alpha)}(t)$ is eventually of one sign. So there exists a sufficiently large $t_4>t_1$ such that either $x^{(\alpha)}(t)>0$ or $x^{(\alpha)}(t)<0$ on $[t_4,\infty)_{\mathbf{T}}$.

If $x^{(\alpha)}(t)<0$, then $x(t)$ is decreasing, and considering $x(t)$ is a positive solution of Eq. (1.1) on $[t_0,\infty)_{\mathbf{T}}$, one can obtain that $\lim_{t\to\infty}x(t)=\beta_1\ge 0$ and $\lim_{t\to\infty}r(t)x^{(\alpha)}(t)=\beta_2\le 0$. We claim $\beta_1=0$. Otherwise, assume $\beta_1>0$. Then there exists t_5 such that $x(t)\ge\beta_1$ on $[t_5,\infty)_{\mathbf{T}}$, and fulfilling α-fractional integral for (2.4) from t to ∞ yields

$$-\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}$$
$$=-\lim_{t\to\infty}\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}+\int_{t}^{\infty}\frac{-q(s)x(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s$$
$$\le-\int_{t}^{\infty}\frac{q(s)x(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s$$
$$\le-\beta_1\int_{t}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s,$$

which is followed by

$$-[r(t)x^{(\alpha)}(t)]^{(\alpha)}\le-\beta_1\Big[\frac{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}{a(t)}\int_{t}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s\Big]. \tag{2.6}$$

Substituting t with τ in (2.6), fulfilling α-fractional integral for (2.6) with respect to τ from t to ∞ yields

$$r(t)x^{(\alpha)}(t)=\lim_{t\to\infty}r(t)x^{(\alpha)}(t)$$
$$-\beta_1\int_{t}^{\infty}\Big(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s\Big)\Delta^\alpha\tau$$
$$=\beta_2-\beta_1\int_{t}^{\infty}\Big(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s\Big)\Delta^\alpha\tau$$
$$\le-\beta_1\int_{t}^{\infty}\Big(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s\Big)\Delta^\alpha\tau,$$

which implies

$$x^{(\alpha)}(t)\le-\beta_1$$

$$[\frac{1}{r(t)}\int_t^\infty(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_\tau^\infty\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s)\Delta^\alpha\tau]. (2.7)$$

Substituting t with ξ in (2.7), fulfilling α-fractional integral for (2.7) with respect to ξ from t_5 to t yields

$$x(t)-x(t_t)\leq-\beta_1\int_{t_5}^t[\frac{1}{r(\xi)}\int_\xi^\infty(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}$$
$$\int_\tau^\infty\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s)\Delta^\alpha\tau]\Delta^\alpha\xi. \quad (2.8)$$

By (2.8) and (2.3) we have $\lim_{t\to\infty}x(t)=-\infty$, which leads to a contradiction. So it holds that $\beta_1=0$. The proof is completed.

Lemma 2.2. Suppose $-\frac{\widetilde{p}}{a}\in\mathcal{R}_+$, and assume that x is a positive solution of Eq. (1.1) such that

$$[r(t)x^{(\alpha)}(t)]^{(\alpha)}>0,\ x^{(\alpha)}(t)>0 \text{ on } [t_1,\infty)_{\mathbf{T}},$$

where $t_1\geq t_0$ is sufficiently large. Then we have

$$x^{(\alpha)}(t)\geq\frac{\delta_1(t,t_1)}{r(t)}[\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}].$$

Proof. From Lemma 2.1 one can see that $\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}$ is decreasing on $[t_1,\infty)$. So

$$r(t)x^{(\alpha)}(t)\geq r(t)x^{(\alpha)}(t)-r(t_1)x^{(\alpha)}(t_1)$$
$$=\int_{t_1}^t\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)a(s)[r(s)x^{(\alpha)}(s)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(s,t_0)a(s)}\Delta^\alpha s$$
$$\geq\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}\int_{t_1}^t\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}\Delta^\alpha s$$
$$=\delta_1(t,t_1)\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)},$$

and then

$$x^{(\alpha)}(t)\geq\frac{\delta_1(t,t_1)}{r(t)}[\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}].$$

The proof is completed.

Theorem 2.3. Assume (2.1), (2.2), (2.3) hold, $-\frac{\widetilde{p}}{a}\in\mathcal{R}_+$, $\phi,\ \varphi$ are two given nonnegative functions on $\mathbf{T}$, and for all sufficiently large t_1, there exists $t_2>t_1$ such that

$$\lim_{t\to\infty}\sup\{\int_{t_2}^t\{q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\phi(s)[a(s)\varphi(s)]^{(\alpha)}$$
$$+\frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}$$
$$-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}\}\Delta^\alpha s\}$$
$$=\infty. \quad (2.9)$$

Then every solution of Eq. (1.1) is oscillatory or tends to zero.

Proof. Assume (1.1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbf{T}}$. Without loss of generality, assume $x(t)>0$ on $[t_1,\infty)_{\mathbf{T}}$, for some sufficiently large t_1. By Lemma 2.1 (ii) it holds either $x^{(\alpha)}(t)>0$ on $[t_2,\infty)_{\mathbf{T}}$ for some sufficiently large $t_2>t_1$ or $\lim_{t\to\infty}x(t)=0$.

