A Numerical Method for the Fractional Variational Problems Based on Chebyshev Cardinal Functions

Jianke Zhang, Xucong Tian, Chang Zhou, Xiaobao Yang

Abstract—**In this paper, an efficient numerical method for solving a class of fractional variational problems (FVPs) is discussed. Firstly, an operational matrix of the fractional derivative in the Caputo sense is derived for the Chebyshev cardinal functions. The choice of Chebyshev cardinal functions provides flexibility for the method, and the boundary conditions can be easily applied. Secondly, a direct method based on the Chebyshev cardinal functions is proposed to obtain the approximate solution of the fractional variational problems. Finally, applicability and simplicity of the proposed method has been shown with some illustrative examples.**

Index Terms—**fractional variational problems, Caputo fractional derivative, Chebyshev cardinal functions, numerical method**

I. Introduction

THE fractional-order variational problem is a special class of fractional-order optimal control problems. Recently, a large number of papers have published on fractional order variational problems, mainly involving Euler-Lagrange equations [3-5] and numerical methods [6-11] using Riemann-Liouville or Caputo derivatives. Under different boundary conditions, the transversal conditions of Euler-Lagrange equations and fractional-order variational problems prevails. Agrawal [1,2] once generalized the Euler-Lagrange equations and transversal conditions of the functionals with Riemann-Liouville and Caputo derivatives under different boundary conditions.

However, how to solve the fractional variational problems is the most important. There are two major methods, one is to use the Euler-Lagrange equations, which needs necessary optimality condition to simplify the variational problem, so as to further study the differential equation. After that, we can use analytical or numerical methods to solve differential equations and obtain the solutions to the original problem. This method is called the indirect methods. For example, spectral methods [6-7][12], Ritz's methods [13], neural networks methods [14] and pseudospectral schemes [15].

In spectral methods, we either approximate admissible functions by all possible linear combinations $y_N(t) = \sum_{i=i}^{N} a_i\phi_i(t)$, with constant coefficients a_i and a set of known base functions ϕ_i, or we approximate the admissible functions with such combinations. We can use the appropriate discrete approximation of the Lagrangian, and replace the integral with a sum to convert the main problem to the optimization of functions of multiple parameters. Spectral approximation [7], [12] requires that the solution of the considered fractional differential equation must be sufficiently smooth so that the desired accuracy can be achieved. In particular, most of the works related to spectral methods are based on polynomials. Since the analytical solutions of fractional-order differential equations is usually not smooth even for well-behaved inputs, their regularity is limited in the usual Sobolev space.When the solution is smooth enough, the Chebyshev polynomials [16-19] can be used as an approximate basis for the spectral methods of fractional differential models.

At present, research on the use of approximation cardinal functions [16-19] to deal with various problems has attracted more and more attention. In this paper, we intend to extend the application of the Chebyshev cardinal functions to solve fractional variational problems. By using the cardinality property of these cardinal functions, a new method for calculating the nonlinear equations is proposed, which greatly simplifies the problem. In addition, a new direct method based on the Chebyshev cardinal functions is proposed to obtain the approximate solution of the fractional variational problems. The proposed method approximates the solution of the mentioned problem by minimizing the integral over linear combinations set of certain basis functions. The cardinal functions are select to make them linearly independent and satisfy the homogeneous initial or boundary conditions.

The structure of this paper is as follows. In Section II, we review the basic definitions and properties of the fractional calculus theory. In section III, we briefly review the operational matrix of fractional derivative of the Chebyshev cardinal functions. In section IV, a numerical method is proposed to solve the fractional variational problems. In Section V, several numerical examples are given and solved by this proposed method. In the end, Section VI gives the conclusion.

II. Preliminaries

In this section, we express some basic definitions of fractional calculus theory. Moreover, we briefly review the rudiments of the Chebyshev cardinal functions.

A. Preliminaries on fractional calculus theory

Some basic definitions and properties of the fractional calculus theory are introduced as follows.

