Relationship between Segment Edges and Thresholds on Segmentation Generated by Minimum Spanning Trees

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Abstract—**An image can be viewed as a connected graph. The minimum spanning tree of the connected graph can be built and not always unique. If the edge whose weight is greater than a threshold of minimum spanning tree is removed, it will form some of the connected components. One of the connected components will form a segment of an image. The image segmentation generated by different minimum spanning tree is unique. The purpose of this study is to determine the relationship between segment edges and thresholds on segmentation generated by minimum spanning trees. This research provides several results. A segment corresponding to a threshold of β is a composite of several segments corresponding to a threshold of α (smaller than β). So the boundary of a segment corresponding to the threshold of β is the pieces of the boundary curve of the smaller segments corresponding to a threshold of α (smaller than β). Let G_D, G_T, G_S be the set of segments formed after the sides whose weight are greater than β are discarded from grid graph, triangular graph, and super grid graph, respectively. A segment in G_T is a union of several segments of G_D. So that the boundary of a segment on G_T is the boundary pieces of several segments of its forming. A segment in G_S is a union of several segments of G_T. So that the boundary of a segment on G_S is the boundary pieces of several segments of its forming.**

Keywords: Boundary, connected component, minimum spanning tree, segmentation

1 Introduction

An image can be viewed as a weighted undirected grid graph, where the pixels are the points on the graph, and the difference from the color intensity value of two adjacent pixels is the weight of the side connecting the two pixels [1]. Some ways of presenting images in the graph are among others: grid graph, triangular grid graph, and super grid graph [2] as shown Figure 1. Grid graph is a graph whose sides are the connecting side of each pixel of digital images with four pixels around them: pixels on top, bottom, left, and right. The triangular grid graph is a graph whose sides are the connecting side of each pixel of digital images with six pixels around them, namely: upper left pixel, bottom right, up, down, left, and right. The super grid graph is a graph whose sides are the connecting side of each pixel of the image with eight or all of the surrounding pixels. Multiscale graph-based segmentation (MGS) algorithms use a popular graph-cut approach that presents digital images as grid graphs [3].

Segmentation is widely used in computer vision [4][5][6][7][8][9]. Image segmentation is done in various approaches such as developing area approach, boundary approach and graph approach. One approach is the method of developing regions meanshift. This method uses a series of vectors to the point of convergence [10]. The edge functions in Matlab are commonly used to develop the boundary approach area [11][12].

Many algorithms in stereo vision use image segmentation to be part of the steps [13]. Multiscale graph-based segmentation (MGS) algorithms use a popular graph-cut approach that presents digital images as grid graphs [3]. In fact, almost all stereo vision uses segmentation as part of its method, among others: belief propagation methods [14], [15], [16], [17]. Methods that rely on segment boundaries and metric similarities [18], [19] also use segmentation. Optimization methods with global constraints [20], [21] also use image segmentation in their steps.

All segmentation steps previously mentioned still have weaknesses. The selection of threshold or bandwidth greatly influences the results of this segmentation. An example is the shifting method. This method still has weaknesses, namely: the wider segment boundary formed by the addition of the threshold value is not always a combination of the segment boundary element of the smaller threshold [22]. This weakness is due to the fact that the proposed method can not exclude outlier data. To improve it, Peter [23] uses the minimum spanning tree approach of the graph to replace the square in locally matching. This method is called labeling in the graph [24].

One of the approaches in image segmentation is the use

*Manuscript received March 21, 2019; revised January 3, 2020.
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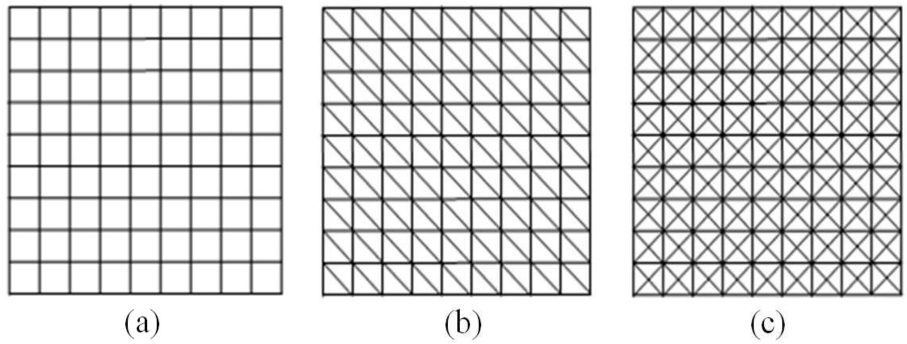

