A New Coefficient of the Conjugate Gradient Method with the Sufficient Descent Condition and Global Convergence Properties

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***Abstract*—Conjugate gradient methods are the most famous methods for solving unconstrained, large-scale optimization. In this article, we propose a new coefficient of the conjugate gradient method for solving unconstrained minimization problems. The new method is a modification of NPRP (2009) coefficient. The sufficient descent condition and global convergence of the new method are given under the exact line search and the strong Wolfe line search with $\sigma \in (0, \frac{1}{8})$. The numerical results show that the new method has good performance in solving unconstrained minimization problems.**

***Index Terms*—Conjugate gradient method, unconstrained minimization problem, sufficient descent condition, global convergence, exact line search, strong Wolfe line search.**

I. Introduction

In many engineering and scientific applications, solutions numerous optimization problems have often been an important issue for researchers. One optimization method that is easy to apply is the conjugate gradient method. For example, in [1] a image restoration problem was formulated into a optimization large scale problem. As well as in [2] a signal recovery problem solved by forming optimization problems. In [3], Wan et al. used biobjective optimization model to solve the portfolio management, and others (see, e.g., [4]-[11]). In application, there are two types, namely the constrained and the unconstrained optimization problems.

In this paper, we consider the unconstrained minimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function and $\mathbb{R}$ is real numbers. We focus the discussion on conjugate gradient method to solve problem (1). The conjugate gradient method is an iterative method which has the form of an iterative formulas [12] as follows

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k, \quad k = 0, 1, 2, ... \tag{2}$$

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where $\mathbf{x}_0$ is initial point and $\mathbf{x}_k$ is the k-th approximation to a solution, $\alpha_k > 0$ is a step length, and $\mathbf{d}_k$ is a search direction which is defined by

$$\mathbf{d}_k = \begin{cases} -\mathbf{g}_k, & k = 0 \\ -\mathbf{g}_k + \beta_k \mathbf{d}_{k-1}, & k \geq 1 \end{cases} \tag{3}$$

where $\mathbf{g}_k = \mathbf{g}(\mathbf{x}_k) = \nabla f(\mathbf{x}_k)$ is a gradient of function f at point $\mathbf{x}_k$ and β_k is a coefficient of the conjugate gradient method [13].

In conjugate gradient method, selection of step length α_k is important work, because the step length can affect the convergence properties. Some of known methods used to compute the step length include the exact line search and inexact line search. In this paper, we use the exact line search [13] which is

$$f(\mathbf{x}_k + \alpha_k \mathbf{d}_k) = \min_{\alpha \geq 0} f(\mathbf{x}_k + \alpha_k \mathbf{d}_k), \tag{4}$$

and inexact line search that is; the strong Wolfe line search defined as follows:

$$f(\mathbf{x}_k + \alpha_k \mathbf{d}_k) \leq f(\mathbf{x}_k) + \delta \alpha_k \mathbf{g}_k^T \mathbf{d}_k, \tag{5}$$

$$\left|\mathbf{g}(\mathbf{x}_k + \alpha_k \mathbf{d}_k)^T \mathbf{d}_k\right| \leq -\sigma \mathbf{g}_k^T \mathbf{d}_k \tag{6}$$

where $\mathbf{g}_k^T$ is transpose $\mathbf{g}_k$ and $0 < \delta < \sigma < 1$ [14].

There are many different coefficient β_k of conjugate gradient methods as follows:

$$\beta_k^{FR} = \frac{\|\mathbf{g}_k\|^2}{\|\mathbf{g}_{k-1}\|^2}, \tag{7}$$

$$\beta_k^{CD} = -\frac{\|\mathbf{g}_k\|^2}{\mathbf{d}_{k-1}^T \mathbf{g}_{k-1}}, \tag{8}$$

$$\beta_k^{DY} = \frac{\|\mathbf{g}_k\|^2}{\mathbf{d}_{k-1}^T (\mathbf{g}_k - \mathbf{g}_{k-1})}, \tag{9}$$

$$\beta_k^{PRP} = \frac{\mathbf{g}_k^T (\mathbf{g}_k - \mathbf{g}_{k-1})}{\|\mathbf{g}_{k-1}\|^2}, \tag{10}$$

$$\beta_k^{WYL} = \frac{\mathbf{g}_k^T (\mathbf{g}_k - \frac{\|\mathbf{g}_k\|}{\|\mathbf{g}_{k-1}\|} \mathbf{g}_{k-1})}{\|\mathbf{g}_{k-1}\|^2}, \tag{11}$$

$$\beta_k^{RMIL} = \frac{\mathbf{g}_k^T (\mathbf{g}_k - \mathbf{g}_{k-1})}{\|\mathbf{d}_{k-1}\|^2}, \tag{12}$$

where $\|.\|$ represent the Euclidean norm of vectors. The above corresponding methods from (7)-(12) are known as Fletcher and Reeves (FR) in [15], Conjugate Descent (CD) [16], Dai and Yuan (DY) [17], Polak and Ribiere (PRP) [18], Wei, Yao, and Liu (WYL) [19], and Rivaie, Mustafa, Ismail, and Leong (RMIL) [20].

The FR method is a method that has been known for a long time. Zoutendijk in [21] proved that this method fulfills the convergence properties under the exact line search, Guanghui et al. in [22] proved the FR method has the global convergence properties with the strong Wolfe line search. The CD method is very similar to that of the FR method. An important feature of the CD algorithm is that this method will generate a descent search directions in each iteration for the parameter $\sigma < 1$ under the strong Wolfe line search, but its global convergence property is not good. Under the exact line serach the DY method is same with the FR method. The DY method is a modification of the FR method. Under the strong Wolfe line search, this method fulfills the descent condition, but the disadvantage of this method is the bad numerical experiment results.

The PRP method is a modification of the FR method. Under the strong Wolfe line search, Yuan in [23] proved this method has the global convergence property and fullfils the descent condition. The WYL method modified numerator of PRP method. This method fulfills the descent condition and have global convergence under exact line search, the Wolfe Powel line search and Grippo-Lucidi line search. Different from WYL method, the RMIL method modified the denominator of PRP method and retained the numerator. Rivaie et al. proved the convergence properties is established using the exact line search. From the description above, study of sufficient descent condition and global convergence properties is interesting in conjugate gradient method. So in this paper, the sufficient descent condition and global convergence of the new method would be studied. Some new conjugate gradient methods have been proposed, see Dai [24], Rivaie et al. [25], Yousif [26], Yao et al. [27], Waziri et al. [28], Mtagulwa and Kaelo [29], Zhu et al. [30], Awwal et al. [31]-[32], Liu et al. [33] and Malik et al. [34]-[35].

To analyze the global convergence properties of the conjugate gradient method, we state the following two properties which are important in our later analysis [36]. We say that the sufficient descent condition holds if there exist $c > 0$ such that for each search direction $\mathbf{d}_k$:

$$\mathbf{g}_k^T \mathbf{d}_k \leq -c\|\mathbf{g}_k\|^2, \text{ for all } k \geq 0. \tag{13}$$

In addition, we say that a conjugate gradient method is globally convergent if

$$\lim_{k\to\infty} \inf \|\mathbf{g}_k\| = 0. \tag{14}$$

In this paper, we propose a new conjugate gradient method whose performance is compared with classical formulas of RMIL, FR, CD, DY, WYL and NPRP. Section II presents a new method with an algorithm designed to solve unconstrained optimization problems. In Section III, we shall present the sufficient descent condition and the global convergence proof of our new method. Significant numerical results and discussions will be presented in Section IV. Finally, the conclusions are presented in Section V.

II. A new Coefficient Conjugate Gradient Method

Recently, Zhang in [37] proposed a modified coefficient of the WYL method. The method is called NPRP method with coefficient as follows:

$$\beta_k^{NPRP} = \frac{\|\mathbf{g}_k\|^2 - \frac{\|\mathbf{g}_k\|}{\|\mathbf{g}_{k-1}\|}\left|\mathbf{g}_k^T \mathbf{g}_{k-1}\right|}{\|\mathbf{g}_{k-1}\|^2}. \tag{15}$$

The NPRP method satisfies the sufficient descent condition and met the global convergent properties under the strong Wolfe line search with greater parameter $\sigma \in \left(0, \frac{1}{2}\right)$. As well as the numerical results show that the NPRP method is efficient for the given problems from the CUTE library. In this article, we propose a new conjugate gradient coefficient by modifying the coefficient of NPRP method. The proposed modification substituted the term $\|\mathbf{g}_{k-1}\|^2$ in the denominator of NPRP by $\|\mathbf{d}_{k-1}\|^2$ as denominator RMIL method, add a negative $\left|\mathbf{g}_k^T \mathbf{g}_{k-1}\right|$ in the numerator and always has a non negative value, so we define it as

$$\beta_k^{MMSIS} = \begin{cases} A & \text{, if } B \\ 0 & \text{, otherwise} \end{cases} \tag{16}$$

where

$$\begin{aligned} A &= \frac{\|\mathbf{g}_k\|^2 - \frac{\|\mathbf{g}_k\|}{\|\mathbf{g}_{k-1}\|}\left|\mathbf{g}_k^T \mathbf{g}_{k-1}\right| - \left|\mathbf{g}_k^T \mathbf{g}_{k-1}\right|}{\|\mathbf{d}_{k-1}\|^2}, \\ B &= \|\mathbf{g}_k\|^2 > \left(\frac{\|\mathbf{g}_k\|}{\|\mathbf{g}_{k-1}\|} + 1\right)\left|\mathbf{g}_k^T \mathbf{g}_{k-1}\right| \end{aligned}$$

and MMSIS is denotes Malik, Mustafa, Sabariah, Ibrahim, Sukono.

