An Improved (G’/G) Method for Conformable Fractional Differential Equations in Mathematical Physics

Qinghua Feng*

***Abstract*—In this paper, an improved (G’/G) method is applied for seeking exact solutions of fractional differential equations in the sense of the conformable fractional derivative. The present method is a generalization of the known (G’/G) method. Based on a fractional complex transformation, certain fractional differential equation can be converted into another ordinary differential equation of integer order, and then can be solved subsequently based on the homogeneous balance principle and the sub-equation. As of applications of this method, we apply it to solve the (2+1)-dimensional space-time fractional Nizhnik-Novikov-Veselov System and the time fractional seventh-order Kaup-Kupershmidt equation. As a result, some exact solutions with new general forms are successfully found for them.**

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***Index Terms*—improved (G’/G) method; fractional differential equations; exact solutions; fractional complex transformation**

I. Introduction

Recently, fractional calculus has played an important role in many researching domains such as fluid mechanics [1], bioengineering [2], finance [3] and so on. Fractional differential equations have attracted much attention in a variety of applied sciences including physics, dynamical systems, control systems, engineering and so on. Furthermore, they are employed in social sciences such as food supplement, climate, and economics. Recently there have been an increasing attention to fractional differential equations due to their frequent appearance in wide applications in various fields. Many practical problems lead to the necessity of seeking exact solutions and numerical solutions for fractional differential equations. Many powerful and efficient methods have been proposed so far, such as the $(\frac{G'}{G})$ method [4-7], the variational iterative method [8-11], the fractional Nikiforov-Uvarov method [12], the modified Kudryashov method [13-16], the exp method [17,18], the first integral method [19-21], the sub-equation method [22-25], the coupled fractional reduced differential transform method [26], the Bernstein polynomials method [27], the residual power series method [28], the Jacobi elliptic function method [29], the finite difference method [30-32] and so on. Based on these methods, a variety of fractional differential equations have been investigated.

In this paper, based on the properties of the conformable fractional derivative [33], we apply an improved (G’/G) method to seek exact solutions for fractional differential equations, which is an extension of the known (G’/G) method [5]. The present method belongs to the categories of the sub-equation methods, and a sub-equation with more general form than that in the (G’/G) method is used here. By use of a fractional complex transformation and the properties of conformable fractional calculus, certain fractional partial differential equation can be converted into another ordinary differential equation of integer order. Then the solutions of the converted ordinary differential equation are supposed to be expressed in a polynomial involving under-determined coefficients, and the degree of the polynomial can be determined due to the homogeneous balance principle. By substituting this polynomial into the ordinary differential equation, a series of algebraic equations involving the under-determined coefficients can be derived. Solving these equations, the coefficients can be determined, and then the exact solutions can be obtained subsequently.

For the conformable fractional derivative and the conformable fractional integral, we have the following definitions and properties.

Definition 1 [33, Definition 2.1]. The conformable fractional derivative of order α is defined by

$$D_t^\alpha f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}. \tag{1}$$

Definition 2 [33, Definition 3.1]. The conformable fractional integral of order α on the interval $[0,t]$ is defined by

$$I^\alpha f(t)=\int_0^t s^{\alpha-1}f(s)ds. \tag{2}$$

Propsition 1. For the conformable fractional derivative, the following conclusions hold.

$$D_t^\alpha(t^\gamma)=\gamma t^{\gamma-\alpha},\ D_t^\alpha f(t)=t^{1-\alpha}f'(t),\ D^\alpha C=0. \tag{3}$$

$$\begin{cases} D_t^\alpha(I^\alpha f(t))=f(t),\\ I_t^\alpha(D_t^\alpha f(t))=f(t)-f(0). \end{cases} \tag{4}$$

$$\begin{cases} D_t^\alpha f[g(t)]=f'_g[g(t)]D_t^\alpha g(t),\\ D_t^\alpha[f(t)g(t)]=f(t)D^\alpha g(t)+g(t)D^\alpha f(t),\\ D_t^\alpha[af(t)+bg(t)]=aD^\alpha f(t)+bD^\alpha g(t),\\ D_t^\alpha(\frac{f}{g})(t)=\frac{g(t)D^\alpha f(t)-f(t)D^\alpha g(t)}{g^2(t)}. \end{cases} \tag{5}$$

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Q. Feng is with the School of Mathematics and Statistics, Shandong University Of Technology, Zibo, Shandong, 255049 China *e-mail: fqhua@sina.com

We note that Proposition 1 can be easily proved due to the definitions of the conformable fractional derivative and the conformable fractional integral.

The rest of this paper is organized as follows. In Section 2, we give the description of the improved (G'/G) method for seeking exact solutions of fractional differential equations. In Section 3, we apply the improved (G'/G) method to construct exact solutions for the time fractional seventh-order Kaup-Kupershmidt equation and the (2+1)-dimensional space-time fractional Nizhnik-Novikov-Veselov System and the time fractional seventh-order Kaup-Kupershmidt equation. Some conclusions are presented at the end of the paper.

II. Description of the improved (G'/G) method for fractional differential equations

In this section we give the description of the improved (G'/G) method for seeking exact solutions of fractional differential equations.