Figure 1: The shape of the Graph of the Image: (a) Grid Graph, (b) Triangular Grid Graph, (c) Super Grid Graph.

of a minimum spanning tree. The pixels of the image are seen as a points, while the side/edge is seen as the difference from the color intensity value of two adjacent pixels. Furthermore, the side-weight value of the minimum spanning tree that is greater than the threshold will be discarded [23], so that sub-tree will be formed, where one sub tree is seen as one segment in the image. Searching for isomorphic sub-trees [25] will help us in terms of looking for some of the same segments.

The spanning tree of a graph is not unique. This can be shown using the Kruskal algorithm. With the minimum spanning tree method, we can generate segmentation of an image.

An increase in a threshold value will increase the extent of each segments of the image. Is a segment, which corresponds to a threshold value β, the union of several segments corresponding to a smaller threshold value than β? This is the problem that will be investigated in this paper.

This paper consists of five sections. The first section is an introduction that provides the background and issues discussed in this paper. The second section discusses the definition of trees and tree properties required in this paper. The third section contains the minimum spanning tree. The fourth section is results and discussion. This section is the answer to the problem stated in the first section. The different aspects of the theorem about the minimum spanning tree and segmentation is the beginning of this section. The discussion concludes with the statement that the boundary of a segment corresponding to the threshold of β is the pieces of the boundary curve of the smaller segments corresponding to a threshold smaller than β. This paper will end with conclusions.

2 Tree

A tree is a connected graph that does not contain a circle. We will notice the important properties of the tree [26], [27] that are useful in this paper. The first property of the tree to be discussed is the relation between the number of points with the number of sides of a tree. Let T be a graph with n points and m sides. T is a tree if and only if T is connected and $n = m + 1$.

Furthermore, a simple path connecting the two points will be the basis of the next discussion which says that the addition of one side to a tree will form a circle. This will be stated in the following properties. The graph T is a tree if and only if every two different points are connected by exactly one simple path. If two points, which are not neighbors, of a tree are connected then they will form exactly one circle. This is easily proven. Let u and v be two non-neighboring points in the tree. There is exactly one path P from point u to point v. So $\langle \mathrm{P}, \{v, u\}\rangle$ is a simple and closed path. Hence, a simple path $\langle P, \{v, u\}\rangle$ forms exactly one circle.

The last properties states that each connected graph G has a spanning tree. This properties guarantees the existence of a spanning tree. This means that we can always build a spanning tree from the graph if the graph is connected. The sum of the spanning trees of graph G with n points is $\kappa(G) = det(J + PP^t)/n^2$, where J is a square matrix whose all elements are one and P is an adjacency matrix. The properties says that the spanning tree of the graph is not unique.

3 Minimum Spanning Tree

This section begins with a discussion of the Kruskal algorithm. After that the properties of the two minimum spanning trees will be discussed.

Kruskal Algorithm

Input : The undirected graph G, with the weight of $w : E(G) \to R$.

Output : Spanning tree with minimum weight of G

1. Sort the sides such that $w(e_1) \leq w(e_2) \leq ... \leq w(e_n)$.
2. Set $T := (V(G), \emptyset)$.
3. For $i := 1$ to m do:
   If $T + e_i$ does not contain the circle then

set $T := T + e_i$.

The Kruskal algorithm produces a minimum tree that spans simple connected graphs. This statement can be proved by two parts [28], [29], [30]. But we will present almost different evidence in proving the following theorem.