Algorithm 1. *(MMSIS Method)*

Step 1. *Choose an initial point $\mathbf{x}_0 \in \mathbb{R}^n$. Given the stopping criteria $\epsilon > 0$, for the strong Wolfe line search given parameter σ and δ.*

Step 2. *Calculate $\|\mathbf{g}_k\|$, if $\|\mathbf{g}_k\| \leq \epsilon$ then stop, $\mathbf{x}_k$ is optimal point. Else, go to Step 3.*

Step 3. *Calculate β_k using (16).*

Step 4. *Calculate search direction $\mathbf{d}_k$ using (3).*

Step 5. *Calculate step length α_k using the exact line search (4) or the strong Wolfe line search (5) and (6).*

Step 6. *Set $k := k+1$ and calculate the next iteration $\mathbf{x}_{k+1}$ using (2).*

Step 7. *Go to Step 2.*

III. Convergence Analysis

In this section we show that under the exact line search and the strong Wolfe line search the MMSIS method will satisfy sufficient descent condition and global convergence properties.

This lemma is important to prove the sufficient descent condition and global convergence, so that the proof will be easier.

Lemma 1. *For any $k \geq 0$, the relation*

$$0 \leq \beta_k^{MMSIS} \leq \frac{\|\mathbf{g}_k\|^2}{\|\mathbf{d}_{k-1}\|^2}. \tag{17}$$

always holds.

Proof: According to (16), there are two cases:

- Case 1: for $\|\mathbf{g}_k\|^2 > \left(\frac{\|\mathbf{g}_k\|}{\|\mathbf{g}_{k-1}\|}+1\right)\left|\mathbf{g}_k^T\mathbf{g}_{k-1}\right|$, then $\beta_k^{MMSIS} > 0$. Further, we obtain

$$\begin{aligned}\beta_k^{MMSIS} &= \frac{\|\mathbf{g}_k\|^2 - \frac{\|\mathbf{g}_k\|}{\|\mathbf{g}_{k-1}\|}\left|\mathbf{g}_k^T\mathbf{g}_{k-1}\right| - \left|\mathbf{g}_k^T\mathbf{g}_{k-1}\right|}{\|\mathbf{d}_{k-1}\|^2} \\ &\leq \frac{\|\mathbf{g}_k\|^2}{\|\mathbf{d}_{k-1}\|^2}.\end{aligned}$$

- Case 2: for $\|\mathbf{g}_k\|^2 \leq \left(\frac{\|\mathbf{g}_k\|}{\|\mathbf{g}_{k-1}\|}+1\right)\left|\mathbf{g}_k^T\mathbf{g}_{k-1}\right|$, then $\beta_k^{MMSIS} = 0$.

So that $0 \leq \beta_k^{MMSIS} \leq \frac{\|\mathbf{g}_k\|^2}{\|\mathbf{d}_{k-1}\|^2}$. The proof is completed. ■

A. Convergence Analysis Based on the Exact Line Search

The following theorem shows that the MMSIS method under the exact line search fulfills a sufficient descent condition.

Theorem 1. *Suppose a conjugate gradient method with search direction (3), where α_k is calculated by exact line search (4), and β_k^{MMSIS} given by (16), then condition (13) holds for all $k \geq 0$.*

Proof: If $k = 0$, we have $\mathbf{d}_0 = -\mathbf{g}_0$, then $\mathbf{g}_0^T\mathbf{d}_0 = -\left(\sqrt{\mathbf{g}_0^T\mathbf{g}_0}\right)^2 = -\|\mathbf{g}_0\|^2$. Hence, the condition (13) fulfilled for $k = 0$. For $k \geq 1$ multiply (3) by $\mathbf{g}_k^T$, we obtain

$$\begin{aligned}\mathbf{g}_k^T\mathbf{d}_k &= -\mathbf{g}_k^T\mathbf{g}_k + \beta_k^{MMSIS}\mathbf{g}_k^T\mathbf{d}_{k-1} \\ &= -\|\mathbf{g}_k\|^2 + \beta_k^{MMSIS}\mathbf{g}_k^T\mathbf{d}_{k-1}.\end{aligned}$$

Using the conjugacy condition $\mathbf{g}_k^T\mathbf{d}_{k-1} = 0$. Thus

$$\mathbf{g}_k^T\mathbf{d}_k = -\|\mathbf{g}_k\|^2, \quad (18)$$

so the condition (13) fulfilled. Hence, we say that the sufficient descent condition holds for all $k \geq 0$. ■

Now, we will be proving that the MMSIS method fulfills the properties of the global convergence under the exact line search. The following assumptions are needed in analyze convergence properties. We assume that the objective function f satisfies the following.

Assumption 1. *The level set $\Omega = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \leq f(\mathbf{x}_0)\}$ at $\mathbf{x}_0$ is bounded; namely there exist a constant $r > 0$ such that $\|\mathbf{x}\| \leq r$ for all $\mathbf{x} \in \Omega$.*

Assumption 2. *In any neighborhood Ω_0 of Ω, f is continuous and differentiable, and its gradient $\mathbf{g}(\mathbf{x})$ is Lipschitz continuous with Lipschitz constant $L > 0$; i.e.,*

$$\|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|, \quad \textit{for all} \;\; \mathbf{x}, \mathbf{y} \in \Omega_0. \quad (19)$$

We need the following lemma to prove the global convergence properties.

Lemma 2. *Suppose that Assumption 1 and Assumption 2 holds. Consider any conjugate gradient method with (2)-(3), where step length α_k calculate by the exact line search (4) and the strong Wolfe line search (5) and (6), then, the following so called Zoutendijk condition holds*

$$\sum_{k=0}^{\infty} \frac{(\mathbf{g}_k^T\mathbf{d}_k)^2}{\|\mathbf{d}_k\|^2} < \infty. \quad (20)$$

This lemma has been proven by Zoutendijk in [21].

Theorem 2. *Consider the conjugate gradient method in the form (2)-(3), where β_k is calculated by (16) and α_k is determined by the exact line search (4). Suppose that Assumption 1, Assumption 2 and sufficient descent condition hold. Then*

$$\lim_{k=0} \inf \|\mathbf{g}_k\| = 0. \quad (21)$$

Hence, the MMSIS method is have the globally convergent.

Proof: To prove this theorem, we use contradiction. Let that the condition (21) not true. By definition inferior limit, we have, there exist constant θ such that

$$\|\mathbf{g}_k\| \geq \theta, \text{ for every } k \geq 0. \quad (22)$$

From (3), we can write

$$\mathbf{d}_k + \mathbf{g}_k = \beta_{k+1}^{MMSIS}\mathbf{d}_k.$$

Squaring both sides yields:

$$\|\mathbf{d}_k\|^2 = (\beta_k^{MMSIS})^2\|\mathbf{d}_{k-1}\|^2 - 2\mathbf{g}_k^T\mathbf{d}_k - \|\mathbf{g}_k\|^2.$$

Dividing both sides by $(\mathbf{g}_k^T\mathbf{d}_k)^2$, then we obtain

$$\begin{aligned}\frac{\|\mathbf{d}_k\|^2}{(\mathbf{g}_k^T\mathbf{d}_k)^2} &= \frac{(\beta_k^{MMSIS})^2\|\mathbf{d}_{k-1}\|^2}{(\mathbf{g}_k^T\mathbf{d}_k)^2} - \frac{2}{\mathbf{g}_k^T\mathbf{d}_k} - \frac{\|\mathbf{g}_k\|^2}{(\mathbf{g}_k^T\mathbf{d}_k)^2} \\ &= \frac{(\beta_k^{MMSIS})^2\|\mathbf{d}_{k-1}\|^2}{(\mathbf{g}_k^T\mathbf{d}_k)^2} - \left(\frac{1}{\|\mathbf{g}_k\|} + \frac{\|\mathbf{g}_k\|}{\mathbf{g}_k^T\mathbf{d}_k}\right)^2 \\ &\quad + \frac{1}{\|\mathbf{g}_k\|^2} \\ &\leq \frac{(\beta_k^{MMSIS})^2\|\mathbf{d}_{k-1}\|^2}{(\mathbf{g}_k^T\mathbf{d}_k)^2} + \frac{1}{\|\mathbf{g}_k\|^2}.\end{aligned}$$