Now we consider the case $x^{(\alpha)}(t)>0$ on $[t_2,\infty)_{\mathbf{T}}$. To this end, we define the generalized Riccati function:

$$\omega(t)=\phi(t)a(t)[\frac{(r(t)x^{(\alpha)}(t))^{(\alpha)}}{x(t)e_{-\frac{\widetilde{p}}{a}}(t,t_0)}+\varphi(t)].$$

Then by Lemma 2.1 (i) one has $\omega(t)\geq0$, and by Theorem 1.12 (ii) and Theorem 1.11 one can deduce that

$$\omega^{(\alpha)}(t)=\frac{\phi(t)}{x(t)}[\frac{a(t)(r(t)x^{(\alpha)}(t))^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}]^{(\alpha)}$$
$$+[\frac{\phi(t)}{x(t)}]^{(\alpha)}\frac{a(\sigma(t))(r(\sigma(t))x^{(\alpha)}(\sigma(t)))^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}$$
$$+\phi(t)[a(t)\varphi(t)]^{(\alpha)}+\phi^{(\alpha)}(t)a(\sigma(t))\varphi(\sigma(t))$$
$$=\frac{\phi(t)}{x(t)}\frac{1}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}$$
$$\{e_{-\frac{\widetilde{p}}{a}}(t,t_0)(a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)})^{(\alpha)}$$
$$-(e_{-\frac{\widetilde{p}}{a}}(t,t_0))^{(\alpha)}a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}\}$$
$$+[\frac{x(t)\phi^{(\alpha)}(t)-x^{(\alpha)}(t)\phi(t)}{x(t)x(\sigma(t))}]\frac{a(\sigma(t))(r(\sigma(t))x^{(\alpha)}(\sigma(t)))^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}$$
$$+\phi(t)[a(t)\varphi(t)]^{(\alpha)}+\phi^{(\alpha)}(t)a(\sigma(t))\varphi(\sigma(t))$$
$$=\frac{\phi(t)}{x(t)}[\frac{(a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)})^{(\alpha)}+p(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}]$$
$$+\frac{\phi^{(\alpha)}(t)}{\phi(\sigma(t))}\omega(\sigma(t))$$
$$-[\frac{\phi(t)x^{(\alpha)}(t)}{x(t)}]\frac{a(\sigma(t))(r(\sigma(t))x^{(\alpha)}(\sigma(t)))^{(\alpha)}}{x(\sigma(t))e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}$$
$$+\phi(t)[a(t)\varphi(t)]^{(\alpha)}$$
$$=-q(t)\frac{\phi(t)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}+\frac{\phi^{(\alpha)}(t)}{\phi(\sigma(t))}\omega(\sigma(t))$$
$$-[\frac{\phi(t)x^{(\alpha)}(t)}{x(t)}]\frac{a(\sigma(t))(r(\sigma(t))x^{(\alpha)}(\sigma(t)))^{(\alpha)}}{x(\sigma(t))e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}$$
$$+\phi(t)[a(t)\varphi(t)]^{(\alpha)}.$$

From Lemma 2.2 one furthermore has

$$\omega^{(\alpha)}(t)\leq-q(t)\frac{\phi(t)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}+\frac{\phi^{(\alpha)}(t)}{\phi(\sigma(t))}\omega(\sigma(t))$$
$$-(\frac{\phi(t)}{x(t)})\frac{\delta_1(t,t_2)}{r(t)}[\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}]$$
$$\frac{a(\sigma(t))(r(\sigma(t))x^{(\alpha)}(\sigma(t)))^{(\alpha)}}{x(\sigma(t))e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}+\phi(t)[a(t)\varphi(t)]^{(\alpha)}$$
$$\leq-q(t)\frac{\phi(t)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}+\frac{\phi^{(\alpha)}(t)}{\phi(\sigma(t))}\omega(\sigma(t))$$
$$-(\frac{\phi(t)}{x(\sigma(t))})\frac{\delta_1(t,t_2)}{r(t)}[\frac{a(\sigma(t))[r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}]$$

$$\frac{a(\sigma(t))(r(\sigma(t))x^{(\alpha)}(\sigma(t)))^{(\alpha)}}{x(\sigma(t))e_{\widetilde{-\frac{p}{a}}}(\sigma(t),t_0)}+\phi(t)[a(t)\varphi(t)]^{(\alpha)}$$
$$=-q(t)\frac{\phi(t)}{e_{\widetilde{-\frac{p}{a}}}(\sigma(t),t_0)}+\frac{\phi^{(\alpha)}(t)}{\phi(\sigma(t))}\omega(\sigma(t))$$
$$-[\frac{\phi(t)\delta_1(t,t_2)}{r(t)}][\frac{a(\sigma(t))(r(\sigma(t))x^{(\alpha)}(\sigma(t)))^{(\alpha)}}{x(\sigma(t))e_{\widetilde{-\frac{p}{a}}}(\sigma(t),t_0)}]^2$$
$$+\phi(t)[a(t)\varphi(t)]^{(\alpha)}$$
$$=-q(t)\frac{\phi(t)}{e_{\widetilde{-\frac{p}{a}}}(\sigma(t),t_0)}+\frac{\phi^{(\alpha)}(t)}{\phi(\sigma(t))}\omega(\sigma(t))-[\frac{\phi(t)\delta_1(t,t_2)}{r(t)}]$$
$$[\frac{\omega(\sigma(t))}{\phi(\sigma(t))}-a(\sigma(t))\varphi(\sigma(t))]^2+\phi(t)[a(t)\varphi(t)]^{(\alpha)}$$
$$=-q(t)\frac{\phi(t)}{e_{\widetilde{-\frac{p}{a}}}(\sigma(t),t_0)}+\phi(t)[a(t)\varphi(t)]^{(\alpha)}$$
$$-\frac{\phi(t)\delta_1(t,t_2)a^2(\sigma(t))\varphi^2(\sigma(t))}{r(t)}$$
$$+[\frac{\phi^{(\alpha)}(t)}{\phi(\sigma(t))}+2\frac{\phi(t)\delta_1(t,t_2)a(\sigma(t))\varphi(\sigma(t))}{r(t)\phi(\sigma(t))}]\omega(\sigma(t))$$
$$-\frac{\phi(t)\delta_1(t,t_2)}{r(t)\phi^2(\sigma(t))}\omega^2(\sigma(t))$$
$$\leq -q(t)\frac{\phi(t)}{e_{\widetilde{-\frac{p}{a}}}(\sigma(t),t_0)}+\phi(t)[a(t)\varphi(t)]^{(\alpha)}$$
$$-\frac{\phi(t)\delta_1(t,t_2)a^2(\sigma(t))\varphi^2(\sigma(t))}{r(t)}$$
$$+\frac{[\phi^{(\alpha)}(t)r(t)+2\phi(t)\delta_1(t,t_2)a(\sigma(t))\varphi(\sigma(t))]^2}{4r(t)\phi(t)\delta_1(t,t_2)}. \quad (2.10)$$