Theorem 3.1.[31] *The Kruskal algorithm produces a minimum spanning tree of connected weighted graphs.*
Proof: Suppose G is a connected weighted graph, with n points, and T is the sub graph produced by the Kruskal Algorithm. Due to the addition of the e_i side of the tree, ie: $T := T + e_i$ in step 3 of the algorithm, which does not contain a circle, then T is the spanning tree of graph G. According to the properties that if two points (which are not neighbors) of a tree are connected then they will form exactly one circle, the sides of the graph can be written as

$$E(T) = \{e_1, e_2, ..., e_{n-1}\},$$

where $w(e_1) \leq w(e_2) \leq ... \leq w(e_{n-1})$. So the weight of the T tree is

$$w(T) = \sum_{i=1}^{n-1} w(e_i).$$

Next we will show that T is the minimum spanning tree of G. Suppose that T is not the minimum spanning tree of G. Thus, from among all the minimum spanning trees of G, there is a minimum spanning tree, called H, which is most similar to T, ie: the number of common sides between H and T is more than that of other minimum spanning trees and T. Since the minimum spanning tree H and T must not be identical, T has at least one side of T which is not the side of H. Suppose e_j, for a $j = 1, 2, ..., n-1$, is the first side of T which is not the side of H. Suppose $H_1 = H + e_j$, then, according to Lemma 2.3 , H_1 has exactly one circle, called C. Since T has no circle, there is a side e_0 in C that is not in T. Since e_0 is the side in the circle C, then $T_1 = H_1 - e_0$ is the spanning tree and

$$w(T_1) = w(H) + w(e_j) - w(e_0).$$

Since H is the minimum spanning tree, then $w(H) \leq w(T_1)$. Consequently $w(e_0) \leq w(e_j)$. According to the Kruskal Algorithm, the e_j side is a next side, with minimum weights, such that the sub graph

$$\langle\{e_1, e_2, ..., e_{j-1}\}\rangle \cup \{e_j\}$$

does not contain the circle. But $\langle\{e_1, e_2, ..., e_{j-1}, e_0\}\rangle$ is a sub graph of H that does not contain the circle. This means $w(e_j) \leq w(e_0)$. Thus $w(T_1) = w(H)$. Therefore, T_1 is a minimum spanning tree of G. But T_1 and T have more common sides sides than H and T. This contradicts the assumption that H and T have more number equal sides than other minimum spanning trees and T. So T is the minimum spanning tree of G. **Q.E.D**

According to Mayr [32], every minimum spanning tree can be generated using the Kruskal Algorithm. Thus all minimum spanning trees can be viewed as the result of the Kruskal Algorithm.

We will start the theorem which discusses the singularity of the circle inside the tree if the two non-adjacent points are connected. This circle has very interesting properties.

Theorem 3.2. [33] *Let $G(V, E)$ a weighted graph and S, T be the minimum spanning tree of G. If $e \in E(S) \setminus E(T)$, then $T+e$ contains exactly one circle C such that C contain side $e \in E(T) \setminus E(S)$ where $w(e) = w(e), E(C - e) \subseteq E(T), E(C - e) \subseteq E(S)$. For every $e_i \in C$ apply $w(e_i) \leq w(e) = w(e)$.*
Proof: Let $e_1, e_2, ..., e_{n1}$ be the sides of S, where $w(e_1) \leq w(e_2) \leq ... \leq w(e_{n1})$. Assume that the first member $e_j = e \in E(S) \subseteq E(T)$ is a side of S that is not a side of T. Since T is the minimum spanning tree of G then $T_1 = T + e$ contains exactly one circle C. Thus $E(C - e) \subseteq E(T)$. Since S is also the minimum spanning tree of G that does not contain a circle, then circle C contains the $e \in E(T) \subseteq E(S)$ with the greatest weight. Circle C is unique and $E(C - e) \subseteq E(S)$. Note $S_1 = T_1 - e$ is the spanning tree of G and $w(S_1) = w(T) + w(e) - w(e)$. Since T is the minimum spanning tree of G then $w(S_1) - w(T) \geq 0$ and $w(e) \leq w(e)$. The sequence of sides starting with the smallest side weights $e_1, e_2, ..., e_{j-1}, e_j$, where $e_j = e$, are the sides of S and $e_1, e_2, ..., e_{j-1}, e$ is a sub-graph of T which does not contain a circle . So by applying the Kruskal Algorithm to S, we get $w(e) = w(e_j) < w(e)$. Thus $w(e) = w(e)$. Since e' is the edge with the greatest weight on the circle C then for every $e_i \in C$ we get $w(e_i) \leq w(e) = w(e)$. In the same way, we can do this for each next member of $E(S) \subseteq E(T)$. **Q.E.D**