From Lemma 1, we have $\beta_k^{MMSIS} \geq 0$. So that,

$$\frac{\|\mathbf{d}_k\|^2}{(\mathbf{g}_k^T\mathbf{d}_k)^2} \leq \frac{1}{\|\mathbf{g}_k\|^2}$$

or

$$\frac{(\mathbf{g}_k^T\mathbf{d}_k)^2}{\|\mathbf{d}_k\|^2} \geq \|\mathbf{g}_k\|^2. \quad (23)$$

Applying (22) to (23), we get

$$\frac{(\mathbf{g}_k^T\mathbf{d}_k)^2}{\|\mathbf{d}_k\|^2} \geq \theta^2. \quad (24)$$

Take the summation (24),

$$\sum_{k=0}^{n} \frac{(\mathbf{g}_k^T\mathbf{d}_k)^2}{\|\mathbf{d}_k\|^2} \geq \sum_{k=0}^{n} \theta^2 = \theta^2(n+1).$$

Hence,

$$\sum_{k=0}^{\infty} \frac{(\mathbf{g}_k^T\mathbf{d}_k)^2}{\|\mathbf{d}_k\|^2} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(\mathbf{g}_k^T\mathbf{d}_k)^2}{\|\mathbf{d}_k\|^2} \geq \lim_{n\to\infty} \theta^2(n+1) = \infty. \quad (25)$$

This contradicts Lemma 2. Therefore, (21) true. Furthermore, based on (14), the MMSIS method is globally convergent under the exact line search. The proof is finished. ■

B. Convergence Analysis Based on the Strong Wolfe Line Search

The following theorem is needed to proof the sufficient descent condition of MMSIS method under the strong Wolfe line search.

Theorem 3. *Let sequences $\{\mathbf{g}_k\}$ and $\{\mathbf{d}_k\}$ be generated by Algorithm 1, where α_k calculate by the strong Wolfe line search with $0<\sigma<\frac{1}{8}$. Then*

$$\frac{\|\mathbf{g}_k\|}{\|\mathbf{d}_k\|}<2 \ \text{, for all } k\geq 0. \tag{26}$$

Proof: The proof would be by mathematical induction. For $k=0$, we have $\|\mathbf{d}_0\|=\|\mathbf{g}_0\|$, so that $\frac{\|\mathbf{g}_0\|}{\|\mathbf{d}_0\|}=1<2$. Hence, the condition (26) hold. Assume for $k=n, n\in\mathbb{N}$ is true, so we get

$$\frac{\|\mathbf{g}_n\|}{\|\mathbf{d}_n\|}<2, \tag{27}$$

then it will be proven that for $k=n+1$ is true. If $k=n+1$, from (3) we have

$$\mathbf{d}_{n+1}=-\mathbf{g}_{n+1}+\beta_{n+1}^{MMSIS}\mathbf{d}_n.$$

Multiply by $\mathbf{g}_{n+1}^T$,

$$\mathbf{g}_{n+1}^T\mathbf{d}_{n+1}=-\|\mathbf{g}_{n+1}\|^2+\beta_{n+1}^{MMSIS}\mathbf{g}_{n+1}^T\mathbf{d}_n \ , \tag{28}$$

furthermore,

$$\begin{aligned}\|\mathbf{g}_{n+1}\|^2 &= -\mathbf{g}_{n+1}^T\mathbf{d}_{n+1}+\beta_{n+1}^{MMSIS}\mathbf{g}_{n+1}^T\mathbf{d}_n\\ &\leq \left|\mathbf{g}_{n+1}^T\mathbf{d}_{n+1}\right|+\left|\beta_{n+1}^{MMSIS}\mathbf{g}_{n+1}^T\mathbf{d}_n\right|\\ &= \left|\mathbf{g}_{n+1}^T\mathbf{d}_{n+1}\right|+\left|\beta_{n+1}^{MMSIS}\right|\left|\mathbf{g}_{n+1}^T\mathbf{d}_n\right|\end{aligned} \tag{29}$$

Since $\beta_k^{MMSIS}\geq 0$, based on absolute value properties and the strong Wolfe condition (6), we obtain

$$\|\mathbf{g}_{n+1}\|^2\leq\left|\mathbf{g}_{n+1}^T\mathbf{d}_{n+1}\right|+\beta_{n+1}^{MMSIS}\sigma\left|\mathbf{g}_n^T\mathbf{d}_n\right| \tag{30}$$

Using the Cauchy-Schwartz inequalities and substituting (17) in (30),

$$\|\mathbf{g}_{n+1}\|^2\leq\|\mathbf{g}_{n+1}\|\|\mathbf{d}_{n+1}\|+\sigma\frac{\|\mathbf{g}_{n+1}\|^2}{\|\mathbf{d}_n\|^2}\|\mathbf{g}_n\|\|\mathbf{d}_n\|.$$

That implies,

$$\|\mathbf{g}_{n+1}\|^2\leq\|\mathbf{g}_{n+1}\|\|\mathbf{d}_{n+1}\|+\sigma\|\mathbf{g}_{n+1}\|^2\frac{\|\mathbf{g}_n\|}{\|\mathbf{d}_n\|}. \tag{31}$$

From (27) and dividing the both side of (31) by $\|\mathbf{g}_{n+1}\|$,

$$\|\mathbf{g}_{n+1}\|<\|\mathbf{d}_{n+1}\|+2\sigma\|\mathbf{g}_{n+1}\|$$

or

$$(1-2\sigma)\|\mathbf{g}_{n+1}\|<\|\mathbf{d}_{n+1}\|.$$

Since $0<\sigma<\frac{1}{8}$, then $(1-2\sigma)>0$. Furthermore,

$$\frac{\|\mathbf{g}_{n+1}\|}{\|\mathbf{d}_{n+1}\|}<\frac{1}{1-2\sigma}<2.$$

Hence (26) is true for all $k\geq 0$. The proof is completed. ■

The following theorem shows that the search direction (3) in MMSIS method fulfill the sufficient descent condition.

Theorem 4. *Suppose that the sequences $\{\mathbf{d}_k\}$ and $\{\mathbf{g}_k\}$ be generated by Algorithm 1 and step lenght α_k calculate by the strong Wolfe line search with $0<\sigma<\frac{1}{8}$. Then*

$$\frac{-1}{1-4\sigma}<\frac{\mathbf{g}_k^T\mathbf{d}_k}{\|\mathbf{g}_k\|^2}<\frac{8\sigma-1}{1-4\sigma} \ \text{, for all } k\geq 0. \tag{32}$$

Hence, the sufficient descent condition (13) holds.

Proof: Setting $k=0$ in (3), we have $\mathbf{d}_0=-\mathbf{g}_0$, so that

$$\frac{-1}{1-4\sigma}<\frac{\mathbf{g}_0^T\mathbf{d}_0}{\|\mathbf{g}_0\|^2}=-\frac{\|\mathbf{g}_0\|^2}{\|\mathbf{g}_0\|^2}=-1<\frac{8\sigma-1}{1-4\sigma}.$$

Therefore for $k=0$, $\mathbf{d}_k$ fulfill condition (32). Assume that (32) is true for $k=n, n\in\mathbb{N}$. So we get

$$\frac{-1}{1-4\sigma}\leq\frac{\mathbf{g}_n^T\mathbf{d}_n}{\|\mathbf{g}_n\|^2}\leq\frac{8\sigma-1}{1-4\sigma}. \tag{33}$$

We will prove it that (32) is true for $k=n+1$, it is clear from (28) that

$$\mathbf{g}_{n+1}^T\mathbf{d}_{n+1}=-\|\mathbf{g}_{n+1}\|^2+\beta_{n+1}^{MMSIS}\mathbf{g}_{n+1}^T\mathbf{d}_n \ ,$$

Dividing by $\|\mathbf{g}_{n+1}\|^2$ yields

$$\frac{\mathbf{g}_{n+1}^T\mathbf{d}_{n+1}}{\|\mathbf{g}_{n+1}\|^2}=-1+\beta_{n+1}^{MMSIS}\frac{\mathbf{g}_{n+1}^T\mathbf{d}_n}{\|\mathbf{g}_{n+1}\|^2}\frac{\|\mathbf{g}_n\|^2}{\|\mathbf{g}_n\|^2}. \tag{34}$$

Multiply the both side of (6) by β_{n+1}^{MMSIS}

$$\left|\beta_{n+1}^{MMSIS}\mathbf{g}_{n+1}^T\mathbf{d}_n\right|\leq-\sigma\left|\beta_{n+1}^{MMSIS}\right|\mathbf{g}_n^T\mathbf{d}_n. \tag{35}$$