Suppose that a fractional partial differential equation, say in the independent variables $t, x_1, x_2, ..., x_n$, is given by

$$P(u_1, ...u_k,\ D_t^{\alpha}u_1, ..., D_t^{\alpha}u_k, D_{x_1}^{\beta}u_1, ..., D_{x_1}^{\beta}u_k, ..., D_{x_n}^{\gamma}u_1, ..., D_{x_n}^{\gamma}u_k, ...) = 0, \tag{6}$$

where $u_i = u_i(t, x_1, x_2, ..., x_n),\ i = 1, ..., k$ are unknown functions, P is a polynomial in u_i and their various partial derivatives including fractional derivatives.

Step 1. Execute certain fractional complex transformation

$$u_i(t, x_1, x_2, ..., x_n) = U_i(\xi), \quad \xi = \xi(t, x_1, x_2, ..., x_n), \tag{7}$$

such that Eq. (6) can be turned into the following ordinary differential equation of integer order with respect to the variable ξ:

$$\widetilde{P}(U_1, ..., U_k,\ U_1', ..., U_k', U_1'', ..., U_k'',\ ...) = 0. \tag{8}$$

Take the expressions $D_t^{\alpha}u_1$ and $D_{x_i}^{\beta}u_1$ for example. If one use the fractional complex transformation $\xi = c\frac{t^{\alpha}}{\alpha} + k\frac{x_i^{\beta}}{\beta}$. Then due to (3) and (5) in Proposition 1, one can obtain that $D_t^{\alpha}u_1 = U_1'D_t^{\alpha}\xi = cU'(\xi),\ D_{x_i}^{\beta}u_1 = U_1'D_{x_i}^{\beta}\xi = kU'(\xi)$.

Step 2. Suppose that the solution of (8) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$U_j(\xi) = \sum_{i=0}^{m_j} a_{j,i}(\frac{G'}{G})^i,\ j = 1, 2, ..., k, \tag{9}$$

where $a_{j,i},\ i = 0, 1, ..., m_j,\ j = 1, 2, ..., k$ are constants to be determined later, $a_{j,m} \neq 0$, the positive integer m_j can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (8), $G = G(\xi)$ satisfies the second order ODE [34] in the form

$$AGG'' - BGG' - C(G')^2 - EG^2 = 0, \tag{10}$$

where $A,\ B,\ C,\ E$ are real parameters.

Note that by use of Eq. (10), the order of $U_j'(\xi)$ is $m+1$, which is one more that that of $U_j(\xi)$.

Denote

$$\Delta_1 = B^2 + 4E(A - C),\ \Delta_2 = E(A - C). \tag{11}$$

By the General solutions of Eq.(10), we have tye following expressions for $(\frac{G'(\xi)}{G(\xi)})$.

When $B \neq 0,\ \Delta_1 > 0$:

$$(\frac{G'(\xi)}{G(\xi)}) = \frac{B}{2(A-C)} + \frac{\sqrt{\Delta_1}}{2(A-C)} [\frac{C_1 \sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)} + C_2 \cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}{C_1 \cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)} + C_2 \sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}], \tag{12}$$

where $C_1,\ C_2$ are arbitrary constants.

When $B \neq 0,\ \Delta_1 < 0$:

$$(\frac{G'(\xi)}{G(\xi)}) = \frac{B}{2(A-C)} + \frac{\sqrt{-\Delta_1}}{2(A-C)} [\frac{-C_1 \sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)} + C_2 \cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}{C_1 \cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)} + C_2 \sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}], \tag{13}$$

where $C_1,\ C_2$ are arbitrary constants.

When $B \neq 0,\ \Delta_1 = 0$:

$$(\frac{G'(\xi)}{G(\xi)}) = \frac{B}{2(A-C)} + \frac{C_2}{C_1 + C_2\xi}, \tag{14}$$

where $C_1,\ C_2$ are arbitrary constants.

When $B = 0,\ \Delta_2 > 0$:

$$(\frac{G'(\xi)}{G(\xi)}) = \frac{\sqrt{\Delta_2}}{(A-C)} [\frac{C_1 \sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)} + C_2 \cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}{C_1 \cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)} + C_2 \sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}], \tag{15}$$

where $C_1,\ C_2$ are arbitrary constants.

When $B = 0,\ \Delta_2 < 0$:

$$(\frac{G'(\xi)}{G(\xi)}) = \frac{\sqrt{-\Delta_2}}{(A-C)} [\frac{-C_1 \sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)} + C_2 \cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}{C_1 \cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)} + C_2 \sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}], \tag{16}$$

where $C_1,\ C_2$ are arbitrary constants.

Step 3. Substituting (9) into (8) and using (10), collecting all terms with the same order of $(\frac{G'}{G})$ together, the left-hand side of (8) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_{j,i},\ i = 0, 1, ..., m,\ j = 1, 2, ..., k$.

Step 4. Solving the equations system in Step 3, and using (12)-(16), we can construct a variety of exact solutions for Eq. (6).

Remark 1. If we take $A = 1,\ B = -\lambda, C = 0,\ E = -\mu$ in Eq. (10), then it reduces to $G'' + \lambda G' + \mu G = 0$, which is the used equation in the known (G'/G) method. So the present method is a generalization of the (G'/G) method.

III. Applications of the improved (G'/G) method

In this section, we apply the improved (G'/G) method described in Section II to seek exact solutions for some space-time and time fractional differential equations.