Corollary 3.3. [33] *Let S and T be the different minimum spanning trees of graph G. Let C be a simple circle containing sides e_s and e_t on $S + T$ such that $e_s \in E(S) \subseteq E(T)$ and $e_t \in E(T) \subseteq E(S)$. Then every $e_i \in E(C)$ apply $w(e_i) \leq w(e_s) = w(e_t)$.*
Proof: Theorem 3.2 is used for $S + e_t$ and for $T + e_s$. **Q.E.D**

Corollary 3.4. [33] *Two different minimum spanning trees from the weighted graph G(V, E) are only distinguished by equal-weighted sides.*
Proof: Let S, T be the different spanning tree minimum from a weighted graph $G(V, E)$. Assume $e \in E(S) \subseteq E(T)$, then according to Corollary 3.3 there is $e \in E(T) \subseteq E(S)$, where $w(e) = w(e')$. **Q.E.D**

4 Results and Discussion

Let S be the minimum spanning tree of a graph G and $\alpha \in R$. The **connected components of S corresponding to α** are the set of all sub-trees of S after sides whose weights are greater than α are removed. The connected components of S corresponding to α are denoted by $\Re(G : S, \alpha)$. If $\Re(G : S, \alpha) = \{S_1, S_2, ..., S_p\}$ and $\Re(G : T, \alpha) = \{T_1, T_2, ..., T_q\}$ then it is easy to show that $\cup_{i=1}^{p} V(S_i) = \cup_{j=1}^{q} V(T_j) = V(G)$.

Lemma 4.1. *Let G(V, E) a weighted graph and S, T be the minimum spanning tree of G. Let* $\Re(G : S, \alpha) = \{S_1, S_2, ..., S_p\}$ *and* $\Re(G : T, \alpha) = \{T_1, T_2, ..., T_q\}$. *If* $V(S_i) \cap V(T_j) \neq \emptyset$ *then* $S_i + T_j$ *is connected. If* $V(S_i) \cap V(T_j) \neq \emptyset$ *and* $S_i + T_j$ *do not contain the circle then* $S_i = T_j$.
Proof: Let $V(S_i) \cap V(T_j) \in \emptyset$ and $x_0 \in V(S_i) \cap V(T_j)$. Let $x_1 \in V(S_i)$ and $x_2 \in V(T_j)$. Since S_i is connected then there is a walk from x_1 to x_0, and T_i is connected then there is a walk from x_0 to x_2. So there is a path from x_1 to x_2 passing x_0. So $S_i + T_j$ is connected. Suppose that $V(S_i) \cap V(T_j) \neq \emptyset$ and $S_i + T_j$ do not contain a circle then according to the Kruskal Algorithm $S_i + T_j$ is the sub-tree of S, and also the sub-tree of T. Since all the weights of the sides are smaller than α then $S_i + T_j$ is one component in S and also one component in T. Thus $S_i + T_j = S_i = T_j$. **Q.E.D**

Lemma 4.2. *Let G(V, E) a weighted graph and S, T be the minimum spanning tree of G. Let* $\Re(G : S, \alpha) = \{S_1, S_2, ..., S_p\}$ *and* $\Re(G : T, \alpha) = \{T_1, T_2, ..., T_q\}$. *If there is* $e_s \in E(S_i) \subseteq E(T_j)$ *then* $S_i + T_j$ *contains a circle* C *and there is* $e_t \in E(T_j) \subseteq E(S_i)$ *also side on* C *such that for each* $e_k \in E(C)$ *apply* $w(e_k) \leq w(e_s) = w(e_s)$, *and* $V(C) \subseteq V(S_i)$, $V(C) \subseteq V(T_j)$.
Proof: Suppose $e_s \in E(S_i) \subseteq E(T_j)$. Since $w(e_s) < \alpha$ and according to the Kruskal algorithm, $T_j + e_s$ contains a circle C. According to Theorem 3.2, there is $e_k \in E(C)$ and for each e_k $E(C)$ apply $w(e_k) \leq w(e_s) = w(es) < \alpha$. So $E(T_j - e_t) = E(S_i - e_s)$ and $V(T_j - e_t) = V(S_i - e_s)$. Since $V(C-e) = V(C-e) = V(C)$, then $V(e) \subseteq V(C) = V(C-e) \subseteq V(T_j)$ and $V(e) \subseteq V(C) = V(C-e) \subseteq V(S_i)$. So $V(S_i) = V(T_j)$. **Q.E.D**