From (34), (35) and using the absolute value properties

$$-1+\sigma\left|\beta_{n+1}^{MMSIS}\right|\frac{\|\mathbf{g}_n\|^2}{\|\mathbf{g}_{n+1}\|^2}\frac{\mathbf{g}_n^T\mathbf{d}_n}{\|\mathbf{g}_n\|^2}\leq\frac{\mathbf{g}_{n+1}^T\mathbf{d}_{n+1}}{\|\mathbf{g}_{n+1}\|^2}$$

and

$$\frac{\mathbf{g}_{n+1}^T\mathbf{d}_{n+1}}{\|\mathbf{g}_{n+1}\|^2}\leq-1-\sigma\left|\beta_{n+1}^{MMSIS}\right|\frac{\|\mathbf{g}_n\|^2}{\|\mathbf{g}_{n+1}\|^2}\frac{\mathbf{g}_n^T\mathbf{d}_n}{\|\mathbf{g}_n\|^2}.$$

Using (17), then

$$-1+\sigma\frac{\|\mathbf{g}_{n+1}\|^2}{\|\mathbf{d}_n\|^2}\frac{\|\mathbf{g}_n\|^2}{\|\mathbf{g}_{n+1}\|^2}\frac{\mathbf{g}_n^T\mathbf{d}_n}{\|\mathbf{g}_n\|^2}\leq\frac{\mathbf{g}_{n+1}^T\mathbf{d}_{n+1}}{\|\mathbf{g}_{n+1}\|^2}$$

and

$$\frac{\mathbf{g}_{n+1}^T\mathbf{d}_{n+1}}{\|\mathbf{g}_{n+1}\|^2}\leq-1-\sigma\frac{\|\mathbf{g}_{n+1}\|^2}{\|\mathbf{d}_n\|^2}\frac{\|\mathbf{g}_n\|^2}{\|\mathbf{g}_{n+1}\|^2}\frac{\mathbf{g}_n^T\mathbf{d}_n}{\|\mathbf{g}_n\|^2}.$$

That implies

$$-1+\sigma\frac{\mathbf{g}_n^T\mathbf{d}_n}{\|\mathbf{g}_n\|^2}\frac{\|\mathbf{g}_n\|^2}{\|\mathbf{d}_n\|^2}\leq\frac{\mathbf{g}_{n+1}^T\mathbf{d}_{n+1}}{\|\mathbf{g}_{n+1}\|^2}\leq-1-\sigma\frac{\mathbf{g}_n^T\mathbf{d}_n}{\|\mathbf{g}_n\|^2}\frac{\|\mathbf{g}_n\|^2}{\|\mathbf{d}_n\|^2}.$$

From (33) and based on Theorem 3, we obtain

$$-1+\sigma\left(\frac{-1}{1-4\sigma}\right)2^2<\frac{\mathbf{g}_{n+1}^T\mathbf{d}_{n+1}}{\|\mathbf{g}_{n+1}\|^2}<-1-\sigma\left(\frac{-1}{1-4\sigma}\right)2^2.$$

Hence,

$$\frac{-1}{1-4\sigma}<\frac{\mathbf{g}_{n+1}^T\mathbf{d}_{n+1}}{\|\mathbf{g}_{n+1}\|^2}<\frac{8\sigma-1}{1-4\sigma}.$$

This shows that the result hold for $k=n+1$. Furthermore, (32) is true.
Denotes $c=\frac{8\sigma-1}{4\sigma-1}$ and $0<\sigma<\frac{1}{8}$, then, $0<c<1$, and from (32), we get

$$(c-2)<\mathbf{g}_k^T\mathbf{d}_k<-c\|\mathbf{g}_k\|^2, \ \text{for all } k\geq 0.$$

This implies that (13) holds. The proof is completed. ■

The following theorem would be used to prove the global convergence of MMSIS method.

Theorem 5. *Consider the conjugate gradient method in the form (2)-(3), where β_k is calculated by (16) and α_k is determined by the strong Wolfe line search (5) and (6) with*

$0 < \sigma < \frac{1}{8}$. *Suppose that Assumption 1, Assumption 2 and the sufficient descent condition (13) hold . Then*

$$\lim_{k=0} \inf \|\mathbf{g}_k\| = 0. \tag{36}$$

Hence, the MMSIS method is have the globally convergent.

Proof: We proof by contradiction. Assume that (36) does not hold, then there exists a constant η such that

$$\|\mathbf{g}_k\| \geq \eta, \text{ for all } k \geq 0. \tag{37}$$

That implies

$$\frac{1}{\|\mathbf{g}_k\|^2} \leq \frac{1}{\eta^2}, \text{ , for all } k \geq 0. \tag{38}$$

Rewriting (3) as

$$\mathbf{d}_k + \mathbf{g}_k = \beta_k^{MMSIS} \mathbf{d}_{k-1}.$$

Squaring both side of the equation above gives

$$\|\mathbf{d}_k\|^2 = -\|\mathbf{g}_k\|^2 - 2\mathbf{g}_k^T \mathbf{d}_k + \left(\beta_k^{MMSIS}\right)^2 \|\mathbf{d}_{k-1}\|^2. \tag{39}$$

From (32), we have

$$\frac{16\sigma - 2}{4\sigma - 1}\|\mathbf{g}_k\|^2 < -2\mathbf{g}_k^T \mathbf{d}_k < \frac{2}{1-4\sigma}\|\mathbf{g}_k\|^2, \text{ for all } k \geq 0,$$

so we obtain $-2\mathbf{g}_k^T \mathbf{d}_k < \frac{2}{1-4\sigma}\|\mathbf{g}_k\|^2$ and using (39),

$$\|\mathbf{d}_k\|^2 < -\|\mathbf{g}_k\|^2 + \frac{2}{1-4\sigma}\|\mathbf{g}_k\|^2 + \left(\beta_k^{MMSIS}\right)^2 \|\mathbf{d}_{k-1}\|^2,$$

furthermore it,

$$\|\mathbf{d}_k\|^2 < \frac{1}{1-4\sigma}\|\mathbf{g}_k\|^2 + \left(\beta_k^{MMSIS}\right)^2 \|\mathbf{d}_{k-1}\|^2.$$

Substituting (17), we have

$$\|\mathbf{d}_k\|^2 < \frac{1}{1-4\sigma}\|\mathbf{g}_k\|^2 + \left(\frac{\|\mathbf{g}_k\|^2}{\|\mathbf{d}_{k-1}\|^2}\right)^2 \|\mathbf{d}_{k-1}\|^2.$$

Dividing both sides by $\|\mathbf{g}_k\|^4$, we get

$$\frac{\|\mathbf{d}_k\|^2}{\|\mathbf{g}_k\|^4} < \frac{1}{1-4\sigma}\frac{1}{\|\mathbf{g}_k\|^2} + \frac{1}{\|\mathbf{d}_{k-1}\|^2} \tag{40}$$

From (26), by squaring the both sides, we obtain

$$\frac{1}{\|\mathbf{d}_k\|^2} < \frac{4}{\|\mathbf{g}_k\|^2}, \text{ for all } k \geq 0. \tag{41}$$

Apply (41) to (40),

$$\frac{\|\mathbf{d}_k\|^2}{\|\mathbf{g}_k\|^4} < \frac{1}{1-4\sigma}\frac{1}{\|\mathbf{g}_k\|^2} + \frac{4}{\|\mathbf{g}_{k-1}\|^2}. \tag{42}$$

Using (38) and (42) together, we have

$$\frac{\|\mathbf{d}_k\|^2}{\|\mathbf{g}_k\|^4} < \frac{1}{1-4\sigma}\frac{1}{\eta^2} + \frac{4}{\eta^2}, \text{ for all } k \geq 0.$$

That implies

$$\frac{\|\mathbf{g}_k\|^4}{\|\mathbf{d}_k\|^2} > \frac{1-4\sigma}{5-16\sigma}\eta^2, \text{ for all } k \geq 0. \tag{43}$$

Since (43) is true for all $k \geq 0$, then

$$\sum_{k=0}^{n} \frac{\|\mathbf{g}_k\|^4}{\|\mathbf{d}_k\|^2} > \sum_{k=0}^{n} \frac{1-4\sigma}{5-16\sigma}\eta^2 = (n+1)\frac{1-4\sigma}{5-16\sigma}\eta^2.$$

Furthermore,

$$\sum_{k=0}^{\infty} \frac{\|\mathbf{g}_k\|^4}{\|\mathbf{d}_k\|^2} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\|\mathbf{g}_k\|^4}{\|\mathbf{d}_k\|^2} > \lim_{n\to\infty} (n+1)\frac{1-4\sigma}{5-16\sigma}\eta^2 = \infty.$$

This contradicts Lemma 2. So, we have $\lim_{k=0} \inf \|\mathbf{g}_k\| = 0$. Based on (14), the MMSIS method is globally convergent. Hence, the proof is completed. ■

IV. Numerical Experiments

In this section, we report the numerical results of MMSIS, RMIL, FR, CD, DY, WYL and NPRP methods for some unconstrained problems in Table I to show the efficiency of each method. We used some of the test functions considered by Andrei in [38] under low, medium, and high dimension as in paper Yousif [26] and Malik et al. [34], [35] , namely 2, 3, 4, 10, 50, 100, 500, 1000, 5000 and 10000. The function used is an artificial function. Artificial functions are functions used to detect algorithmic behavior under different conditions such as local optimal functions, bowel shaped functions, plate shaped functions, valley shaped function, unimodal functions, and other functions. Most of initial point were suggested by Andrei [38] and the others we take randomly.