A. (2+1)-dimensional space-time fractional Nizhnik-Novikov-Veselov System

Consider the (2+1)-dimensional space-time fractional Nizhnik-Novikov-Veselov System [35]

$$\begin{cases} D_t^{\alpha}u + aD_x^{3\beta}u + bD_y^{3\gamma}u + cD_x^{\beta}u \\ +dD_y^{\gamma}u = 3aD_x^{\beta}(uv) + 3bD_y^{\gamma}(uw), \\ D_x^{\beta}u = D_y^{\gamma}v, \\ D_y^{\gamma}u = D_x^{\beta}w, \end{cases} ,0 < \alpha,\ \beta,\ \gamma \leq 1, \tag{17}$$

which is a fractional extension of the following (2+1)-dimensional Nizhnik-Novikov-Veselov system [36]:

$$\begin{cases} u_t + au_{xxx} + bu_{yyy} + cu_x + du_y = 3a(uv)_x + 3b(uw)_y, \\ u_x = v_y, \\ u_y = w_x. \end{cases} \tag{18}$$

In order to apply the improved (G'/G) method, we suppose $u(x,y,t) = U(\xi),\ v(x,y,t) = V(\xi),\ w(x,y,t) = W(\xi)$, where $\xi = \frac{m}{\Gamma(1+\alpha)}t^{\alpha} + \frac{k}{\Gamma(1+\beta)}x^{\beta} + \frac{l}{\Gamma(1+\gamma)}y^{\gamma} + \xi_0$, $m,\ k,\ l,\ \xi_0$ are all constants with $k,\ l,\ m \neq 0$. By use of (3) and (5), one can obtain

$$\begin{cases} D_t^{\alpha}u = D_t^{\alpha}U(\xi) = U'(\xi)D_t^{\alpha}\xi = mU'(\xi), \\ D_x^{\beta}u = D_x^{\beta}U(\xi) = U'(\xi)D_x^{\beta}\xi = kU'(\xi), \\ D_y^{\gamma}u = D_y^{\gamma}U(\xi) = U'(\xi)D_y^{\gamma}\xi = lU'(\xi), \end{cases}$$

So Eqs. (17) can be turned into the following forms

$$\begin{cases} mU' + ak^3U''' + bl^3U''' + ckU' + dlU' = \\ \quad 3ak(UV)' + 3bl(UW)', \\ kU' = lV', \\ lU' = kW'. \end{cases} \tag{19}$$

Integrating (19) once, one has

$$\begin{cases} mU + ak^3U'' + bl^3U'' + ckU + dlU \\ \quad = 3akUV + 3blUW + g_1, \\ kU = lV + g_2, \\ lU = kW + g_3, \end{cases} \tag{20}$$

where g_1, g_2, g_3 are the integration constants.

Suppose that the solutions of (20) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$\begin{cases} U(\xi) = \sum_{i=0}^{m_1} a_i(\frac{G'}{G})^i, \\ V(\xi) = \sum_{i=0}^{m_2} b_i(\frac{G'}{G})^i, \\ W(\xi) = \sum_{i=0}^{m_3} c_i(\frac{G'}{G})^i. \end{cases} \tag{21}$$

where a_i, b_i, c_i are constants, and $G = G(\xi)$ satisfies Eq. (10). Balancing the order of U'' and UV, the order of U and V, the order of U and V in (20), one can obtain $m_1 = m_2 = m_3 = 2$. So

$$\begin{cases} U(\xi) = a_0 + a_1(\frac{G'}{G})^1 + a_2(\frac{G'}{G})^2, \\ V(\xi) = b_0 + b_1(\frac{G'}{G})^1 + b_2(\frac{G'}{G})^2, \\ W(\xi) = c_0 + c_1(\frac{G'}{G})^1 + c_2(\frac{G'}{G})^2. \end{cases} \tag{22}$$

Substituting (22) into (20), using Eq. (10) and collecting all the terms with the same power of $(\frac{G'}{G})$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations, yields the following results.

Case 1:

$$a_2 = \frac{2lk(-C+A)^2}{A^2},\ a_1 = -\frac{2klB(-C+A)}{A^2},$$

$$a_0 = -\frac{2klE(-C+A)}{A^2},\ b_2 = \frac{2k^2(-C+A)^2}{A^2},$$

$$b_1 = -\frac{2k^2B(-C+A)}{A^2},\ b_0 = -\frac{2k^2E(-C+A)}{A^2},$$

$$c_2 = \frac{2l^2(-C+A)^2}{A^2},\ c_1 = -\frac{2l^2B(-C+A)}{A^2},$$

$$c_0 = -\frac{2l^2E(-C+A)}{A^2},$$

$$m = -\frac{1}{A^2}(dlA^2 + ckA^2 + 4bl^3AE + 4ak^3AE + bl^3B^2$$
$$-4ak^3CE + ak^3B^2 - 4bl^3CE),\ g_1 = g_2 = g_3 = 0.$$

Case 2:

$$a_2 = \frac{2lk(A^2 - 2CA + C^2)}{A^2},\ a_1 = -\frac{2klB(-C+A)}{A^2},$$

$$a_0 = -\frac{lk(2AE - 2CE - B^2)}{3A^2} b_2 = \frac{2k^2(A^2 - 2CA + C^2)}{A^2},$$

$$b_1 = -\frac{2k^2B(-C+A)}{A^2},\ b_0 = -\frac{k^2(2AE - 2CE - B^2)}{3A^2},$$

$$c_2 = \frac{2l^2(A^2 - 2CA + C^2)}{A^2},\ c_1 = -\frac{2l^2B(-C+A)}{A^2},$$

$$c_0 = -\frac{l^2(2AE - 2CE - B^2)}{3A^2},$$

$$m = -\frac{1}{A^2}(dlA^2 + ckA^2 - 4bl^3AE - 4ak^3AE - bl^3B^2$$
$$+4ak^3CE - ak^3B^2 + 4bl^3CE),\ g_1 = g_2 = g_3 = 0.$$

Substituting the results above into Eqs. (22) and combining with (12)-(16) we can obtain corresponding exact solutions for Eqs. (18).