Theorem 4.3. *Let* $G(V, E)$ *be a weighted connected graph. Let* $\alpha \in R$, *and S, T be the minimum spanning tree of graph G. Let* $\Re(G : S, \alpha) = \{S_1, S_2, ..., S_p\}$ *and* $\Re(G : T, \alpha) = \{T_1, T_2, ..., T_q\}$. *Thus* $p = q$, *and if* $V(S_i) \cap V(T_j) \neq \emptyset$ *then* $V(S_i) = V(T_j)$.
Proof: We will first prove the statement that if $V(S_i) \cap V(T_j) \neq \emptyset$ then $V(S_i) = V(T_j)$. In the case of $E(S_i) = E(T_j)$, $V(S_i) = V(T_j)$ is proven. So we only prove in the case of $E(S_i) \subseteq E(T_j)$. No less generality, since $V(S_i) \cap V(T_j) \neq \emptyset$ then suppose there is only one $e \in E(S_i) \subseteq E(T_j)$. According to Lemma 4.2, $V(C) \subseteq V(S_i)$ and $V(C) \subseteq V(Tj)$. So $V(S_i) = V(T_j)$.
Furthermore, since

$$V(S) = \cup_{i=1}^{q} V(S_i) = \cup_{j=1}^{p} V(T_j) = V(T)$$

is finite set. For $i = 1, 2, ..., q$, suppose the S_i component is connected from the tree S. Select $p_i \in S_i$. Since $V(S) = V(T)$, then there is a connected component of T which contains $p_i \in T_j$ for $j = 1, 2, ..., p$. Since $V(S_i) \in V(T_j) \neq \emptyset$ then $V(S_i) = V(T_j)$. So we get $q \leq p$. In the same way for $j = 1, 2, ..., p$, take the T_j of the connected component of the S. Select $p_j \in T_j$. Since $V(S) = V(T)$, then there is a connected component of S which contains $p_j \in S_i$ for $i = 1, 2, ..., q$. Since $V(S_i) \cap V(T_j) \neq \emptyset$ then $V(S_i) = V(T_j)$. Thus we obtained $p \leq q$. Hence $p = q$. **Q.E.D**

Image can be viewed as a connected graph. Spanning tree of a connected graph can be built and not always unique. So the minimum spanning tree of the connected graph can be built and not always unique. If the edge whose weight is greater than a threshold of minimum spanning tree is removed, it will form some of the connected components. One of the connected components will form a segment of an image. A collection of all these segments will form the segmentation of the image. Such a segmentation of an image is formed by using a minimum spanning tree for a threshold. Suppose that there are two segmentation of an image produced by two different minimum spanning trees for the same threshold value. Then the two segmentations are the same.

The following theorem says that the points of a sub-tree corresponding to a threshold of β are the union of the points of the sub-trees for a threshold smaller than β.

Theorem 4.4. *Let* $\alpha, \beta \in R$, *and* $\alpha < \beta$. *Let* $\Re(G : S, \alpha) = \{S_1, S_2, ..., S_p\}$ *and* $\Re(G : T, \beta) = \{T_1, T_2, ..., T_q\}$. *For each* $T_i \in \Re(G : T, \beta), i = 1, 2, ..., q$, *there are* $\{S_{i,1}, S_{i,2}, ..., S_{i,n_i}\} \subseteq \Re(G : S, \alpha)$ *such that* $V(T_i) = \cup_{j=1}^{n_i} V(S_{i,j})$.
Proof: Let $T_i \in \Re(G : T, \beta)$, for $i = 1, 2, ..., q$. According to Theorem 4.3, there is $S_i \in \Re(G : S, \alpha)$ such that $V(T_i) = V(S'_i)$. The sides whose weight is greater than α are removed from the S'_i, so we obtain the connected components labeled with $S_{i,1}, S_{i,2}, ..., S_{i,n_i}$. Thus $V(T_i) = \cup_{j=1}^{n_i} V(S_{i,j})$. Suppose that the G graph is constructed from the image. The image can be segmented using a minimum spanning tree corresponding to a threshold. A segment in the image is viewed as a sub-tree corresponding to a threshold. Then the Theorem 4.4 says that a segment corresponding to a threshold of β is a composite of several segments corresponding to a threshold of α (smaller than β). So the boundary of a segment corresponding to the threshold of β is the pieces of the boundary curve of the smaller segments corresponding to a threshold of α (smaller than β). **Q.E.D**