Definition 1 [22]. A real function $f(t)$, $t > 0$, is said to be in the space C_α, $\alpha \in \mathbb{R}$, if there exists a real number

Manuscript received January 15th, 2020; revised June 10th, 2020. This work is supported by the National Natural Science Foundation of China (Grant No.11701446, 11601420, 11401469), the Natural Science Foundation of Shaanxi Province (2018JM1055), New Star Team of Xi'an University of Posts and Telecommunications, Construction of Special Funds for Key Disciplines in Shaanxi Universities.

Jianke Zhang is with the School of Science, Xi'an University of Post and Telecommunications, Xi'an 710121, China.(e-mail: jiankezh@163.com).

Xucong Tian is with the School of Science, Xi'an University of Post and Telecommunications, Xi'an 710121, China.(e-mail: 564883341@qq.com).

Chang Zhou is with the School of Science, Xi'an University of Post and Telecommunications, Xi'an 710121, China.(e-mail: maytheday@163.com).

Xiaobao Yang is with the School of Modern Posts & Institute of Posts, Xi'an University of Post and Telecommunications, Xi'an 710121, China.(e-mail: y78h11b09@163.com).

$p > \alpha$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty)$, and it is said to be in the space C_α^m, $m \in \mathbb{N}\bigcup\{0\}$, if and only if $f^{(m)}(t) \in C_\alpha$.

Definition 2 [22]. For an arbitrary function $f(t) \in C_\alpha$, $\alpha \in \mathbb{R}$, the Riemann-Liouville fractional integral operator of order $\alpha > 0$ is defined as

$$I^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}ds, & \alpha > 0, t > 0, \\ f(t), & \alpha = 0, \end{cases} \tag{1}$$

where $\Gamma(\alpha)$ is the well-konwn Gamma function.

For the Riemann-Liouville fractional integral we have

$$1. I^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}t^{\alpha+\beta}, \quad \beta > -1,$$
$$2. I^\alpha(\lambda f(t) + \mu g(t)) = \lambda I^\alpha f(t) + \mu I^\alpha g(t),$$

where λ and μ are real constants.

Definition 3 [22]. The Riemann-Liouville fractional derivative of $f(t) \in C_\alpha$, of order α is defined as

$$D^\alpha f(t) = \frac{d^m}{dt^m}I^{m-\alpha}f(t), \quad m-1 < \alpha < m, m \in \mathbb{N}. \tag{2}$$

Definition 4 [22]. The Caputo fractional derivative of $f(t) \in C_\alpha^m$, $m \in \mathbb{N} \cup 0$, is defined as

$$D_*^\alpha f(t) = \begin{cases} I^{m-\alpha}f^{(m)}(t), & m-1 < \alpha < m, m \in \mathbb{N}, \\ \frac{d^m}{dt^m}f(t), & \alpha = m. \end{cases} \tag{3}$$

Some properties ([22]) of the operator D_α are as follows,

$$1. D^\alpha I^\alpha f(t) = f(t),$$
$$2. D^\alpha t^\beta = \begin{cases} 0, & \alpha \in N_0, \beta < \alpha, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}t^{\beta-\alpha}, & otherwise, \end{cases}$$
$$3. D^\alpha C = 0,$$
$$4. D^\alpha(\lambda_1 f(t) + \lambda_2 g(t)) = \lambda_1 D^\alpha f(t) + \lambda_2 D^\alpha g(t), \tag{4}$$

where λ_1, λ_2 and C are real constants. Also, if $\alpha \in \mathbb{R}$, $n-1 < \alpha \leq n$, $n \in N$ then

$$D^\alpha(f(t)) = I^{n-\alpha}D^n f(t). \tag{5}$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of problem.