From Case 1 we get the following exact solutions, where $C_1,\ C_2$ are arbitrary constants, and

$$\xi = \frac{t^{\alpha}}{\Gamma(1+\alpha)}[-\frac{1}{A^2}(dlA^2 + ckA^2 + 4bl^3AE + 4ak^3AE + bl^3B^2 - 4ak^3CE + ak^3B^2 - 4bl^3CE)] + \frac{k}{\Gamma(1+\beta)}x^{\beta} + \frac{l}{\Gamma(1+\gamma)}y^{\gamma} + \xi_0.$$

Family 1: when $B \neq 0,\ \Delta_1 > 0$:

$$u_1(x,y,t) = -\frac{2klE(-C+A)}{A^2} - \frac{2klB(-C+A)}{A^2}\{\frac{B}{2(A-C)} + \frac{\sqrt{\Delta_1}}{2(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)} + C_2\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)} + C_2\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}]\}^1 + \frac{2lk(-C+A)^2}{A^2}\{\frac{B}{2(A-C)}$$

$$+\frac{\sqrt{\Delta_1}}{2(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}]\}^2,$$

$$v_1(x,y,t)=-\frac{2k^2E(-C+A)}{A^2}-\frac{2k^2B(-C+A)}{A^2}\{\frac{B}{2(A-C)}$$

$$+\frac{\sqrt{\Delta_1}}{2(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}]\}^1$$

$$+\frac{2k^2(-C+A)^2}{A^2}\{\frac{B}{2(A-C)}+\frac{\sqrt{\Delta_1}}{2(A-C)}$$

$$[\frac{C_1\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}]\}^2,$$

$$w_1(x,y,t)=-\frac{2l^2E(-C+A)}{A^2}-\frac{2l^2B(-C+A)}{A^2}\{\frac{B}{2(A-C)}$$

$$+\frac{\sqrt{\Delta_1}}{2(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}]\}^1$$

$$+\frac{2l^2(-C+A)^2}{A^2}\{\frac{B}{2(A-C)}+\frac{\sqrt{\Delta_1}}{2(A-C)}$$

$$[\frac{C_1\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}]\}^2. \tag{23}$$

Family 2: when $B\neq 0,\ \Delta_1<0$:

$$u_2(x,y,t)=-\frac{2klE(-C+A)}{A^2}-\frac{2klB(-C+A)}{A^2}\{\frac{B}{2(A-C)}$$

$$+\frac{\sqrt{-\Delta_1}}{2(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}]\}^1$$

$$+\frac{2lk(-C+A)^2}{A^2}\{\frac{B}{2(A-C)}+\frac{\sqrt{-\Delta_1}}{2(A-C)}$$

$$[\frac{-C_1\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}]\}^2,$$

$$v_2(x,y,t)=-\frac{2k^2E(-C+A)}{A^2}-\frac{2k^2B(-C+A)}{A^2}\{\frac{B}{2(A-C)}$$

$$+\frac{\sqrt{-\Delta_1}}{2(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}]\}^1$$

$$+\frac{2k^2(-C+A)^2}{A^2}\{\frac{B}{2(A-C)}+\frac{\sqrt{-\Delta_1}}{2(A-C)}$$

$$[\frac{-C_1\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}]\}^2,$$

$$w_2(x,y,t)=-\frac{2l^2E(-C+A)}{A^2}-\frac{2l^2B(-C+A)}{A^2}\{\frac{B}{2(A-C)}$$

$$+\frac{\sqrt{-\Delta_1}}{2(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}]\}^1$$

$$+\frac{2l^2(-C+A)^2}{A^2}\{\frac{B}{2(A-C)}+\frac{\sqrt{-\Delta_1}}{2(A-C)}$$

$$[\frac{-C_1\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}]\}^2. \tag{24}$$

Family 3: when $B\neq 0,\ \Delta_1=0$:

$$u_3(x,y,t)=-\frac{2klE(-C+A)}{A^2}-\frac{2klB(-C+A)}{A^2}\{\frac{B}{2(A-C)}$$

$$+\frac{C_2}{C_1+C_2\xi}\}^1+\frac{2lk(-C+A)^2}{A^2}\{\frac{B}{2(A-C)}+\frac{C_2}{C_1+C_2\xi}\}^2,$$

$$v_3(x,y,t)=-\frac{2k^2E(-C+A)}{A^2}-\frac{2k^2B(-C+A)}{A^2}\{\frac{B}{2(A-C)}$$

$$+\frac{C_2}{C_1+C_2\xi}\}^1+\frac{2k^2(-C+A)^2}{A^2}\{\frac{B}{2(A-C)}+\frac{C_2}{C_1+C_2\xi}\}^2,$$

$$w_3(x,y,t)=-\frac{2l^2E(-C+A)}{A^2}-\frac{2l^2B(-C+A)}{A^2}\{\frac{B}{2(A-C)}$$

$$+\frac{C_2}{C_1+C_2\xi}\}^1+\frac{2l^2(-C+A)^2}{A^2}\{\frac{B}{2(A-C)}+\frac{C_2}{C_1+C_2\xi}\}^2. \tag{25}$$