Substituting t with s in (2.10), fulfilling α-fractional integral for (2.10) with respect to s from t_2 to t yields

$$\int_{t_2}^{t}\{q(s)\frac{\phi(s)}{e_{\widetilde{-\frac{p}{a}}}(\sigma(s),t_0)}-\phi(s)[a(s)\varphi(s)]^{(\alpha)}$$
$$+\frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}$$
$$-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}\}\Delta^{\alpha}s$$
$$\leq \omega(t_2)-\omega(t)\leq\omega(t_2),$$

which contradicts the condition (2.9), and thus the proof is completed.

Corollary 2.4. in the case $\mathbf{T}=\mathbf{R}$, if we assume that

$$\int_{t_0}^{\infty}\frac{e_{\widetilde{-\frac{p}{a}}}(s,t_0)}{a(s)}s^{\alpha-1}ds=\infty, \quad (2.11)$$
$$\int_{t_0}^{\infty}\frac{1}{r(s)}s^{\alpha-1}ds=\infty, \quad (2.12)$$
$$\int_{t_0}^{\infty}[\frac{\xi^{\alpha-1}}{r(\xi)}\int_{\xi}^{\infty}(\frac{e_{\widetilde{-\frac{p}{a}}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{\widetilde{-\frac{p}{a}}}(\sigma(s),t_0)}ds)\tau^{\alpha-1}d\tau]d\xi$$
$$=\infty, \quad (2.13)$$

and for all sufficiently large t_1, there exists t_2 such that

$$\lim_{t\to\infty}\sup\{\int_{t_2}^{t}\{q(s)\frac{\phi(s)}{e_{\widetilde{-\frac{p}{a}}}(s,t_0)}-\phi(s)s^{1-\alpha}[a(s)\varphi(s)]'$$
$$+\frac{\phi(s)\delta_1(s,t_2)a^2(s)\varphi^2(s)}{r(s)}$$
$$-\frac{[s^{1-\alpha}\phi'(s)r(s)+2\phi(s)\delta_1(s,t_2)a(s)\varphi(s)]^2}{4r(s)\phi(s)\delta_1(s,t_2)}\}s^{\alpha-1}ds\}$$
$$=\infty, \quad (2.14)$$

where ϕ, φ are two given nonnegative functions on $\mathbf{R}$, then every solution of Eq. (1.1) is oscillatory or tends to zero.

Corollary 2.5. Let $\mathbf{T}=\mathbf{Z}$ and $-\widetilde{\frac{p}{a}}\in\mathcal{R}_+$. Assume that

$$\sum_{s=t_0}^{\infty}\frac{e_{\widetilde{-\frac{p}{a}}}(s,t_0)}{a(s)}s^{\alpha-1}=\infty, \quad (2.15)$$
$$\sum_{s=t_0}^{\infty}\frac{1}{r(s)}s^{\alpha-1}=\infty, \quad (2.16)$$
$$\sum_{\xi=t_0}^{\infty}[\frac{\xi^{\alpha-1}}{r(\xi)}\sum_{\tau=\xi}^{\infty}\tau^{\alpha-1}(\frac{e_{\widetilde{-\frac{p}{a}}}(\tau,t_0)}{a(\tau)}\sum_{s=\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{\widetilde{-\frac{p}{a}}}(s+1,t_0)})]$$
$$=\infty, \quad (2.17)$$

and for all sufficiently large t_1, there exists t_2 such that

$$\lim_{t\to\infty}\sup\{\sum_{s=t_2}^{t-1}\{q(s)\frac{\phi(s)}{e_{\widetilde{-\frac{p}{a}}}(s+1,t_0)}-\phi(s)s^{1-\alpha}[a(s+1)$$
$$\varphi(s+1)-a(s)\varphi(s)]+\frac{\phi(s)\delta_1(s,t_2)a^2(s+1)\varphi^2(s+1)}{r(s)}-$$
$$\frac{[s^{1-\alpha}(\phi(s+1)-\phi(s))r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}\}\}$$
$$=\infty, \quad (2.18)$$

where ϕ, φ are two given nonnegative functions on $\mathbf{Z}$. Then every solution of Eq. (1.1) is oscillatory or tends to zero.