B. Chebyshev cardinal functions and their properties

Chebyshev cardinal functions of order N in [-1,1] are defined as [18], [19]

$$\phi_j(x) = \frac{T_{N+1}(x)}{T'_{N+1}(x_j)(x-x_j)}, \quad j = 1, 2, \cdots, N+1, \tag{6}$$

where $T_{N+1}(x)$ is the first kind Chebyshev polynomial of order $N+1$ in [-1,1] defined by

$$T_{N+1}(x) = \cos((N+1)\arccos(x)), \tag{7}$$

and $x_j, j = 1, 2, \cdots, N+1$, are the zeros of $T_{N+1}(x)$ defined by $\cos(2j-1)\pi/(2N+2), j = 1, 2, \cdots, N+1$. We change the variable $t = (x+1)/2$ to use these functions on $[0,1]$. Now any function $f(t)$ on $[0,1]$ can be approximated as

$$f(t) \approx \sum_{j=1}^{N+1} f(t_j)\phi_j(t) = F^T\Phi_N(t), \tag{8}$$

and $t_j, j = 1, 2, \cdots, N+1$, are the shifted points of $x_j, j = 1, 2, \cdots, N+1$ as follows

$$t_j = -\frac{b-a}{2}\cos(\frac{(2j-1)\pi}{2(N+1)}) + \frac{b+a}{2}, \quad j = 1, 2, \cdots, N+1. \tag{9}$$

$$F = [f(t_1), f(t_2), \cdots, f(t_{N+1})]^T,$$
$$\Phi_N(t) = [\phi_1(t), \phi_2(t), \cdots, \phi_{N+1}(t)]^T. \tag{10}$$

Note that the functions $\phi_j(t)$ satisfy in the relation

$$\phi_j(t_i) = \delta_{j,i} = \begin{cases} 1, & j = i, \\ 0, & j \neq i, \end{cases} \quad i, j = 1, \cdots, N+1.$$

So we have

$$\Phi_N(t_j) = e_j, \quad j = 1, \cdots, N+1, \tag{11}$$

where e_i is the ith column of unit matrix of order $N+1$.

Also, an arbitrary function $g(x,t)$ on $([0,1] \times [0,1])$ can be approximated by Chebyshev cardinal functions

$$g(x,t) \approx \Phi_N^T(t) \cdot \mathbf{G} \cdot \Phi_N(x) \tag{12}$$

where

$$[\mathbf{G}]_{i,j} = g(t_j, t_i), \quad i, j = 1, 2, \cdots, N+1. \tag{13}$$

III. Analysis of the Methods

In this section, we briefly review the method of solving fractional differential equations with the Chebyshev cardinal functions.

A. The operational matrix of derivative

The differentiation of vector Φ_N in Eq.(10) can be expressed as

$$\Phi'_N = \mathbf{D}\Phi_N, \tag{14}$$

where $\mathbf{D}$ is $(N+1) \times (N+1)$ operational matrix of derivative for Chebyshev cardinal functions.

It is shown [18], [19] that the matrix $\mathbf{D}$ is the form

$$\mathbf{D}_\alpha = \begin{pmatrix} \phi'_1(t_1) & \cdots & \phi'_1(t_{N+1}) \\ \vdots & \ddots & \vdots \\ \phi'_{N+1}(t_1) & \cdots & \phi'_{N+1}(t_{N+1}) \end{pmatrix}, \tag{15}$$

where

$$\phi'_j(t_j) = \sum_{\substack{i=1 \\ i\neq j}}^{N+1} \frac{1}{t_j - t_i}, \quad j = 1, \cdots, N+1.$$

and if $j \neq k$, then

$$\phi'_j(t_k) = \frac{\beta}{T'_{N+1}(t_j)} \sum_{\substack{l=1 \\ l\neq k,j}}^{N+1} (t_k - t_l), \quad j, k = 1, \cdots, N+1. \tag{16}$$

and

$$\beta = 2^{2N+1}/L^{N+1}. \tag{17}$$

Note that

$$\frac{T_{N+1}(t)}{t - t_j} = \beta \sum_{\substack{k=1 \\ k\neq j}}^{N+1} (t - t_k). \tag{18}$$