Family 4: when $B=0,\ \Delta_2>0$:

$$u_4(x,y,t)=-\frac{2klE(-C+A)}{A^2}-\frac{2klB(-C+A)}{A^2}\{\frac{\sqrt{\Delta_2}}{(A-C)}$$

$$[\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}]\}^1+\frac{2lk(-C+A)^2}{A^2}$$

$$\{\frac{\sqrt{\Delta_2}}{(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}]\}^2,$$

$$v_4(x,y,t)=-\frac{2k^2E(-C+A)}{A^2}-\frac{2k^2B(-C+A)}{A^2}\{\frac{\sqrt{\Delta_2}}{(A-C)}$$

$$[\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}]\}^1+\frac{2k^2(-C+A)^2}{A^2}$$

$$\{\frac{\sqrt{\Delta_2}}{(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}]\}^2,$$

$$w_4(x,y,t)=-\frac{2l^2E(-C+A)}{A^2}-\frac{2l^2B(-C+A)}{A^2}\{\frac{\sqrt{\Delta_2}}{(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}]\}^1+\frac{2l^2(-C+A)^2}{A^2}\{\frac{\sqrt{\Delta_2}}{(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}]\}^2. \quad (26)$$

Family 5: when $B=0,\ \Delta_2<0$:

$$u_5(x,y,t)=-\frac{2klE(-C+A)}{A^2}-\frac{2klB(-C+A)}{A^2}\{\frac{\sqrt{-\Delta_2}}{(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}]\}^1+\frac{2lk(-C+A)^2}{A^2}\{\frac{\sqrt{-\Delta_2}}{(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}]\}^2,$$

$$v_5(x,y,t)=-\frac{2k^2E(-C+A)}{A^2}-\frac{2k^2B(-C+A)}{A^2}\{\frac{\sqrt{-\Delta_2}}{(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}]\}^1+\frac{2k^2(-C+A)^2}{A^2}\{\frac{\sqrt{-\Delta_2}}{(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}]\}^2,$$

$$w_5(x,y,t)=-\frac{2l^2E(-C+A)}{A^2}-\frac{2l^2B(-C+A)}{A^2}\{\frac{\sqrt{-\Delta_2}}{(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}]\}^1+\frac{2l^2(-C+A)^2}{A^2}\{\frac{\sqrt{-\Delta_2}}{(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}]\}^2. \quad (27)$$

Remark 2. If we put $A=1,\ B=-\lambda,\ C=0,\ E=-\mu$, then the solutions (23)-(25) reduce to the results established in [35, Eqs. (24)-(26)].

From Case 2 we get the following exact solutions, where $C_1,\ C_2$ are arbitrary constants, and

$$\xi=\frac{t^\alpha}{\Gamma(1+\alpha)}[-\frac{1}{A^2}(dlA^2+ckA^2-4bl^3AE-4ak^3AE-bl^3B^2+4ak^3CE-ak^3B^2+4bl^3CE)]+\frac{k}{\Gamma(1+\beta)}x^\beta+\frac{l}{\Gamma(1+\gamma)}y^\gamma+\xi_0.$$

Family 6: when $B\neq 0,\ \Delta_1>0$:

$$u_6(x,y,t)=-\frac{lk(2AE-2CE-B^2)}{3A^2}-\frac{2klB(-C+A)}{A^2}\{\frac{B}{2(A-C)}+\frac{\sqrt{\Delta_1}}{2(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}]\}^1+\frac{2lk(A^2-2CA+C^2)}{A^2}\{\frac{B}{2(A-C)}+\frac{\sqrt{\Delta_1}}{2(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}]\}^2,$$

$$v_6(x,y,t)=-\frac{k^2(2AE-2CE-B^2)}{3A^2}-\frac{2k^2B(-C+A)}{A^2}\{\frac{B}{2(A-C)}+\frac{\sqrt{\Delta_1}}{2(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}]\}^1+\frac{2k^2(A^2-2CA+C^2)}{A^2}\{\frac{B}{2(A-C)}+\frac{\sqrt{\Delta_1}}{2(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}]\}^2,$$

$$w_6(x,y,t)=-\frac{l^2(2AE-2CE-B^2)}{3A^2}-\frac{2l^2B(-C+A)}{A^2}\{\frac{B}{2(A-C)}+\frac{\sqrt{\Delta_1}}{2(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}]\}^1+\frac{2l^2(A^2-2CA+C^2)}{A^2}\{\frac{B}{2(A-C)}+\frac{\sqrt{\Delta_1}}{2(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}]\}^2. \quad (28)$$