Let $G = (V, E)$ graph. Graph $G_1 = (V_1, E_1)$ is called **a sub-side graph** of G if $V_1 = V$ and $E_1 \subseteq E$. A segment of an image can be seen as a set of points from a sub-tree S. Suppose $s_1, s_2 \in V(S)$ are points that are in the same segment then there is a path from s_1 to s_2. Suppose x and y are two points on graph G. If there is not path from x to y then x and y are in different segments. This idea will be used to prove the Theorem 4.5 which relates to Super Grid graph in Figure 1.

Theorem 4.5. *Suppose* G_D, G_T *are a sub-side graph of Super Grid graph and* G_D *is a sub-side graph of* G_T. *Let* $\Re(G_D : D, \beta) = \{D_1, D_2, ..., D_p\}$ *and* $\Re(G_T : T, \beta) = \{T_1, T_2, ..., T_q\}$. *For each* $D_i \in \Re(G_D : D, \beta)$, *there is*

$T_j \in \Re(G_T : T, \beta)$ *such that* $V(D_i) \subseteq V(T_j)$. *So* $V(T_j) = \{d \in V(G_D) | d \in V(D_i), D_i \in \Re(G_D : D, \beta), V(D_i) \subseteq V(T_j)\}$.
Proof: Because G_D is a sub-side graph of G_T then $V(G_D) = V(G_T)$ and $E(G_D) \subseteq E(G_T)$. We prove that for every $D_i \in \Re(G_D : D, \beta)$, there is $T_j \in \Re(G_T : T, \beta)$ such that $V(D_i) \subseteq V(T_j)$. This theorem will be proven by the contradiction with fact that D, and T are the minimum spanning trees of G_D and G_T graphs, respectively. Suppose there is $D_i \in \Re(G_D : D, \beta)$ such that there is not $T_j \in \Re(G_T : T, \beta)$ such that $V(D_i) \subseteq V(T_j)$. Without reducing generality, suppose $V(D_i) = (V(D_i) \cap V(T_j)) \cup (V(D_i) \cap V(T_k))$ for $i, j = 1, 2, ..., q$ and $j \neq k$. Select $d_1, d_2 \in V(D_i)$ adjacent points such that $d_1 \in V(T_j)$, $d_2 \in V(T_k)$. Because $d_1, d_2 \in V(D_i)$, then the weight of the side (d_1, d_2) is smaller than β. Because $d_1, d_2 \in V(T)$, there is a path $(d_2, t_1), (t_1, t_2), (t_2, t_3), ..., (t_m, d_1)$ in T. Because the path $(d_2, t_1), (t_1, t_2), (t_2, t_3), ..., (t_m, d_1)$ connect two different segments, T_j and T_k segments, then, according to segment formation, there are sides $(t_i, t_{(i+1)})$ whose weight is greater than β. Paths $(d1, d2), (d_2, t_1), (t_1, t_2), (t_2, t_3), ..., (t_m, d_1)$ are closed paths. Then circle C can be formed from the part of the closed path which contains sides $(d_1, d_2), (t_i, t_{(i+1)})$. So $T - (t_i, t_{(i+1)}) + (d_1, d_2)$ is a tree that spans G_T because $E(G_D) \subseteq E(G_T)$. Because $w(d_1, d_2) < \beta$ and $w(t_i, t_{(i+1)}) > \beta$ then $w(T - (t_i, t_{(i+1)}) + (d_1, d_2)) < w(T)$. This means T is not the minimum of spanning tree, i.e. a contradiction. So the true statement is that every $D_i \in \Re(G_D : D, \beta)$, there is $T_j \in \Re(G_T : T, \beta)$ such that $V(D_i) \subseteq V(T_j)$.
We prove $T_j = \{d \in V(G_D) | d \in V(D_i), D_i \in \Re(G_D : D, \beta), V(D_i) \subseteq V(T_j)\}$. It is clear that $\{d \in V(G_D) | d \in V(D_i), D_i \in \Re(G_D : D, \beta), V(D_i) \subseteq V(T_j)\} \subseteq V(T_j)$. Let $d \in V(T_j)$. Because $V(G_D) = V(G_T)$, there is $D_i \in \Re(G_D : D, \beta)$ such that $d \in V(D_i)$. We get $V(D_i) \subseteq V(T_j)$. So $V(T_j) \subseteq \{d \in V(G_D) | d \in V(D_i), D_i \in \Re(G_D : D, \beta), V(D_i) \subseteq V(T_j)\}$. **Q.E.D**