B. The operational matrix of fractional derivative

The fractional differentiation of vector $\Phi_N(t)$ in Eq.(9) can be expressed as [17], [19]

$$D^{(\alpha)}\Phi_N = \mathbf{D}_\alpha \Phi_N, \tag{19}$$

where $\mathbf{D}_\alpha$ is $(N+1)\times(N+1)$ operational matrix of fractional derivative for Chebyshev cardinal functions.The matrix $\mathbf{D}_\alpha$ can be obtained by the following process. Let

$$D^{(\alpha)}\Phi_N(t) = [\phi_1^\alpha(t), \phi_2^\alpha(t), \cdots, \phi_{N+1}^\alpha(t)]^T. \tag{20}$$

Note that

$$\frac{T_{N+1}(t)}{t-t_j} = \beta \times \prod_{\substack{k=1\\k\neq j}}^{N+1}(t-t_k). \tag{21}$$

Using Eqs.(2), (10) and (21) the function $\phi_j^\alpha(t)$ can be approximated as

$$\phi_j^\alpha(t) = \beta \times \frac{1}{T'_{N+1}(t_j)}\Big(\prod_{\substack{k=1\\k\neq j}}^{N+1}(t-t_k)\Big)^{(\alpha)} \tag{22}$$

Let

$$\mathbf{T} = [1, t, t^2, \cdots, t^N]^T \tag{23}$$

then Eq.(9) results in

$$\Phi_N(t) = [\phi_1(t), \phi_2(t), \cdots, \phi_{N+1}(t)]^T = \mathbf{A}\mathbf{T}, \tag{24}$$

where $\mathbf{A}$ is $(N+1)\times(N+1)$ operational matrix of coefficient for Chebyshev cardinal functions.

Because of orthogonality of $\phi_j(t), j = 1, \cdots, N+1$, this matrix is invertible. From Eq.(4) and for $0 \leq \alpha < 1$, we get

$$\begin{aligned} D_t^{(\alpha)}\mathbf{T} = [0, &\frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}, \frac{\Gamma(3)}{\Gamma(3-\alpha)}t^{2-\alpha}, \\ &\cdots, \frac{\Gamma(N+1)}{\Gamma(N+1-\alpha)}t^{N-\alpha}]^T \\ &= t^{-\alpha}\mathbf{D}_1\mathbf{T}, \end{aligned} \tag{25}$$

where $\mathbf{D}_1$ is $(N+1)\times(N+1)$ matrix of the following form

$$\mathbf{D}_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \frac{\Gamma(N+1)}{\Gamma(N+1-\alpha)} \end{pmatrix}. \tag{26}$$

If $1 \leq \alpha < 2$ then the second rows of $\mathbf{D}_1$ is zero and etc. Using Eq.(23) we have

$$D_t^\alpha \Phi_N(t) = \mathbf{A} D_t^\alpha \mathbf{T} = t^{-\alpha}\mathbf{A}\mathbf{D}_1\mathbf{T}. \tag{27}$$

$\mathbf{A}$ is invertible, so

$$D_t^\alpha \Phi_N(t) = t^{-\alpha}\mathbf{A}\mathbf{D}_1\mathbf{A}^{-1}\mathbf{A}\mathbf{T} = t^{-\alpha}\mathbf{A}\mathbf{D}_1\mathbf{A}^{-1}\Phi_N(t). \tag{28}$$

Hence

$$D_t^\alpha \Phi_N(t) = \mathbf{D}_\alpha \Phi_N(t), \tag{29}$$

where

$$\mathbf{D}_\alpha = t^{-\alpha}\mathbf{A}\mathbf{D}_1\mathbf{A}^{-1}.$$

IV. The Fractional Variational Problem

In this section, we give a numerical technique for obtain the extremal values of functionals of the general form

$$\begin{aligned} J[y] = \int_0^1 F(t, y(t), D^\beta y(t), D^\alpha y(t))dt, \\ n-1 < \alpha \leq n, \quad 0 \leq \beta \leq \alpha, \end{aligned} \tag{30}$$

with the boundary conditions

$$y^{(j)}(0) = \kappa_j, \quad y^{(j)}(1) = \eta_j, \qquad j = 0, 1, \cdots, n-1. \tag{31}$$

Here F is a linear or nonlinear function. To develop the formulation for the general form(30), we follow the following steps:

1) We use Eqs.(8), (23), (24) and (25) to approximate the function $y_m(t)$ as

$$y_m(t) = \mathbf{C}^T\Phi_N(x), \tag{32}$$

where $\mathbf{C}$ is $(N+1)$ unknown vector as $\mathbf{C} = [C_1, C_2, \cdots, C_{N+1}]^T$ and should be found.