Based on the results above, furthermore we prove and establish some Philos type oscillation criteria for Eq. (1.1).

Theorem 2.6. Assume (2.1)-(2.3) hold, and $-\frac{p}{a}\in\mathcal{R}_+$. Define $\mathbf{D}=\{(t,s)|t\geq s\geq t_0\}$. If there exists a function $H\in C_{rd}(\mathbf{D},\mathbf{R})$ such that

$$H(t,t)=0,\ for\ t\geq t_0,$$
$$H(t,s)>0,\ for\ t>s\geq t_0, \quad (2.19)$$

and H has a nonpositive continuous $\alpha-$ partial fractional derivative $H_s^{(\alpha)}(t,s)$ with respect to the second variable, and

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\{\int_{t_0}^{t}H(t,s)[q(s)\frac{\phi(s)}{e_{\widetilde{-\frac{p}{a}}}(\sigma(s),t_0)}$$

$$-\phi(s)(a(s)\varphi(s))^{(\alpha)}+\frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}$$
$$-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}]\Delta^\alpha s\}=\infty, \tag{2.20}$$

where t_2 is sufficiently large. Then every solution of Eq. (1.1) is oscillatory or tends to zero.

Proof. Assume (1.1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbf{T}}$. Without loss of generality, we may assume $x(t)>0$ on $[t_1,\infty)_{\mathbf{T}}$, where t_1 is sufficiently large. By Lemma 2.1 (ii) we have either $x^{(\alpha)}(t)>0$ on $[t_2,\infty)_{\mathbf{T}}$ for some sufficiently large t_2 or $\lim_{t\to\infty}x(t)=0$.

Now we assume $x^{(\alpha)}(t)>0$ on $[t_2,\infty)_{\mathbf{T}}$. Let $\omega(t)$ be defined as in Theorem 2.3. By (2.10) we have

$$q(t)\frac{\phi(t)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}-\phi(t)(a(t)\varphi(t))^{(\alpha)}$$
$$+\frac{\phi(t)\delta_1(t,t_2)a^2(\sigma(t))\varphi^2(\sigma(t))}{r(t)}$$
$$-\frac{[\phi^{(\alpha)}(t)r(t)+2\phi(t)\delta_1(t,t_2)a(\sigma(t))\varphi(\sigma(t))]^2}{4r(t)\phi(t)\delta_1(t,t_2)}$$
$$\leq-\omega^{(\alpha)}(t). \tag{2.21}$$

Substituting t with s in (2.21), multiplying both sides by $H(t,s)$ and fulfilling α-fractional integral with respect to s from t_2 to t, together with Theorem 1.14 one can obtain that

$$\int_{t_2}^t H(t,s)\{q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\phi(s)(a(s)\varphi(s))^{(\alpha)}$$
$$+\frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}$$
$$-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}\}\Delta^\alpha s$$
$$\leq-\int_{t_2}^t H(t,s)\omega^{(\alpha)}(s)\Delta^\alpha s$$
$$=H(t,t_2)\omega(t_2)+\int_{t_2}^t H_s^{(\alpha)}(t,s)\omega(\sigma(s))\Delta^\alpha s$$
$$\leq H(t,t_2)\omega(t_2)\leq H(t,t_0)\omega(t_2),$$

where in the last two steps we have used the fact that the function $H(t,s)$ is decreasing with respect to the second variable due to $H_s^{(\alpha)}(t,s)$ is nonpositive. Then

$$\int_{t_0}^t H(t,s)[q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\phi(s)(a(s)\varphi(s))^{(\alpha)}$$
$$+\frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}$$
$$-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}]\Delta^\alpha s$$
$$=\int_{t_0}^{t_2} H(t,s)[q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\phi(s)(a(s)\varphi(s))^{(\alpha)}$$
$$+\frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}$$
$$-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}]\Delta^\alpha s$$
$$+\int_{t_2}^t H(t,s)[q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\phi(s)(a(s)\varphi(s))^{(\alpha)}$$
$$+\frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}$$
$$-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}]\Delta^\alpha s$$
$$\leq H(t,t_0)\omega(t_2)+H(t,t_0)\int_{t_0}^{t_2}|q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}$$
$$-\phi(s)(a(s)\varphi(s))^{(\alpha)}+\frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}$$
$$-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}|\Delta^\alpha s.$$

Furthermore,

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\{\int_{t_0}^t H(t,s)[q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}$$
$$-\phi(s)(a(s)\varphi(s))^{(\alpha)}+\frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}$$
$$-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}]\Delta^\alpha s$$
$$\leq\omega(t_2)+\int_{t_0}^{t_2}|q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\phi(s)(a(s)\varphi(s))^{(\alpha)}$$
$$+\frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}$$
$$-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}|\Delta^\alpha s$$
$$<\infty,$$

which contradicts (2.20), and then the proof is completed.