Family 7: when $B\neq 0,\ \Delta_1<0$:

$$u_7(x,y,t)=-\frac{lk(2AE-2CE-B^2)}{3A^2}-\frac{2klB(-C+A)}{A^2}\{\frac{B}{2(A-C)}+\frac{\sqrt{-\Delta_1}}{2(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}]\}^1+\frac{2lk(A^2-2CA+C^2)}{A^2}\{\frac{B}{2(A-C)}+\frac{\sqrt{-\Delta_1}}{2(A-C)}$$

$$[\frac{-C_1\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}]\}^2,$$

$$v_7(x,y,t)=-\frac{k^2(2AE-2CE-B^2)}{3A^2}-\frac{2k^2B(-C+A)}{A^2}$$
$$\{\frac{B}{2(A-C)}+\frac{\sqrt{-\Delta_1}}{2(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}]\}^1$$
$$+\frac{2k^2(A^2-2CA+C^2)}{A^2}\{\frac{B}{2(A-C)}+\frac{\sqrt{-\Delta_1}}{2(A-C)}$$
$$[\frac{-C_1\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}]\}^2,$$

$$w_7(x,y,t)=-\frac{l^2(2AE-2CE-B^2)}{3A^2}-\frac{2l^2B(-C+A)}{A^2}$$
$$\{\frac{B}{2(A-C)}+\frac{\sqrt{-\Delta_1}}{2(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}]\}^1$$
$$+\frac{2l^2(A^2-2CA+C^2)}{A^2}\{\frac{B}{2(A-C)}+\frac{\sqrt{-\Delta_1}}{2(A-C)}$$
$$[\frac{-C_1\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}+C_2\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}]\}^2. \quad (29)$$

Family 8: when $B\neq 0,\ \Delta_1=0$:

$$u_8(x,y,t)=-\frac{lk(2AE-2CE-B^2)}{3A^2}$$
$$-\frac{2klB(-C+A)}{A^2}\{\frac{B}{2(A-C)}+\frac{C_2}{C_1+C_2\xi}\}^1$$
$$+\frac{2lk(A^2-2CA+C^2)}{A^2}\{\frac{B}{2(A-C)}+\frac{C_2}{C_1+C_2\xi}\}^2,$$

$$v_8(x,y,t)=-\frac{k^2(2AE-2CE-B^2)}{3A^2}$$
$$-\frac{2k^2B(-C+A)}{A^2}\{\frac{B}{2(A-C)}+\frac{C_2}{C_1+C_2\xi}\}^1$$
$$+\frac{2k^2(A^2-2CA+C^2)}{A^2}\{\frac{B}{2(A-C)}+\frac{C_2}{C_1+C_2\xi}\}^2,$$

$$w_8(x,y,t)=-\frac{l^2(2AE-2CE-B^2)}{3A^2}$$
$$-\frac{2l^2B(-C+A)}{A^2}\{\frac{B}{2(A-C)}+\frac{C_2}{C_1+C_2\xi}\}^1$$
$$+\frac{2l^2(A^2-2CA+C^2)}{A^2}\{\frac{B}{2(A-C)}+\frac{C_2}{C_1+C_2\xi}\}^2. \quad (30)$$

Family 9: when $B=0,\ \Delta_2>0$:

$$u_9(x,y,t)=-\frac{lk(2AE-2CE-B^2)}{3A^2}-\frac{2klB(-C+A)}{A^2}$$
$$\{\frac{\sqrt{\Delta_2}}{(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}]\}^1$$
$$+\frac{2lk(A^2-2CA+C^2)}{A^2}\{\frac{\sqrt{\Delta_2}}{(A-C)}$$
$$[\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}]\}^2,$$

$$v_9(x,y,t)=-\frac{k^2(2AE-2CE-B^2)}{3A^2}-\frac{2k^2B(-C+A)}{A^2}$$
$$\{\frac{\sqrt{\Delta_2}}{(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}]\}^1$$
$$+\frac{2k^2(A^2-2CA+C^2)}{A^2}\{\frac{\sqrt{\Delta_2}}{(A-C)}$$
$$[\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}]\}^2,$$

$$w_9(x,y,t)=-\frac{l^2(2AE-2CE-B^2)}{3A^2}-\frac{2l^2B(-C+A)}{A^2}$$
$$\{\frac{\sqrt{\Delta_2}}{(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}]\}^1$$
$$+\frac{2l^2(A^2-2CA+C^2)}{A^2}\{\frac{\sqrt{\Delta_2}}{(A-C)}$$
$$[\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}+C_2\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}]\}^2. \quad (31)$$

Family 10: when $B=0,\ \Delta_2<0$:

$$u_{10}(x,y,t)=-\frac{lk(2AE-2CE-B^2)}{3A^2}-\frac{2klB(-C+A)}{A^2}$$
$$\{\frac{\sqrt{-\Delta_2}}{(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}]\}^1$$
$$+\frac{2lk(A^2-2CA+C^2)}{A^2}\{\frac{\sqrt{-\Delta_2}}{(A-C)}$$
$$[\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}]\}^2,$$

$$v_{10}(x,y,t)=-\frac{k^2(2AE-2CE-B^2)}{3A^2}-\frac{2k^2B(-C+A)}{A^2}$$

$$\{\frac{\sqrt{-\Delta_2}}{(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}]\}^1$$
$$+\frac{2k^2(A^2-2CA+C^2)}{A^2}\{\frac{\sqrt{-\Delta_2}}{(A-C)}$$
$$[\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}]\}^2,$$

$$w_{10}(x,y,t)=-\frac{l^2(2AE-2CE-B^2)}{3A^2}-\frac{2l^2B(-C+A)}{A^2}$$
$$\{\frac{\sqrt{-\Delta_2}}{(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}]\}^1$$
$$+\frac{2l^2(A^2-2CA+C^2)}{A^2}\{\frac{\sqrt{-\Delta_2}}{(A-C)}$$
$$[\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}+C_2\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}]\}^2. \quad (32)$$