2) Now, using Eqs.(9) and (10) we can obtain $\mathbf{C}$ and Φ_N.

3) Substitute Eq.(32) in Eq.(30), then the general form of Eq.(30) is transformed to the following approximated form

$$J[y_m] = \int_0^1 F(t, y_m(t), D^\beta y_m(t), D^\alpha y_m(t))dt. \tag{33}$$

4) Approximate the boundary conditions, as

$$\begin{aligned} y^{(j)}(0) - \kappa_j &\cong y_m^{(j)}(0) - \kappa_j = 0, \\ y^{(j)}(1) - \eta_j &\cong y_m^{(j)}(1) - \eta_j = 0, \\ j &= 0, 1, \cdots, n-1. \end{aligned} \tag{34}$$

where

$$y^{(j)}(t) \cong y_m(t) = \mathbf{C}^T\Phi_N(x), \quad j = 0, 1, \cdots, n-1. \tag{35}$$

Then, let

$$\begin{aligned} G^T =&[y_m(0) - \kappa_0, y_m^{(1)}(0) - \kappa_1, \\ &\cdots, y_m^{(n-1)} - \kappa_{n-1}, y_m(1) - \eta_0, \\ &\cdots, y_m^{(n-1)}(1) - \eta_{n-1}], \end{aligned} \tag{36}$$

where G is a $2n \times 1$ vector. Consider

$$\begin{aligned} &J^*[a_0, a_1, \cdots, a_m, \mu_1, \mu_2, \cdots, \mu_{2n}] \\ &= J[a_0, a_1, \cdots, a_m] + G^T\mu, \end{aligned} \tag{37}$$

where

$$\mu = [\mu_1, \mu_2, \cdots, \mu_{2n}]^T, \tag{38}$$

is the unknown Lagrange multiplier vector.

5) The necessary conditions for the extremum of functions of the general form Eq.(30) are

$$\begin{aligned} \frac{\partial J^*}{\partial a_i} = 0, \quad i = 0, 1, \cdots, m, \\ \frac{\partial J^*}{\partial \mu_i} = 0, \quad i = 1, 2, \cdots, 2n, \end{aligned} \tag{39}$$

which give a system of $m+2n+1$ algebraic equations with $m+2n+1$ unknowns.

6) Solve the resulting algebraic system by Newton's iterative method, to obtain the unknown coefficient $a_0, a_1, \cdots, a_m$, then the approximation of the function y which gives the extremes of Eq.(30) is

$$y_m(t) = \mathbf{C}^T.\Phi_N(x) \tag{40}$$

V. Numerical examples

In this section, we provide some numerical results to test the new method. The computations were performed in Maple18 on a personal computer.

Example 1. Consider the following FVP: find the extremum of the fractional [20]

$$J[y] = \int_0^1 \left(D^{0.5}y(t) - \frac{2}{\Gamma(2.5)}t^{1.5} \right)^2 dt, \quad (41)$$

under the following boundary conditions

$$y(0) = 0, \quad y(1) = 1. \quad (42)$$

The exact solution of this problem is $y(t) = t^2$.

Here, we solved this problem using the method proposed in Sec.IV. When solving this problem, we achieve the approximate solution of $y(t)$ at $N = 4$. In [20], the authors introduced a direct numerical solution of this problem by using the finite difference method together with the Euler-like direct method (E-LDM) and did not achieve any accurate approximations of the function $y(t)$ unless a large number of N (N is increased up to 30) is used. In Table I, we present a comparison between the maximum absolute errors (MAEs) of the function $y(t)$ achieved here with those achieved in [20]. Finally, Fig.1 plots the absolute error function (AEF) of the function $y(t)$ at $N = 4$ for obtaining the level of accuracy of the new technique.