All codes are written in Matlab R2019a and ran on personal laptop with specifications; processor intel Core i7, 16 GB RAM memory and operating system Windows 10 pro 64 bit. All methods have the same stopping criteria with $\epsilon = 10^{-6}$ and have been implemented under the strong Wolfe line search with $\sigma = 0.001$ and $\delta = 0.0001$. The numerical results are said to fail if the number of iterations exceeds 10000 and no solution is reached, written in Table II and Table III. We symbolize N is number of iterations and C is CPU time.

The numerical results under the exact line search in Table II show that the MMSIS method has an ability to solve all problem, the RMIL method which only can solve 97%, the FR method 93%, the CD method 94%, the DY method 88%, the WYL method 94%, and the NPRP method 95%.

The numerical results under the strong Wolfe line search in Table III show that the MMSIS method successfully reaches the solution point up to 100%, the FR method and CD method 93%, the DY method 90%, the WYL method 97% and the NPRP method 96%.

Based on the numerical results data in Table II and Table III, we can compare which method is more efficient by looking at the performance profile curve. We employ the performance profiles by Dolan and More in [39] to analyze the performance of each method. The formulas used to describe the performance profile will be explained as follows. Suppose S is set of n_s solvers, P is set of n_p test functions. For each solver $s \in S$ and function $p \in P$, consider $a_{p,s}$ is the number of iterations or CPU time required to solve function $p \in P$ by solver $s \in S$. Then the solvers comparison is based on the performance ratio as follows

$$r_{p,s} = \frac{a_{p,s}}{\min\{a_{p,s} : p \in P \text{ and } s \in S\}}$$

The overall evaluation of the solvers output is then obtained from the output profile feature as follows:

$$\rho_s(\tau) = \frac{1}{n_p} size\{p \in P : r_{p,s} \leq \tau\}$$

with $\rho_s(\tau)$ is the probability for solvers that a performance ratio $\rho_s(\tau)$ is within a factor $\tau \in R$ of the best possible ratio. In general, solvers with high vaules of $\rho_s(\tau)$ or in the upper right of the image represent the best solver. The performance profile results are illustrated in Figs. 1-4.

The left side of the figure represents the percentage of the test problem for which the method is robust and the fastest;the right side of the figure shows the percentage of test problems that have been successfully resolved.

TABLE I: List of the test functions, dimension, and initial point.

Problem	Function	Dimension	Initial point	Problem	Function	Dimension	Initial point
1	Ext. White & Holst	1000	(-1.2,1,...,-1.2,1)	50	Ext. Maratos	10	(-1,...,-1)
2	Ext. White & Holst	1000	(10,...,10)	51	Six hump camel	2	(-1,2)
3	Ext. White & Holst	10000	(-1.2,1,...,-1.2,1)	52	Six hump camel	2	(-5,10)
4	Ext. White & Holst	10000	(5,...,5)	53	Three hump camel	2	(-1,2)
5	Ext. Rosenbrock	1000	(-1.2,1,...,-1.2,1)	54	Three hump camel	2	(2,-1)
6	Ext. Rosenbrock	1000	(10,...,10)	55	Booth	2	(5,5)
7	Ext. Rosenbrock	10000	(-1.2,1,...,-1.2,1)	56	Booth	2	(10,10)
8	Ext. Rosenbrock	10000	(5,...,5)	57	Trecanni	2	(-1,0.5)
9	Ext. Freudenstein & Roth	4	(0.5,-2,0.5,-2)	58	Trecanni	2	(-5,10)
10	Ext. Freudenstein & Roth	4	(5,5,5,5)	59	Zettl	2	(-1,2)
11	Ext. Beale	1000	(1,0.8,...,1,0.8)	60	Zettl	2	(10,10)
12	Ext. Beale	1000	(0.5,...,0.5)	61	Shallow	1000	(0,...,0)
13	Ext. Beale	10000	(-1,...,-1)	62	Shallow	1000	(10,...,10)
14	Ext. Beale	10000	(0.5,...,0.5)	63	Shallow	10000	(-1,...,-1)
15	Ext. Wood	4	(-3,-1,-3,-1)	64	Shallow	10000	(-10,...,-10)
16	Ext. Wood	4	(5,5,5,5)	65	Generalized Quartic	1000	(1,...,1)
17	Raydan 1	10	(1,...,1)	66	Generalized Quartic	1000	(20,...,20)
18	Raydan 1	10	(10,...,10)	67	Quadratic QF2	50	(0.5,...,0.5)
19	Raydan 1	100	(-1,...,-1)	68	Quadratic QF2	50	(30,...,30)
20	Raydan 1	100	(-10,...,-10)	69	Leon	2	(2,2)
21	Ext. Tridiagonal 1	500	(2,...,2)	70	Leon	2	(8,8)
22	Ext. Tridiagonal 1	500	(10,...,10)	71	Gen. Tridiagonal 1	10	(2,...,2)
23	Ext. Tridiagonal 1	1000	(1,...,1)	72	Gen. Tridiagonal 1	10	(10,...,10)
24	Ext. Tridiagonal 1	1000	(-10,...,-10)	73	Gen. Tridiagonal 2	4	(1,1,1,1)
25	Diagonal 4	500	(1,...,1)	74	Gen. Tridiagonal 2	4	(10,10,10,10)
26	Diagonal 4	500	(-20,...,-20)	75	POWER	10	(1,...,1)
27	Diagonal 4	1000	(1,...,1)	76	POWER	10	(10,...,10)
28	Diagonal 4	1000	(-30,...,-30)	77	Quadratic QF1	50	(1,...,1)
29	Ext. Himmelblau	1000	(1,...,1)	78	Quadratic QF1	50	(10,...,10)
30	Ext. Himmelblau	1000	(20,...,20)	79	Quadratic QF1	500	(1,...,1)
31	Ext. Himmelblau	10000	(-1,...,-1)	80	Quadratic QF1	500	(-5,...,-5)
32	Ext. Himmelblau	10000	(50,...,50)	81	Ext.quad.pen.QP2	100	(1,...,1)
33	FLETCHCR	10	(0,...,0)	82	Ext.quad.pen.QP2	100	(10,...,10)
34	FLETCHCR	10	(10,...,10)	83	Ext.quad.pen.QP2	500	(10,...,10)
35	Ext. Powel	100	(3,-1,0,1,...,1)	84	Ext.quad.pen.QP2	500	(50,...,50)
36	Ext. Powel	100	(5,...,5)	85	Ext.quad.pen.QP1	4	(1,1,1,1)
37	NONSCOMP	2	(3,3)	86	Ext.quad.pen.QP1	4	(10,10,10,10)
38	NONSCOMP	2	(10,10)	87	Quartic	4	(10,10,10,10)
39	Ext. DENSCHNB	10	(1,...,1)	88	Quartic	4	(15,15,15,15)
40	Ext. DENSCHNB	10	(10,...,10)	89	Matyas	2	(1,1)
41	Ext. DENSCHNB	100	(10,...,10)	90	Matyas	2	(20,20)
42	Ext. DENSCHNB	100	(-50,...,-50)	91	Colville	4	(2,2,2,2)
43	Ext. Penalty	10	(1,2,3,...,10)	92	Colville	4	(10,10,10,10)
44	Ext. Penalty	10	(-10,...,-10)	93	Dixon and Price	3	(1,1,1)
45	Ext. Penalty	100	(5,...,5)	94	Dixon and Price	3	(10,10,10)
46	Ext. Penalty	100	(-10,...,-10)	95	Sphere	5000	(1,...,1)
47	Hager	10	(1,...,1)	96	Sphere	5000	(10,...,10)
48	Hager	10	(-10,...,-10)	97	Sum Squares	50	(0,1,...,0,1)
49	Ext. Maratos	10	(1.1,0.1)	98	Sum Squares	50	(10,...,10)

TABLE II: Numerical results under the exact line search.