Remark 3. If we put $A=1,\ B=-\lambda,\ C=0,\ E=-\mu$, then the solutions (28)-(30) reduce to the results established in [35, Eqs. (27)-(29)]. Furthermore, letting $A=1,\ B=-\lambda,\ C=0,\ E=-\mu,\ \alpha=1,\ k=l=1$, then the solutions (23)-(25), (28)-(30) reduce to the results established in [36] despite the slight difference of the forms of constants. So the solutions (23)-(25), (28)-(30) are of more general forms than those in [35,36].

Remark 4. The solutions (26), (27), (31), (32) are new solutions for the (2+1)-dimensional space-time fractional Nizhnik-Novikov-Veselov System to our best knowledge.

B. Time fractional seventh-order Kaup-Kupershmidt equation

Now we apply the present method to solve time fractional differential equations. Consider the following time fractional seventh-order Kaup-Kupershmidt equation [5]

$$D_t^\alpha u+2016u^3u_x+630(u_x)^3+2268uu_xu_{xx}+504u^2u_{xxx}$$
$$+252u_{xx}u_{xxx}+147u_xu_{xxxx}+42uu_{xxxxx}+u_{xxxxxxx}=0. \quad (33)$$

In order to use the improved (G'/G) method to solve Eq. (33), we let $u(x,t)=U(\xi)$, where $\xi=x-\frac{c}{\Gamma(1+\alpha)}t^\alpha+\xi_0$, $m,\ k,\ l,\ \xi_0$ are all constants with $k,\ l,\ m\neq 0$. By use of (3) and (5), one can obtain $D_t^\alpha u=D_t^\alpha U(\xi)=U'(\xi)D_t^\alpha\xi=-cU'(\xi)$. So Eq. (33) can be turned into the following form

$$-cU'+2016U^3U'+630(U')^3+2268UU'U''+504U^2U'''$$
$$+252U''U'''+147U'U^{(4)}+42UU^{(5)}+U^{(7)}=0. \quad (34)$$

Suppose that the solutions of (34) can be expressed by a polynomial in $(\frac{G'}{G})$ as $U(\xi)=\sum_{i=0}^{m}a_i(\frac{G'}{G})^i$, where $a_i,\ i=0,1,...,m$ are constants, and $G=G(\xi)$ satisfies Eq. (10). Balancing the order of $U''U'''$ and $U^{(7)}$ in (34), one can obtain $m=2$. So one has

$$U(\xi)=a_0+a_1(\frac{G'}{G})^1+a_2(\frac{G'}{G})^2. \quad (35)$$

Substituting (35) into (34), using Eq. (10) and collecting all the terms with the same power of $(\frac{G'}{G})$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations, one can obtain the following results.

Case 1:

$$a_0=\frac{1}{24A^2}(8AE-8EC-B^2),\ a_1=\frac{(A-C)B}{2A^2},$$
$$a_2=-\frac{1}{2A^2}(A^2-2AC+C^2),$$
$$c=-\frac{1}{48A^6}(64A^3E^3-192CA^2E^3+48A^2B^2E^2$$
$$-96CAB^2E^2+192C^2AE^3+12AB^4E-64C^3E^3$$
$$+B^6-12CB^4E+48C^2B^2E^2).$$

Case 2:

$$a_0=-\frac{B^2}{24A^2},\ a_1=\frac{(A-C)B}{2A^2},\ a_2=-\frac{A^2-2AC+C^2}{2A^2},$$
$$c=-\frac{B^6}{48A^6},\ E=0.$$

Remark 5. If we take $A=1,\ B=-\lambda, C=0,\ E=-\mu$ in Cases 1-2, then the results reduce to the results in [5], which were obtained by use of the (G'/G) method. So our results are of more general forms than those obtained from the (G'/G) method.

Substituting the results above into Eq. (35) and combining with (12)-(16), one can obtain corresponding exact solutions for the time fractional seventh-order Kaup-Kupershmidt equation.

From Case 1 one can get the following exact solutions, where $C_1,\ C_2$ are arbitrary constants, and

$$\xi=x+\frac{1}{48A^6\Gamma(1+\alpha)}(64A^3E^3-192CA^2E^3+48A^2B^2E^2$$
$$-96CAB^2E^2+192C^2AE^3+12AB^4E-64C^3E^3+B^6$$
$$-12CB^4E+48C^2B^2E^2)t^\alpha+\xi_0.$$

Family 1: when $B\neq 0,\ \Delta_1>0$:

$$u_1(x,t)=\frac{1}{24A^2}(8AE-8EC-B^2)+\frac{(A-C)B}{2A^2}\{\frac{B}{2(A-C)}$$
$$+\frac{\sqrt{\Delta_1}}{2(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}]\}^1$$
$$-\frac{1}{2A^2}(A^2-2AC+C^2)\{\frac{B}{2(A-C)}+\frac{\sqrt{\Delta_1}}{2(A-C)}$$
$$[\frac{C_1\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}+C_2\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}]\}^2. \quad (36)$$