TABLE I
Comparison between our method with the E-LDM in [20] for Example 1

E-LDM in [20]		Our method	
N	MAEs	N	MAEs
5	0.0264	4	2.816×10^{-4}
10	0.0158	-	-
30	0.0065	-	-

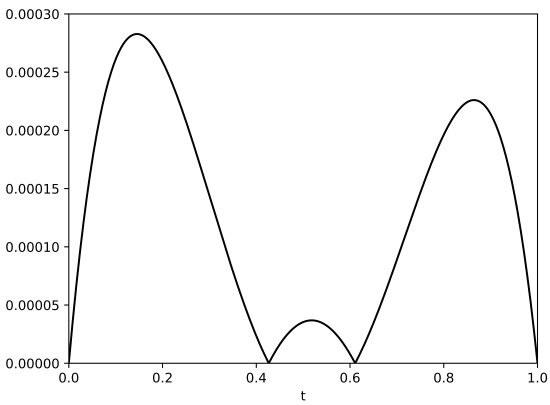

Fig. 1. AEF of $y(t)$ with $N = 4$ for Example 1

Example 2. Consider the following FVP: find the extremum of the functional [21]

$$J[y] = \frac{1}{2}\int_0^1 \left(D^{\alpha}y(t) - \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}t^{\beta-\alpha} \right)^2 dt, \quad (43)$$

under the following boundary conditions

$$y(0) = 0, \quad y(1) = 1. \quad (44)$$

In this case the exact solution is $y(t) = t^{\beta}$.

In [21], the authors consider this problem and applied the Jacobi polynomials for getting its optimal solution. In Table II, we compare the MAEs achieved using our numerical approach with other method in [21] for $\alpha = 0.39, 0.59, 0.79$ and $\beta = 3$ at $N = 4$. In addition, Fig.2 plots the AEF of the function $y(t)$ for $\alpha = 0.59$ and $\beta = 3$ at $N = 4$. At the end, in Table III, we introduce the absolute errors (AEs) achieved using our numerical approach with $N = 4$ and different choices of α and $\beta = 0.8$.

TABLE II
Comparison between our method with the other method in [21] for Example 2

α	$MAES^{[21]}$	$MAEs^{our\ method}$
0.39	1.31×10^{-4}	7.64×10^{-5}
0.59	8.82×10^{-5}	8.84×10^{-5}
0.79	4.34×10^{-5}	5.86×10^{-5}

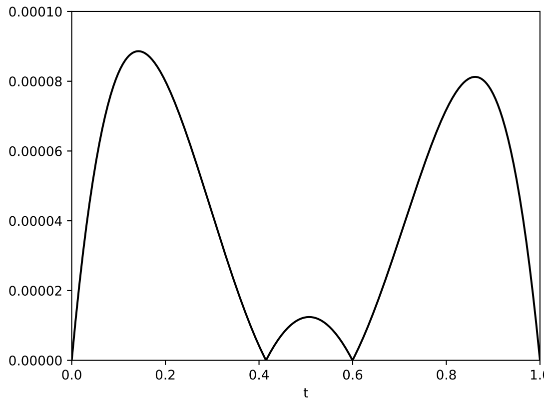

Fig. 2. AEF of $y(t)$ for $\alpha = 0.59$ and $\beta = 3$ at $N = 4$ for Example 2

TABLE III
AEs of $y(t)$ with $N = 4$ and different choices of α and $\beta = 0.8$ for Example 2

t	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
0.0	5.00×10^{-11}	2.00×10^{-11}	1.00×10^{-11}
0.1	4.78×10^{-3}	1.95×10^{-3}	1.25×10^{-3}
0.2	2.92×10^{-3}	5.91×10^{-3}	6.98×10^{-3}
0.3	3.77×10^{-3}	5.68×10^{-3}	6.78×10^{-3}
0.4	8.11×10^{-4}	1.51×10^{-3}	2.33×10^{-3}
0.5	2.34×10^{-3}	2.35×10^{-3}	2.02×10^{-3}
0.6	3.28×10^{-3}	3.19×10^{-3}	3.48×10^{-3}
0.7	1.31×10^{-3}	4.63×10^{-4}	1.31×10^{-3}
0.8	2.36×10^{-3}	4.13×10^{-3}	2.97×10^{-3}
0.9	4.54×10^{-3}	6.46×10^{-3}	5.49×10^{-3}
1.0	6.00×10^{-9}	1.40×10^{-8}	1.00×10^{-9}