Theorem 2.7. Assume that (2.1), (2.2), (2.3) hold, and $-\frac{\widetilde{p}}{a}\in\mathcal{R}_+$. If either of the following two conditions satisfy:

(i). $$\lim_{t\to\infty}\sup\frac{1}{(t-t_0)^m}\{\int_{t_0}^t(t-s)^m[q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}$$
$$-\phi(s)(a(s)\varphi(s))^{(\alpha)}+\frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}$$
$$-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}]\Delta^\alpha s\}$$
$$=\infty,\ m\geq 1, \tag{2.22}$$

(ii). $$\lim_{t\to\infty}\sup\frac{1}{(\ln t-\ln t_0)}\{\int_{t_0}^t(\ln t-\ln s)$$
$$[q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\phi(s)(a(s)\varphi(s))^{(\alpha)}$$
$$+\frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}$$
$$-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}]\Delta^\alpha s\}$$
$$=\infty, \tag{2.23}$$

then every solution of Eq. (1.1) is oscillatory or tends to zero.

The proof of Theorem 2.7 can be reached by setting $H(t,s)=(t-s)^m,\ m\geq 1$ or $H(t,s)=\ln\frac{t}{s}$ in Theorem 2.6.

Remark 1. In the established oscillation criteria above, if we set $\alpha=1$, then the results reduce to corresponding oscillation criteria for dynamic equations on time scales involving integer order derivative.

III. Applications

In this section, we will present some applications for the established results above. First we consider the following fractional differential equation with damping term:

Example 1. $\{\sqrt{t}[t^{-\frac{1}{2}}x^{(\frac{1}{2})}(t)]^{(\frac{1}{2})}\}^{(\frac{1}{2})}+t^{-\frac{5}{2}}[t^{-\frac{1}{2}}x^{(\frac{1}{2})}(t)]^{(\frac{1}{2})}$

$$+t^{-\frac{3}{2}}x(t)=0,\ t\in[2,\infty). \tag{3.1}$$

Related to (1.1), one has $\mathbf{T}=\mathbf{R}$, $\alpha=\frac{1}{2},\ a(t)=\sqrt{t},\ p(t)=t^{-\frac{5}{2}},\ q(t)=t^{-\frac{3}{2}},\ \widetilde{p}(t)=t^{-\frac{1}{2}}p(t)=t^{-3},\ r(t)=t^{-\frac{1}{2}},\ t_0=2$. So $\mu(t)=\sigma(t)-t=0$, which means $-\frac{\widetilde{p}}{a}\in\mathcal{R}_+$. Then $e_{-\frac{\widetilde{p}}{a}}(t,t_0)=e_{-\frac{\widetilde{p}}{a}}(t,2)=\exp(-\int_2^t\frac{\widetilde{p}(s)}{a(s)}ds)$. Moreover,

$$1>\exp(-\int_2^t\frac{\widetilde{p}(s)}{a(s)}ds)\geq 1-\int_2^t\frac{\widetilde{p}(s)}{a(s)}ds=1-\int_2^t s^{-\frac{7}{2}}ds$$
$$=1+\frac{2}{5}[t^{-\frac{5}{2}}-2^{-\frac{5}{2}}]>\frac{3}{5}.$$

So towards (2.1)-(2.2), by Definition 1.8 one can deduce that

$$\int_{t_0}^{\infty}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}\Delta^{\alpha}s=\int_{t_0}^{\infty}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}s^{\alpha-1}\Delta s$$
$$=\int_{t_0}^{\infty}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}s^{\alpha-1}ds$$
$$>\frac{3}{5}\int_2^{\infty}\frac{1}{\sqrt{s}}s^{\alpha-1}ds=\frac{3}{5}\int_2^{\infty}\frac{1}{s}ds=\infty,$$

and

$$\int_{t_0}^{\infty}\frac{1}{r(s)}\Delta^{\alpha}s=\int_{t_0}^{\infty}\frac{1}{r(s)}s^{-\frac{1}{2}}ds=\int_{t_0}^{\infty}1ds=\infty.$$

Furthermore, for (2.3) one has

$$\int_{t_0}^{\infty}[\frac{1}{r(\xi)}\int_{\xi}^{\infty}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^{\alpha}s)\Delta^{\alpha}\tau]\Delta^{\alpha}\xi$$
$$=\int_{t_0}^{\infty}\xi^{\alpha-1}[\frac{1}{r(\xi)}\int_{\xi}^{\infty}\tau^{\alpha-1}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}$$
$$\int_{\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta s)\Delta\tau]\Delta\xi$$
$$=\int_{t_0}^{\infty}\xi^{\alpha-1}[\frac{1}{r(\xi)}\int_{\xi}^{\infty}\tau^{\alpha-1}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}$$
$$\int_{\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}ds)d\tau]d\xi$$
$$=\int_2^{\infty}[\int_{\xi}^{\infty}\tau^{-\frac{1}{2}}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,2)}{\sqrt{\tau}}\int_{\tau}^{\infty}\frac{1}{s^2e_{-\frac{\widetilde{p}}{a}}(s,2)}ds)d\tau]d\xi$$
$$>\frac{3}{5}\int_2^{\infty}[\int_{\xi}^{\infty}(\frac{1}{\tau}\int_{\tau}^{\infty}\frac{1}{s^2}ds)d\tau]d\xi=\frac{3}{5}\int_2^{\infty}[\int_{\xi}^{\infty}\frac{1}{\tau^2}d\tau]d\xi$$
$$=\frac{3}{5}\int_2^{\infty}\frac{1}{\xi}d\xi=\infty.$$

On the other hand, for a sufficiently large t_2, we have

$$\delta_1(t,t_2)=\int_{t_2}^{t}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}\Delta^{\alpha}s=\int_{t_2}^{t}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}s^{\alpha-1}\Delta s$$
$$=\int_{t_2}^{t}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}s^{\alpha-1}ds>\frac{3}{5}\int_{t_2}^{t}\frac{1}{s}ds\to\infty\ (t\to\infty).$$

So there exists a sufficiently large $t_3>t_2$ such that $\delta_1(t,t_2)>1$ for $t\in[t_3,\infty)$.