Suppose G_D, G_T, G_S are grid graph, triangular Graph, and Super Grid graph, respectively, which are formed from an image in Figure 1. We will use Theorem 4.5 to prove Theorem 4.6.

Theorem 4.6. *Suppose* G_D*,* G_T*,* G_S *are grid graph, triangular Graph, and Super Grid graph, respectively, which are formed from an image. Let* $\Re(G_D : D, \beta) = \{D_1, D_2, ..., D_p\}$*,* $\Re(G_T : T, \beta) = \{T_1, T_2, ..., T_q\}$*, and* $\Re(G_S : S, \beta) = \{S_1, S_2, ..., S_r\}$*. So* $V(T_j) = \{d \in V(G_D) | d \in V(D_i), D_i \in \Re(G_D : D, \beta), V(D_i) \subseteq V(T_j)\}$ *and* $V(S_k) = \{d \in V(G_T) | d \in V(T_j), T_j \in \Re(G_T : T, \beta), V(T_j) \subseteq V(S_k)\}$
Proof: Because G_D is a sub-side graph of G_T then according to theorem 4.5 $V(T_j) = \{d \in V(G_D) | d \in V(D_i), D_i \in \Re(G_D : D, \beta), V(D_i) \subseteq V(T_j)\}$. Because G_T is a sub-side graph of G_S then $V(S_k) = \{d \in V(G_T) | d \in V(T_j), T_j \in \Re(G_T : T, \beta), V(T_j) \subseteq V(S_k)\}$ according to theorem 4.5. **Q.E.D**

Let G_D, G_T, G_S be the set of segments formed after the sides whose weight are greater than β are discarded from grid graph, triangular graph, and super grid graph, respectively. Toerema 4.5 says that a segment in G_T is a union of several segments of G_D. So that the boundary of a segment on G_T is the boundary pieces of several segments of its forming. A segment in G_S is a union of several segments of G_T. So that the boundary of a segment on G_S is the boundary pieces of several segments of its forming.

5 Conclusion

Image can be viewed as a connected graph. Spanning tree of a connected graph can be built and not always unique. So the minimum spanning tree of the connected graph can be built and not always unique. If the edge whose weight is greater than a threshold of minimum spanning tree is removed, it will form some of the connected components. One of the connected components will form a segment of an image. A collection of all these segments will form the segmentation of the image. Such a segmentation of an image is formed by using a minimum spanning tree for a threshold. Suppose that there are two segmentation of an image produced by two different minimum spanning trees for the same threshold value. Then the two segmentations are the same.

A segment corresponding to a threshold of β is a composite of several segments corresponding to a threshold of α (smaller than β). So the boundary of a segment corresponding to the threshold of β is the pieces of the boundary curve of the smaller segments corresponding to a threshold of α (smaller than β).

Let G_D, G_T, G_S be the set of segments formed after the sides whose weight are greater than β are discarded from grid graph, triangular graph, and super grid graph, respectively. A segment in G_T is a union of several segments of G_D. So that the boundary of a segment on G_T is the boundary pieces of several segments of its forming. A segment in G_S is a union of several segments of G_T. So that the boundary of a segment on G_S is the boundary pieces of several segments of its forming.

6 Acknowledgement

My special thanks to Dr. Argenes Siburian for editing my paper and giving several valuable suggestions. This research project was sponsored by the Nommensen HKBP University, Medan-Indonesia.

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