Example 3. Consider the following FVP: find the extremum of the functional [12]

$$J[y] = \int_0^1 \left(D^{\alpha}y(t) - \frac{\Gamma(2\alpha+3)}{\alpha+3}t^{\alpha+2} - \Gamma(\alpha+2)t \right)^2 dt, \quad (45)$$

subjected to

$$y(0) = 1, \quad y(1) = 3. \quad (46)$$

with exact solution $y(t) = t^{2\alpha+2} + t^{\alpha+1} + 1$.

In [12], the authors using shifted Legendre orthonormal polynomials to getting its optimal solution. In Table IV, we compare the MAEs achieved using our approach at different values of α with other method in [12]. Obviously, from Fig 3, we can deduce that a good approximation of the function $y(t)$ may be achieved by using the Chebyshev cardinal functions.

TABLE IV
COMPARISON BETWEEN OUR METHOD WITH THE OTHER METHOD IN [12] FOR EXAMPLE 3

α	$MAEs^{[12]}$	$MAEs^{our\ method}$
0.30	3.998×10^{-3}	2.983×10^{-3}
0.60	2.874×10^{-3}	1.830×10^{-3}
0.90	7.767×10^{-4}	4.864×10^{-4}

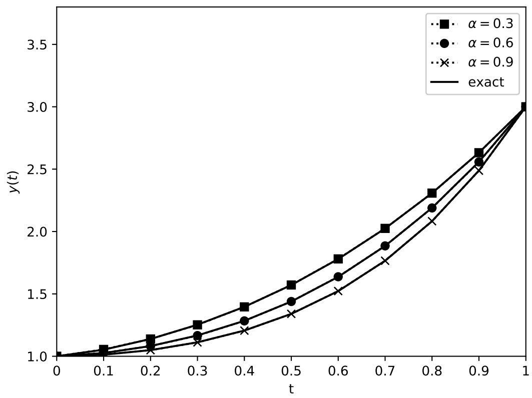

Fig. 3. Comparison of $y(t)$ for $\alpha = 0.30, 0.60, 0.90$ with exact solution at $N = 4$ for Example 3

VI. CONCLUSION

In this paper, we studied an efficient and simple method to solve a class of fractional variational problems. Utilizing the method based on Chebyshev cardinal functions, we derived operational matrix of the fractional integration. We approximately solved the above-mentioned problems by minimizing the integral on the linear combination set of some basis functions. The Chebyshev cardinal functions were selected to make them linearly independent and satisfy the homogeneous initial or boundary conditions. The results show that this new method can solve this kind of problems effectively.

ACKNOWLEDGMENT

The authors would like to thank the referees for careful reading and valuable suggestions.

REFERENCES

[1] Agrawal O P. “A general finite element formulation for fractional variational problems”. *Journal of Mathematical Analysis and Applications*, 2008, 337(1):1-12.

[2] Agrawal O P. “Formulation of Euler-Lagrange equations for fractional variational problems”. *Journal of Mathematical Analysis and Applications*, 2002, 272(1): 368-379.

[3] Zhang J, Yin L, Zhou C. “Fractional Herglotz variational problems with Atangana-Baleanu fractional derivatives”. *Journal of inequalities and applications*, 2018, 2018(1): 1-16.

[4] Zhang J, Ma X, Li L. “Optimality conditions for fractional variational problems with Caputo-Fabrizio fractional derivatives”. *Advances in Difference Equations*, 2017, 2017(1): 357.

[5] Zhang J, Wang G, Zhi X, et al. “Generalized Euler-Lagrange Equations for Fuzzy Fractional Variational Problems under gH-Atangana-Baleanu Differentiability”. *Journal of Function Spaces*, 2018, 2018:1-15.