Problem	MMSIS		RMIL		FR		CD		DY		WYL		NPRP	
	N	C	N	C	N	C	N	C	N	C	N	C	N	C
1	16	0.44	28	0.89	26	0.86	26	0.98	26	1.02	1251	29.80	20	0.54
2	30	0.80	30	0.82	293	9.74	222	7.54	278	9.69	2227	53.52	44	1.11
3	17	4.36	28	7.28	30	8.01	30	7.93	30	8.04	1180	281.66	19	4.79
4	25	6.40	32	8.14	180	48.63	193	53.46	145	39.89	2923	704.48	43	12.72
5	16	0.08	28	0.14	211	0.85	100	0.41	221	0.88	1283	3.73	19	0.09
6	30	0.13	47	0.20	56	0.23	56	0.25	56	0.35	1157	3.35	44	0.18
7	16	0.30	28	0.51	227	3.87	99	1.66	230	3.91	415	9.13	19	0.37
8	26	0.48	27	1.39	206	3.44	219	3.59	224	3.65	922	17.90	29	0.50
9	9	0.05	9	0.06	15	0.07	15	0.06	15	0.08	560	1.03	9	0.05
10	7	0.03	6	0.03	7	0.03	7	0.03	7	0.03	435	0.81	6	0.03
11	13	0.41	52	1.58	75	2.28	75	2.23	75	2.44	315	8.41	26	0.75
12	12	0.38	45	1.37	81	2.53	81	2.55	81	2.49	263	7.13	25	0.69
13	14	3.87	15	4.17	87	24.01	87	24.07	87	24.03	111	29.97	28	7.54
14	12	3.33	45	12.43	87	23.97	87	23.99	87	23.99	450	121.55	25	6.78
15	203	0.47	495	1.17	fail	fail	fail	fail	fail	fail	990	1.83	374	0.78
16	272	0.62	863	1.79	fail	fail	fail	fail	fail	fail	929	2.27	1387	2.82
17	21	0.08	19	0.08	19	0.04	19	0.05	19	0.05	24	0.07	27	0.09
18	75	0.21	84	0.26	fail	fail	fail	fail	fail	fail	208	0.43	47	0.14
19	118	0.42	107	0.38	99	0.32	90	0.28	96	0.31	249	0.67	97	0.32

(Continued on next page)

TABLE II – *Continued*

Problem	MMSIS		RMIL		FR		CD		DY		WYL		NPRP	
	N	C	N	C	N	C	N	C	N	C	N	C	N	C
20	194	0.62	180	0.59	985	2.85	751	2.07	fail	fail	376	0.97	133	0.38
21	12	0.22	202	2.94	453	7.28	452	7.22	453	7.27	3133	46.12	92	1.34
22	139	2.01	189	2.79	7	0.12	90	1.49	90	1.50	2667	38.50	61	0.90
23	12	0.37	181	4.90	517	15.36	517	15.46	517	20.11	4381	118.20	118	3.16
24	198	5.29	115	3.12	8	0.30	32	1.12	78	2.92	3651	97.61	10	0.31
25	5	0.04	3	0.01	5	0.03	5	0.03	5	0.03	46	0.14	5	0.04
26	5	0.03	4	0.02	5	0.03	5	0.03	5	0.03	46	0.14	5	0.03
27	5	0.03	3	0.02	5	0.03	5	0.03	5	0.04	46	0.16	5	0.03
28	5	0.04	4	0.02	5	0.04	5	0.04	5	0.04	64	0.21	5	0.03
29	9	0.07	11	0.05	15	0.09	15	0.08	15	0.08	20	0.08	13	0.08
30	6	0.04	6	0.03	8	0.05	8	0.05	8	0.05	10	0.05	8	0.04
31	10	0.23	9	0.18	27	0.53	26	0.54	27	0.52	12	0.22	15	0.29
32	7	0.17	9	0.21	17	0.34	17	0.36	17	0.34	20	0.34	18	0.33
33	80	0.22	72	0.17	1214	3.18	973	2.51	1208	3.09	198	0.39	86	0.22
34	39	0.12	31	0.08	30	0.12	30	0.12	30	0.10	48	0.12	47	0.14
35	810	3.78	fail	fail	5644	87.64	5640	89.26	5653	90.63	fail	fail	fail	fail
36	264	1.33	fail	fail	5338	25.36	5331	25.35	5334	23.55	fail	fail	fail	fail
37	8	0.04	15	0.04	50	0.15	50	0.16	50	0.16	118	0.25	12	0.17
38	15	0.06	16	0.04	915	4.77	241	0.70	1755	4.83	109	0.23	40	0.12
39	7	0.04	5	0.02	9	0.04	9	0.04	9	0.04	10	0.04	10	0.04
40	10	0.05	10	0.03	13	0.06	13	0.06	13	0.06	13	0.04	10	0.04
41	11	0.05	10	0.03	13	0.06	13	0.06	13	0.07	14	0.05	10	0.04
42	11	0.06	9	0.03	77	0.43	77	0.23	77	0.23	14	0.04	11	0.05
43	22	0.08	20	0.05	13	0.06	13	0.06	13	0.06	48	0.13	22	0.09
44	8	0.04	21	0.06	14	0.06	14	0.07	14	0.07	39	0.10	13	0.04
45	13	0.06	15	0.06	13	0.06	13	0.07	14	0.06	fail	fail	fail	fail
46	10	0.04	fail	fail	31	0.10	39	0.16	fail	fail	fail	fail	fail	fail
47	13	0.06	12	0.06	11	0.03	11	0.03	11	0.03	13	0.16	12	0.20
48	18	0.07	18	0.07	97	0.23	99	0.23	97	0.23	21	0.05	19	0.06
49	53	0.15	42	0.12	4820	8.58	1022	1.92	4096	7.57	fail	fail	31	0.10
50	22	0.08	20	0.07	31	0.11	28	0.10	29	0.09	380	1.05	30	0.09
51	7	0.02	8	0.04	24	0.06	24	0.05	24	0.06	7	0.04	10	0.04
52	6	0.02	6	0.04	11	0.04	11	0.03	11	0.04	9	0.03	9	0.04
53	9	0.03	10	0.18	21	0.20	24	0.08	9	0.20	10	0.05	11	0.04
54	11	0.03	11	0.03	fail	fail	39	0.23	fail	fail	8	0.02	14	0.05
55	4	0.01	3	0.02	3	0.01	3	0.02	3	0.02	5	0.02	3	0.02
56	4	0.02	3	0.02	3	0.02	3	0.02	3	0.02	4	0.01	3	0.02
57	1	0.01	1	0.01	1	0.01	1	0.01	1	0.01	1	0.01	1	0.01
58	5	0.02	6	0.03	14	0.05	14	0.06	14	0.06	13	0.04	9	0.04
59	11	0.04	21	0.08	11	0.04	11	0.05	11	0.04	51	0.15	9	0.04
60	11	0.03	19	0.08	10	0.05	10	0.05	10	0.05	73	0.18	12	0.04
61	8	0.04	26	0.14	18	0.10	18	0.12	18	0.09	76	0.29	14	0.08
62	11	0.05	11	0.07	175	0.65	175	0.71	175	0.66	94	0.32	15	0.07
63	9	0.17	37	0.70	47	0.88	47	0.81	47	0.84	73	1.25	26	0.51
64	9	0.19	35	0.65	43	0.76	43	0.73	43	0.73	68	1.17	22	0.43
65	5	0.02	6	0.05	6	0.05	6	0.05	6	0.05	6	0.04	6	0.06
66	6	0.04	10	0.06	12	0.07	12	0.07	13	0.10	9	0.05	10	0.06
67	87	0.19	77	0.25	116	0.31	117	0.31	116	0.32	151	0.32	85	0.24
68	78	0.18	77	0.23	125	0.41	124	0.32	126	0.33	153	0.35	109	0.27
69	25	0.06	15	0.05	37	0.11	40	0.15	40	0.12	1046	4.18	26	0.08
70	18	0.04	30	0.12	504	1.12	474	2.93	256	1.81	755	2.64	18	0.06
71	24	0.07	22	0.10	27	0.13	27	0.12	27	0.11	28	0.09	23	0.10
72	29	0.08	27	0.11	43	0.17	43	0.17	43	0.17	41	0.11	29	0.10
73	4	0.01	4	0.03	5	0.02	5	0.02	5	0.02	5	0.03	5	0.03
74	11	0.04	7	0.04	fail	fail	fail	fail	fail	fail	20	0.21	23	0.07
75	102	0.20	123	0.31	20	0.05	20	0.05	21	0.05	146	0.33	64	0.17
76	129	0.27	139	0.34	24	0.05	24	0.07	23	0.07	157	0.31	88	0.24
77	69	0.16	69	0.31	38	0.08	38	0.08	38	0.09	126	0.27	88	0.23
78	85	0.20	78	0.22	41	0.10	41	0.10	41	0.09	129	0.27	101	0.25
79	240	1.31	422	3.23	131	0.58	131	0.55	131	0.59	532	3.08	698	4.54
80	424	2.41	538	3.32	137	0.56	150	0.84	137	0.60	525	1.89	777	4.65
81	41	0.14	33	0.15	189	0.90	231	0.77	141	0.75	598	1.53	44	0.19
82	36	0.12	31	0.35	2758	7.67	101	0.45	fail	fail	567	1.48	44	0.18
83	94	0.75	58	0.63	1034	24.57	149	2.06	fail	fail	8007	49.45	88	0.71
84	96	0.80	41	0.54	fail	fail	140	1.76	fail	fail	5972	38.48	112	0.88
85	9	0.03	14	0.06	20	0.06	20	0.05	20	0.05	17	0.05	20	0.08
86	9	0.04	9	0.05	19	0.06	19	0.06	19	0.06	21	0.07	26	0.08
87	114	0.28	807	5.76	271	1.61	271	1.57	271	1.22	1016	4.28	1241	2.77
88	118	0.33	807	2.96	272	1.09	272	1.07	272	1.00	577	3.68	1216	2.65