Setting $\phi(t)=t,\ \varphi(t)=0$ in (2.14), one can obtain that

$$\int_{t_3}^{t}[q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\frac{s^{2-2\alpha}(\phi'(s))^2r(s)}{4\phi(s)\delta_1(s,t_2)}]s^{\alpha-1}ds$$
$$>\int_{t_3}^{t}(\frac{1}{s}-\frac{1}{4s})ds=\int_{t_3}^{t}\frac{3}{4s}ds\to\infty\ (t\to\infty).$$

From the analysis above one can see (2.11)-(2.14) all hold. So it follows from Corollary 2.4 that every solution of Eq. (3.1) is oscillatory or tends to zero.

Next we consider the following fractional difference equation:

Example 2. $\Delta^{(\frac{1}{2})}\{\sqrt{t}\Delta^{(\frac{1}{2})}[t^{-\frac{1}{2}}\Delta^{(\frac{1}{2})}x(t)]\}+t^{-\frac{5}{2}}$

$$\Delta^{(\frac{1}{2})}[t^{-\frac{1}{2}}\Delta^{(\frac{1}{2})}x(t)]+t^{-\frac{3}{2}}x(t)=0,\ t\in[2,\infty)_{\mathbf{Z}}, \tag{3.2}$$

where $\Delta^{(\frac{1}{2})}$ denotes the fractional difference operator of order $\frac{1}{2}$.

Related to (1.1), one has $\mathbf{T}=\mathbf{Z}$, $\alpha=\frac{1}{2},\ a(t)=\sqrt{t},\ p(t)=t^{-\frac{5}{2}},\ q(t)=t^{-\frac{3}{2}},\ \widetilde{p}(t)=t^{-\frac{1}{2}}p(t)=t^{-3},\ r(t)=t^{-\frac{1}{2}},\ t_0=2$. Then $\mu(t)=\sigma(t)-t=1$, and

$$1-\mu(t)\frac{\widetilde{p}(t)}{a(t)}=1-t^{-\frac{7}{2}}\geq 1-t^{-3}\geq 1-\frac{1}{2^3}>0,$$

which means $-\frac{\widetilde{p}}{a}\in\mathcal{R}_+$. So according to [43, Lemma 2] one can obtain that

$$e_{-\frac{\widetilde{p}}{a}}(t,t_0)=e_{-\frac{\widetilde{p}}{a}}(t,2)\geq 1-\int_2^t\frac{\widetilde{p}(s)}{a(s)}\Delta s$$
$$=1-\int_2^t s^{-\frac{7}{2}}\Delta s=1-\sum_{s=2}^{t-1}s^{-\frac{7}{2}}$$
$$\geq 1-\int_1^{t-1}s^{-\frac{7}{2}}ds=1+\frac{2}{5}[(t-1)^{-\frac{5}{2}}-1]>\frac{3}{5},$$

and

$$e_{-\frac{\widetilde{p}}{a}}(t,t_0)\leq\exp(-\int_2^t\frac{\widetilde{p}(s)}{a(s)}\Delta s)<1.$$

To use Corollary 2.5, one needs to verify (2.15)-(2.18). To this end, one has

$$\sum_{s=t_0}^{\infty}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}s^{\alpha-1}=\sum_{s=2}^{\infty}\frac{e_{-\frac{\widetilde{p}}{a}}(s,2)}{a(s)}s^{\alpha-1}$$

$$= \sum_{s=2}^{\infty} \frac{e_{-\frac{\widetilde{p}}{a}}(s,2)}{s} > \frac{3}{5}\sum_{s=2}^{\infty}\frac{1}{s} = \infty,$$

and

$$\sum_{s=t_0}^{\infty} \frac{1}{r(s)} s^{\alpha-1} = \sum_{s=2}^{\infty} 1 = \infty.$$

Furthermore,

$$\sum_{\xi=t_0}^{\infty}[\frac{\xi^{\alpha-1}}{r(\xi)}\sum_{\tau=\xi}^{\infty}\tau^{\alpha-1}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\sum_{s=\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(s+1,t_0)})]$$
$$= \sum_{\xi=t_0}^{\infty}[\frac{\xi^{\alpha-1}}{r(\xi)}\sum_{\tau=\xi}^{\infty}\tau^{\alpha-1}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,2)}{a(\tau)}\sum_{s=\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(s+1,2)})]$$
$$> \frac{3}{5}\sum_{\xi=2}^{\infty}[\sum_{\tau=\xi}^{\infty}(\frac{1}{\tau}\sum_{s=\tau}^{\infty}\frac{1}{s^2})] > \frac{3}{5}\sum_{\xi=2}^{\infty}[\sum_{\tau=\xi}^{\infty}(\frac{1}{\tau}\sum_{s=\tau}^{\infty}\frac{1}{s(s+1)})]$$
$$= \frac{3}{5}\sum_{\xi=2}^{\infty}\sum_{\tau=\xi}^{\infty}\frac{1}{\tau^2} > \frac{3}{5}\sum_{\xi=2}^{\infty}\sum_{\tau=\xi}^{\infty}\frac{1}{\tau(\tau+1)} = \frac{3}{5}\sum_{\xi=2}^{\infty}\frac{1}{\xi} = \infty.$$