[6] Singh H, Pandey R K, Srivastava H M. “Solving Non-Linear Fractional Variational Problems Using Jacobi Polynomials”. *Mathematics*, 2019, 7(3): 224.

[7] Dabiri A, Butcher E A. “Numerical Solution of Multi-Order Fractional Differential Equations with Multiple Delays via Spectral Collocation Methods”. *Applied Mathematical Modelling*, 2017, 56:424-448.

[8] Binti Ismail N F, Phang C. “Numerical Solution for A Class of Fractional Variational Problem via Second Order B-Spline Function”. *Journal of the Indonesian Mathematical Society*, 2019, 1(1): 171-182.

[9] Jianhua Hou, Changqing Yang, and Xiaoguang Lv, “Jacobi Collocation Methods for Solving the Fractional Bagley-Torvik Equation”, *IAENG International Journal of Applied Mathematics*, vol. 50, no.1, pp114-120, 2020.

[10] Linjun Wang, Yan Wu, Yixin Ren, and Xumei Chen, “Two Analytical Methods for Fractional Partial Differential Equations with Proportional Delay,” *IAENG International Journal of Applied Mathematics*, vol. 49, no.1, pp37-44, 2019

[11] Bin Zheng, “Some New Gronwall-Bellman Type And Volterra-Fredholm Type Fractional Integral Inequalities And Their Applications in Fractional Differential Equations”, *IAENG International Journal of Applied Mathematics*, vol. 48, no.3, pp288-296, 2018

[12] Ezz-Eldien S S, Hafez R M, Bhrawy A H, et al. “New numerical approach for fractional variational problems using shifted Legendre orthonormal polynomials”. *Journal of Optimization Theory and Applications*, 2017, 174(1): 295-320.

[13] Jahanshahi S, Torres D F M. “A simple accurate method for solving fractional variational and optimal control problems”. *Journal of Optimization Theory and Applications*, 2017, 174(1): 156-175.

[14] Sabouri J, Effati S, Pakdaman M. “A neural network approach for solving a class of fractional optimal control problems”. *Neural Processing Letters*, 2017, 45(1): 59-74.

[15] Maleki M, Hashim I, Abbasbandy S, et al. “Direct solution of a type of constrained fractional variational problems via an adaptive pseudospectral method”. *Journal of Computational and Applied Mathematics*, 2015, 283: 41-57.

[16] Heydari M H, Atangana A, Avazzadeh Z, et al. “Numerical treatment of the strongly coupled nonlinear fractal-fractional Schrödinger equations through the shifted Chebyshev cardinal functions”. *Alexandria Engineering Journal*, 2020.

[17] Heydari M H. “Chebyshev cardinal functions for a new class of nonlinear optimal control problems generated by Atangana-Baleanu-Caputo variable-order fractional derivative”. *Chaos, Solitons and Fractals*, 2020, 130: 109401.

[18] Heydari M H, Mahmoudi M R, Shakiba A, et al. “Chebyshev cardinal wavelets and their application in solving nonlinear stochastic differential equations with fractional Brownian motion”. *Communications in Nonlinear Science and Numerical Simulation*, 2018, 64: 98-121.

[19] Irandoust-Pakchin S, Abdi-Mazraeh S, Khani A. “Numerical Solution for a Variable-Order Fractional Nonlinear Cable Equation via Chebyshev Cardinal Functions”. *Computational Mathematics and Mathematical Physics*, 2017, 57(12):2047-2056.

[20] Pooseh S, Almeida R, Torres D F M. “Discrete Direct Methods in the Fractional Calculus of Variations”. *Computers & Mathematics with Applications*, 2012, 66(5):668-676.

[21] Dehghan M, Hamedi E A, Khosravian-Arab H. “A numerical scheme for the solution of a class of fractional variational and optimal control problems using the modified Jacobi polynomials”. *Journal of Vibration and Control*, 2016, 22(6): 1547-1559.

[22] Dumitru B, Kai D, Enrico S. “Fractional calculus: models and numerical methods”. *World Scientific*, 2012.