(Continued on next page)

TABLE II – *Continued*

Problem	MMSIS		RMIL		FR		CD		DY		WYL		NPRP	
	N	C	N	C	N	C	N	C	N	C	N	C	N	C
89	1	0.00	1	0.01	1	0.01	1	0.01	1	0.01	1	0.01	1	0.01
90	1	0.01	1	0.01	1	0.00	1	0.00	1	0.01	1	0.01	1	0.01
91	357	0.68	1035	4.00	fail	fail	fail	fail	fail	fail	1392	2.91	1591	3.03
92	58	0.14	392	0.87	34	0.10	34	0.11	34	0.11	1468	2.92	578	1.07
93	15	0.04	35	0.11	15	0.05	15	0.03	15	0.03	74	0.16	14	0.06
94	18	0.05	56	0.17	29	0.07	29	0.07	29	0.07	76	0.17	36	0.10
95	1	0.02	1	0.02	1	0.02	1	0.01	1	0.02	1	0.02	1	0.03
96	1	0.02	1	0.02	1	0.02	1	0.02	1	0.02	1	0.01	1	0.02
97	49	0.15	46	0.14	26	0.08	26	0.09	26	0.08	73	0.20	33	0.11
98	80	0.23	81	0.22	42	0.12	42	0.13	41	0.12	183	0.44	102	0.27

TABLE III: Numerical results under the strong Wolfe line search.

Problem	MMSIS		RMIL		FR		CD		DY		WYL		NPRP	
	N	C	N	C	N	C	N	C	N	C	N	C	N	C
1	13	0.04	22	0.06	61	0.12	43	0.10	49	0.10	2861	3.43	17	0.06
2	53	0.17	35	0.10	218	1.08	229	0.95	210	0.92	5763	8.88	55	0.20
3	13	0.32	22	0.44	67	1.00	49	0.76	50	0.81	2129	27.16	17	0.41
4	40	1.10	43	0.97	272	6.30	109	4.10	178	5.25	4134	70.62	42	1.10
5	23	0.04	28	0.04	83	0.13	105	0.14	59	0.11	773	0.50	35	0.06
6	33	0.05	50	0.06	173	0.22	159	0.22	178	0.22	1300	0.93	25	0.04
7	23	0.25	28	0.31	88	0.98	106	1.06	59	0.82	811	4.64	36	0.38
8	23	0.24	46	0.38	244	3.22	201	2.99	196	3.04	2277	15.89	23	0.30
9	8	0.00	15	0.00	30	0.01	fail	fail	fail	fail	657	0.02	9	0.01
10	11	0.01	9	0.01	21	0.00	21	0.00	21	0.00	fail	fail	fail	fail
11	14	0.05	52	0.10	75	0.13	75	0.13	75	0.13	281	0.39	26	0.06
12	12	0.04	45	0.09	81	0.14	81	0.14	81	0.13	166	0.24	25	0.06
13	14	0.26	15	0.27	87	1.16	87	1.18	87	1.18	290	3.83	28	0.47
14	12	0.23	45	0.67	87	1.16	87	1.15	87	1.13	257	3.31	25	0.43
15	195	0.01	500	0.02	fail	fail	fail	fail	fail	fail	784	0.02	281	0.01
16	473	0.02	1085	0.03	fail	fail	fail	fail	fail	fail	1637	0.05	1427	0.05
17	21	0.00	19	0.00	19	0.00	19	0.00	19	0.00	22	0.00	27	0.00
18	34	0.00	fail	fail	2670	0.09	3215	0.11	2613	0.09	65	0.00	29	0.00
19	103	0.04	109	0.03	95	0.03	95	0.03	95	0.03	192	0.05	97	0.03
20	180	0.04	180	0.05	857	0.15	868	0.14	fail	fail	364	0.07	134	0.04
21	12	0.02	28	0.05	452	0.36	452	0.37	452	0.36	720	0.59	22	0.05
22	8	0.02	8	0.03	9	0.03	9	0.02	9	0.02	376	0.34	8	0.02
23	12	0.04	28	0.07	517	0.73	517	0.73	517	0.80	82	0.14	22	0.07
24	8	0.04	5	0.02	9	0.04	9	0.04	9	0.03	151	0.24	8	0.04
25	2	0.00	2	0.00	2	0.00	2	0.00	2	0.00	5	0.01	2	0.00
26	2	0.00	2	0.00	2	0.00	2	0.00	2	0.00	9	0.01	2	0.00
27	2	0.00	2	0.00	2	0.00	2	0.00	2	0.00	5	0.01	2	0.00
28	2	0.01	2	0.00	2	0.00	2	0.00	2	0.00	11	0.01	2	0.00
29	9	0.02	11	0.02	15	0.02	15	0.03	15	0.03	20	0.03	13	0.02
30	6	0.01	7	0.03	9	0.03	9	0.02	9	0.02	11	0.02	10	0.02
31	11	0.12	9	0.10	17	0.23	20	0.25	17	0.22	12	0.17	15	0.13
32	7	0.08	11	0.12	13	0.13	13	0.12	13	0.12	14	0.15	10	0.10
33	87	0.01	72	0.00	1208	0.06	1239	0.06	1208	0.06	124	0.01	85	0.01
34	107	0.02	111	0.01	285	0.02	303	0.02	299	0.03	183	0.01	134	0.01
35	151	0.06	fail	fail	5492	0.96	5607	1.02	5584	1.00	4345	0.80	fail	fail
36	144	0.05	fail	fail	5934	1.08	5980	1.09	6019	1.15	6556	1.27	fail	fail
37	12	0.00	54	0.00	166	0.01	155	0.01	156	0.01	148	0.01	15	0.00
38	14	0.01	21	0.00	93	0.01	93	0.00	93	0.01	446	0.02	15	0.00
39	7	0.00	6	0.00	9	0.00	9	0.00	9	0.00	10	0.00	10	0.00
40	9	0.00	7	0.00	11	0.00	11	0.00	11	0.00	12	0.00	9	0.00
41	10	0.00	7	0.00	11	0.00	11	0.01	11	0.00	12	0.00	9	0.00
42	7	0.00	9	0.00	63	0.02	63	0.02	63	0.02	8	0.00	13	0.01
43	20	0.00	24	0.01	11	0.00	11	0.00	11	0.00	25	0.00	14	0.00
44	9	0.01	26	0.01	19	0.00	19	0.00	19	0.01	35	0.25	14	0.00
45	9	0.00	fail	fail	28	0.01	30	0.01	fail	fail	fail	fail	10	0.00
46	13	0.01	fail	fail	1919	1.42	860	0.25	fail	fail	197	0.05	16	0.01
47	13	0.00	12	0.00	11	0.00	11	0.00	11	0.00	13	0.00	12	0.00
48	18	0.00	18	0.00	97	0.01	97	0.01	97	0.01	21	0.00	19	0.00
49	44	0.01	42	0.00	3001	0.23	9042	0.59	fail	fail	610	0.03	48	0.01
50	33	0.00	37	0.00	128	0.02	122	0.01	128	0.02	173	0.02	37	0.17
51	7	0.00	fail	fail	10	0.00	11	0.00	11	0.00	7	0.00	10	0.00
52	9	0.00	11	0.00	15	0.00	17	0.00	17	0.00	13	0.00	10	0.00
53	13	0.01	11	0.00	fail	fail	14	0.01	fail	fail	11	0.16	12	0.00
54	13	0.01	fail	fail	fail	fail	22	0.01	17	0.01	11	0.00	8	0.00

(Continued on next page)