So (2.15)-(2.17) hold. Moreover, since for a sufficiently large t_2, it holds that

$$\delta_1(t,t_2) = \sum_{s=t_2}^{t-1}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}s^{\alpha-1}$$
$$> \frac{3}{5}\sum_{s=t_2}^{t-1}\frac{1}{s} \to \infty\ (t\to\infty),$$

then there exists $t_3 > t_2$ such that $\delta_1(t,t_2) > 1$ for $t \in [t_3,\infty)_{\mathbf{Z}}$. If we let $\phi(t) = t$, $\varphi(t) = 0$ in (2.18), then one can obtain that

$$\sum_{s=t_3}^{t-1}[q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(s+1,t_0)} - \frac{s^{2-2\alpha}(\phi(s+1)-\phi(s))^2 r(s)}{4\phi(s)\delta_1(s,t_2)}]s^{\alpha-1}$$
$$> \sum_{s=t_3}^{t-1}(\frac{1}{s}-\frac{1}{4s}) = \sum_{s=t_3}^{t-1}\frac{3}{4s} \to \infty\ (t\to\infty).$$

So (2.18) also holds. After an application of Corollary 2.5 one can see that every solution of Eq. (3.2) is oscillatory or tends to zero.

Finally we consider the following fractional $q-$ difference equation:

Example 3. $\Delta^{(\frac{3}{5})}\{t^{0.6}\Delta^{(\frac{3}{5})}[t^{-0.4}\Delta^{(\frac{3}{5})}x(t)]\}$

$$+t^{-2.4}\Delta^{(\frac{3}{5})}[t^{-0.4}\Delta^{(\frac{3}{5})}x(t)] + t^{-1.6}x(t) = 0,$$

$$t \in [\beta,\infty)_{\beta^{\mathbf{Z}}}, \tag{3.3}$$

where $\Delta^{(\frac{3}{5})}$ denotes the fractional difference operator of order $\frac{3}{5}$, $\beta \geq 2$.

Related to (1.1), one has $\mathbf{T} = \beta^{\mathbf{Z}}$, $\alpha = \frac{3}{5}$, $a(t) = t^{0.6}$, $p(t) = t^{-2.4}$, $q(t) = t^{-1.6}$, $\widetilde{p}(t) = t^{-0.4}p(t) = t^{-2.8}$, $r(t) = t^{-0.4}$, $t_0 = \beta$. Then $\mu(t) = \sigma(t)-t = t(\beta-1)$, and considerin g $t \geq \beta$, one has

$$1-\mu(t)\frac{\widetilde{p}(t)}{a(t)} = 1-t(\beta-1)\frac{1}{t^{3.4}} = 1-(\beta-1)\frac{1}{t^{2.4}}$$
$$\geq 1-(\beta-1)\frac{1}{\beta^2} = \frac{\beta^2-\beta+1}{\beta^2} > 0,$$

which means $-\frac{\widetilde{p}}{a} \in \mathcal{R}_+$. So we obtain

$$e_{-\frac{\widetilde{p}}{a}}(t,t_0) = e_{-\frac{\widetilde{p}}{a}}(t,\beta) \geq 1-\int_\beta^t \frac{\widetilde{p}(s)}{a(s)}\Delta s$$
$$= 1-\int_\beta^t \frac{1}{s^{3.4}}\Delta s \geq 1-\int_\beta^t \frac{1}{s^3}\Delta s = 1-(\beta-1)\frac{t^{-2}-\beta^{-2}}{\beta^{-2}-1}$$
$$= \frac{1+(\beta-1)t^{-2}-\beta^{-1}}{1-\beta^{-2}} > \frac{1-\beta^{-1}}{1-\beta^{-2}}$$
$$\geq \frac{1}{2-2\beta^{-2}} = \frac{\beta^2}{2(\beta^2-1)},$$

and

$$e_{-\frac{\widetilde{p}}{a}}(t,t_0) \leq \exp(-\int_q^t \frac{\widetilde{p}(s)}{a(s)}\Delta s) < 1.$$

Now we verify the conditions (2.1)-(2.3).

$$\int_{t_0}^{\infty}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}\Delta^\alpha s = \int_\beta^\infty \frac{e_{-\frac{\widetilde{p}}{a}}(s,\beta)}{a(s)}s^{\alpha-1}\Delta s$$
$$= \int_\beta^\infty \frac{e_{-\frac{\widetilde{p}}{a}}(s,\beta)}{s}\Delta s > \frac{\beta^2}{2(\beta^2-1)}\int_\beta^\infty \frac{1}{s}\Delta s = \infty,$$

and

$$\int_{t_0}^{\infty}\frac{1}{r(s)}\Delta^\alpha s = \int_{t_0}^{\infty}\frac{1}{r(s)}s^{\alpha-1}\Delta s = \int_{t_0}^{\infty}1\Delta s = \infty.$$

Furthermore,

$$\int_{t_0}^t[\frac{1}{r(\xi)}\int_\xi^\infty(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_\tau^\infty\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s)\Delta^\alpha\tau]\Delta^\alpha\xi$$
$$= \int_{t_0}^\infty \xi^{\alpha-1}[\frac{1}{r(\xi)}\int_\xi^\infty\tau^{\alpha-1}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}$$
$$\int_\tau^\infty\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta s)\Delta\tau]\Delta\xi$$
$$= \int_\beta^\infty \xi^{\alpha-1}[\frac{1}{r(\xi)}\int_\xi^\infty\tau^{\alpha-1}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,\beta)}{a(\tau)}$$
$$\int_\tau^\infty\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),\beta)}\Delta s)\Delta\tau]\Delta\xi$$
$$> \frac{\beta^2}{2(\beta^2-1)}\int_\beta^\infty[\int_\xi^\infty(\frac{1}{\tau}\int_\tau^\infty\frac{1}{s^2}\Delta s)\Delta\tau]\Delta\xi$$
$$> \frac{\beta^2}{2(\beta^2-1)}\int_\beta^\infty[\int_\xi^\infty(\frac{1}{\tau}\int_\tau^\infty\frac{1}{s\sigma(s)}\Delta s)\Delta\tau]\Delta\xi$$
$$= \frac{\beta^2}{2(\beta^2-1)}\int_\beta^\infty[\int_\xi^\infty(\frac{1}{\tau}[-\frac{1}{s}]_\tau^\infty)\Delta\tau]\Delta\xi$$
$$= \frac{\beta^2}{2(\beta^2-1)}\int_\beta^\infty[\int_\xi^\infty\frac{1}{\tau^2}\Delta\tau]\Delta\xi$$
$$> \frac{\beta^2}{2(\beta^2-1)}\int_\beta^\infty[\int_\xi^\infty\frac{1}{\tau\sigma(\tau)}\Delta\tau]\Delta\xi$$
$$= \frac{\beta^2}{2(\beta^2-1)}\int_\beta^\infty\frac{1}{\xi}\Delta\xi = \infty.$$

So (2.1)-(2.3) hold. On the other hand, one can see for a sufficiently large t_2 that

$$\delta_1(t,t_2) = \int_{t_2}^t\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}\Delta^\alpha s = \int_{t_2}^t\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}s^{\alpha-1}\Delta s$$
$$> \frac{\beta^2}{2(\beta^2-1)}\int_{t_2}^t\frac{1}{s}\Delta s \to \infty\ (t\to\infty).$$

So there exists $t_3 > t_2$ such that $\delta_1(t, t_2) > 1$ for $t \in [t_3, \infty)_{q^{\mathbf{Z}}}$.

To use Theorem 2.7, let $m = 1,\ \phi(t) = t,\ \varphi(t) = 0$ in (2.22), and one has

$$
\begin{aligned}
&\lim_{t\to\infty} \sup \frac{1}{(t-t_0)} \{ \textstyle\int_{t_3}^{t} [(t-s)q(s) \dfrac{\phi(s)}{e_{-\frac{p}{a}}^{\sim}(\sigma(s), t_0)} \\
&- \frac{(\phi^{(\alpha)}(s))^2 r(s)}{4\phi(s)\delta_1(s,t_2)}] \Delta^{\alpha} s \} \\
&= \lim_{t\to\infty} \sup \frac{1}{(t-t_0)} \{ \textstyle\int_{t_3}^{t} [(t-s)q(s) \dfrac{\phi(s)}{e_{-\frac{p}{a}}^{\sim}(\sigma(s), t_0)} \\
&- \frac{s^{2-2\alpha}(\phi^{\Delta}(s))^2 r(s)}{4\phi(s)\delta_1(s,t_2)}] s^{\alpha-1} \Delta s \} \\
&> \lim_{t\to\infty} \sup \frac{1}{(t-t_0)} \textstyle\int_{t_3}^{t} (t-s) \dfrac{3}{4s} \Delta s \\
&= \lim_{t\to\infty} \sup [\frac{t}{(t-\beta)} \textstyle\int_{t_3}^{t} \dfrac{3}{4s} \Delta s - \dfrac{3(t-t_2)}{4(t-\beta)}] = \infty,
\end{aligned}
$$

which means (2.22) also holds, and by Theorem 2.7 one can deduce that every solution of Eq. (3.3) is oscillatory or tends to zero.

Remark 2. From the examples presented above, one can see that the oscillation criteria established in Section II can be used for the research of oscillation of fractional dynamic equations on various time scales involving fractional differential equations and fractional difference equations.

IV. Conclusions

We have established some new oscillation criteria for a class of fractional dynamic equation with damping term on time scales by use of the properties of fractional calculus and generalized Riccati transformation technique. Oscillation criteria for corresponding dynamic equations on time scales involving integer order derivative are only special cases of our results. The validity of the established results are illustrated by some examples. We note that this approach can be applied to research oscillation of other types of fractional dynamic equation on time scales.

In further research, we will apply the presented method in this paper to research oscillation of fractional delay dynamic equation on time scales such as
$(a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\nu})^{(\alpha)} + p(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\nu} + q(t)f(x(\kappa(t))) = 0,\ t \in \mathbf{T}_0$,
where $x^{(\alpha)}(t)$ denotes the fractional derivative of order α, $\kappa \in C_{rd}(\mathbf{R}, \mathbf{R})$ is the delay function satisfying $\kappa(t) \leq t,\ \kappa^{\Delta}(t) \geq 0$ and $\lim\limits_{t\to\infty} \kappa(t) = \infty$.

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