TABLE III – *Continued*

Problem	MMSIS		RMIL		FR		CD		DY		WYL		NPRP	
	N	C	N	C	N	C	N	C	N	C	N	C	N	C
55	2	0.00	2	0.00	2	0.00	2	0.00	2	0.00	5	0.00	2	0.00
56	2	0.00	2	0.00	2	0.00	2	0.00	2	0.00	4	0.00	2	0.00
57	2	0.00	fail	fail	fail	fail	fail	fail	2	0.00	2	0.00	1	0.00
58	7	0.00	7	0.00	10	0.00	10	0.00	10	0.00	15	0.00	7	0.00
59	11	0.00	21	0.00	11	0.00	11	0.00	11	0.00	34	0.00	9	0.00
60	12	0.00	21	0.00	16	0.00	16	0.00	16	0.00	76	0.00	13	0.00
61	8	0.01	26	0.02	18	0.02	18	0.02	18	0.02	86	0.06	14	0.02
62	10	0.01	17	0.02	98	0.09	89	0.07	96	0.07	70	0.05	35	0.05
63	9	0.08	37	0.23	47	0.29	47	0.27	47	0.38	123	0.73	26	0.23
64	9	0.07	35	0.28	10	0.09	10	0.09	9	0.08	131	0.87	12	0.11
65	5	0.04	fail	fail	7	0.04	7	0.05	7	0.04	7	0.04	6	0.05
66	10	0.03	11	0.05	38	0.19	23	0.10	40	0.19	23	0.08	14	0.08
67	82	0.01	78	0.01	116	0.01	116	0.01	116	0.01	184	0.02	86	0.01
68	82	0.01	77	0.01	1271	0.09	1331	0.09	1176	0.09	149	0.01	109	0.02
69	22	0.00	32	0.00	196	0.01	222	0.02	194	0.01	807	0.03	26	0.00
70	49	0.01	33	0.00	297	0.04	211	0.02	736	0.03	3688	0.10	57	0.00
71	24	0.00	22	0.00	27	0.00	27	0.00	27	0.00	29	0.00	23	0.00
72	30	0.00	28	0.00	43	0.01	43	0.00	43	0.00	41	0.00	29	0.00
73	4	0.00	4	0.00	5	0.00	5	0.00	5	0.00	5	0.01	5	0.00
74	11	0.00	7	0.00	6347	0.20	fail	fail	6384	0.18	28	0.00	14	0.00
75	97	0.01	123	0.01	10	0.00	10	0.00	10	0.00	269	0.02	10	0.00
76	104	0.00	139	0.01	10	0.00	10	0.00	10	0.00	196	0.01	10	0.00
77	71	0.01	69	0.01	38	0.01	38	0.01	38	0.01	107	0.01	38	0.00
78	74	0.01	78	0.01	40	0.01	40	0.01	40	0.01	128	0.02	40	0.01
79	310	0.13	447	0.18	131	0.07	131	0.07	131	0.06	670	0.25	639	0.24
80	432	0.18	500	0.22	137	0.08	137	0.06	137	0.07	649	0.23	716	0.27
81	33	0.03	34	0.02	204	0.14	235	0.15	368	0.17	2030	0.27	50	0.02
82	33	0.03	31	0.02	336	0.18	1320	0.37	398	0.19	1577	0.28	50	0.03
83	58	0.11	57	0.12	1137	3.68	1202	3.55	1188	3.50	7631	3.99	75	0.13
84	62	0.11	fail	fail	1142	3.46	1461	4.13	1216	3.45	7044	5.30	76	0.14
85	10	0.00	14	0.00	20	0.00	20	0.00	20	0.00	15	0.00	19	0.00
86	9	0.00	15	0.00	51	0.00	51	0.00	51	0.00	17	0.00	10	0.01
87	167	0.01	802	0.04	272	0.02	272	0.01	272	0.02	608	0.03	1234	0.06
88	177	0.01	806	0.05	273	0.02	273	0.02	273	0.02	736	0.04	1198	0.06
89	1	0.00	1	0.00	1	0.00	1	0.00	1	0.00	1	0.00	1	0.00
90	1	0.00	1	0.00	1	0.00	1	0.00	1	0.00	1	0.00	1	0.01
91	265	0.02	1032	0.06	fail	fail	fail	fail	fail	fail	1276	0.04	1293	0.04
92	159	0.01	669	0.04	34	0.01	34	0.00	33	0.00	1448	0.05	578	0.04
93	15	0.00	35	0.00	16	0.00	16	0.00	16	0.00	53	0.00	14	0.00
94	26	0.11	46	0.01	23	0.00	25	0.00	25	0.00	66	0.01	47	0.07
95	1	0.01	1	0.01	1	0.01	1	0.01	1	0.01	1	0.01	1	0.01
96	1	0.01	1	0.01	1	0.01	1	0.01	1	0.01	1	0.02	1	0.18
97	53	0.01	46	0.01	25	0.01	25	0.00	25	0.01	73	0.01	25	0.01
98	72	0.01	81	0.01	41	0.01	41	0.01	41	0.01	149	0.02	41	0.83

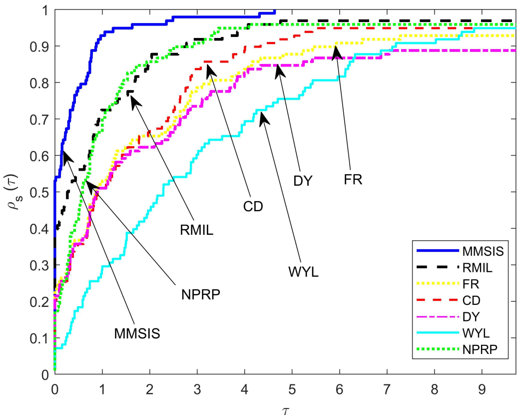

Fig. 1: Performance profile based on the number of iterations under the exact line search.

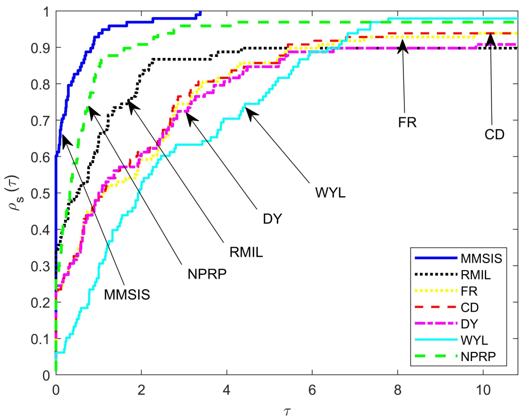

Fig. 2: Performance profile based on the number of iterations under the strong Wolfe line search.

Fig. 1 and Fig. 2 plots the performance profile of all methods for the number of iterations. The MMSIS method is have the top curve from left to right side compared to other methods. So that, the MMSIS have the best performance based on number of iterations than other method. The second method that has good performance is NPRP method, etc.

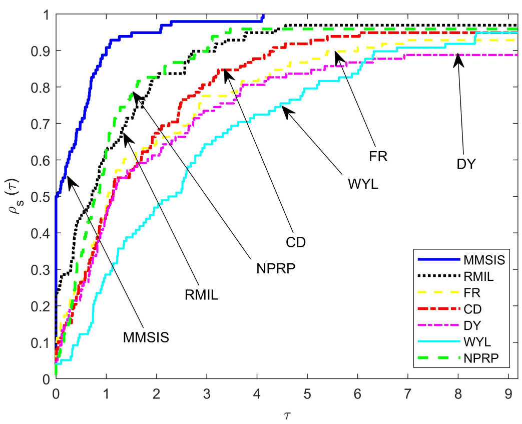

Fig. 3: Performance profile based on the CPU time under the exact line search.

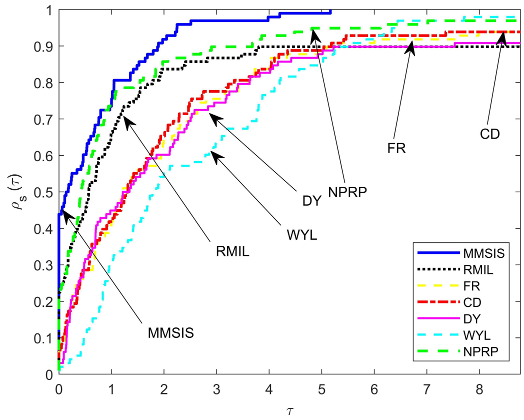

Fig. 4: Performance profile based on the CPU time under the strong Wolfe line search.

Fig. 3 and Fig. 4 plots the performance profile of all methods for the CPU time. This figure show that the MMSIS method have the top curve compared to other methods. Hence, the MMSIS have the best performance based on CPU time than other method. Under the MMSIS method curve is NPRP method curve, this indicates that NPRP method has a good performance after the MMSIS method.

Hence, comparing Figs. 1-4 shows that the MMSIS method outperforms the others method in every case. The top curve is the most efficient method, so the new method is also efficient in terms of numerical result.

V. Conclusion

The conjugate gradient method is an iterative method and this method can be used for solving the optimization problems without constrains in large scales. In this paper, we propose a new conjugate gradient method with coefficient which is a modification of NPRP method. The new method fulfills the sufficient descent condition and have the globally convergent under the exact line search and the strong Wolfe line search with $\sigma \in \left(0, \frac{1}{8}\right)$. Under the test function used, the numerical results show that the new method is more efficient based on number of iterations and CPU time compared to other methods.

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