$Family$ 2: when $B \neq 0,\ \Delta_1 < 0$:

$$u_2(x,t) = \frac{1}{24A^2}(8AE-8EC-B^2) + \frac{(A-C)B}{2A^2}\{\frac{B}{2(A-C)} + \frac{\sqrt{-\Delta_1}}{2(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)} + C_2\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)} + C_2\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}]\}^1 - \frac{1}{2A^2}(A^2-2AC+C^2)\{\frac{B}{2(A-C)} + \frac{\sqrt{-\Delta_1}}{2(A-C)} [\frac{-C_1\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)} + C_2\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_1}\xi}{2(A-C)} + C_2\sin\frac{\sqrt{-\Delta_1}\xi}{2(A-C)}}]\}^2. \quad (37)$$

$Family$ 3: when $B \neq 0,\ \Delta_1 = 0$:

$$u_3(x,t) = \frac{1}{24A^2}(8AE-8EC-B^2) + \frac{(A-C)B}{2A^2}\{\frac{B}{2(A-C)} + \frac{C_2}{C_1+C_2\xi}\}^1 - \frac{1}{2A^2}(A^2-2AC+C^2)\{\frac{B}{2(A-C)} + \frac{C_2}{C_1+C_2\xi}\}^2. \quad (38)$$

$Family$ 4: when $B = 0,\ \Delta_2 > 0$:

$$u_4(x,t) = \frac{1}{24A^2}(8AE-8EC-B^2) + \frac{(A-C)B}{2A^2}\{\frac{\sqrt{\Delta_2}}{(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)} + C_2\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)} + C_2\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}]\}^1 - \frac{1}{2A^2}(A^2-2AC+C^2)\{\frac{\sqrt{\Delta_2}}{(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)} + C_2\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi}{(A-C)} + C_2\sinh\frac{\sqrt{\Delta_2}\xi}{(A-C)}}]\}^2. \quad (39)$$

$Family$ 5: when $B = 0,\ \Delta_2 < 0$:

$$u_5(x,t) = \frac{1}{24A^2}(8AE-8EC-B^2) + \frac{(A-C)B}{2A^2}\{\frac{\sqrt{-\Delta_2}}{(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)} + C_2\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)} + C_2\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}]\}^1 - \frac{1}{2A^2}(A^2-2AC+C^2)\{\frac{\sqrt{-\Delta_2}}{(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)} + C_2\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi}{(A-C)} + C_2\sin\frac{\sqrt{-\Delta_2}\xi}{(A-C)}}]\}^2. \quad (40)$$

From Case 2 one can get the following exact solutions.

$Family$ 6: when $B \neq 0,\ \Delta_1 > 0$:

$$u_6(x,t) = -\frac{B^2}{24A^2} + \frac{(A-C)B}{2A^2}\{\frac{B}{2(A-C)} + \frac{\sqrt{\Delta_1}}{2(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)} + C_2\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)} + C_2\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}]\}^1 - \frac{A^2-2AC+C^2}{2A^2}\{\frac{B}{2(A-C)} + \frac{\sqrt{\Delta_1}}{2(A-C)} [\frac{C_1\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)} + C_2\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_1}\xi}{2(A-C)} + C_2\sinh\frac{\sqrt{\Delta_1}\xi}{2(A-C)}}]\}^2, \quad (41)$$

where $C_1,\ C_2$ are arbitrary constants, and

$$\xi = x + \frac{B^6}{48A^6\Gamma(1+\alpha)}t^\alpha + \xi_0.$$

In (36)-(37), by taking special parameters, we obtain solitary wave solution expressed in hyperbolic function and periodic wave solution in trigonometric function as shown in Figs. 1-2.

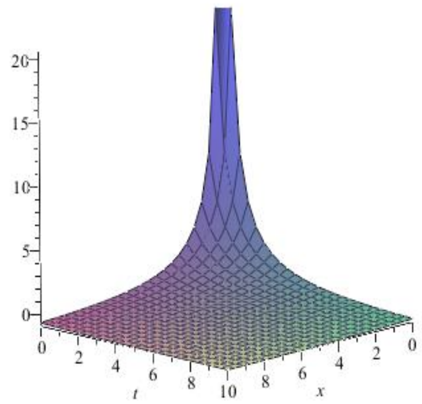

Fig. 1. The solitary wave solution u_1 with $C_1 = 1$, $C_2 = 0$, $A = 2$, $B = C = E = 1$, $\xi_0 = 0$, α=0.5

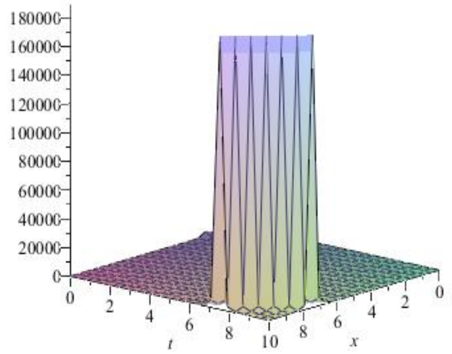

Fig. 2. The periodic wave solution u_2 with $C_1 = 1$, $C_2 = 0$, $A = 2$, $B = C = E = 1$, $\xi_0 = 0$, α=0.5

IV. Conclusions

An improved (G'/G) method have been applied to seek exact solutions for fractional differential equations, which is an extension of the known (G'/G) method. For illustrating the validity of the method, we apply it to solve the (2+1)-dimensional space-time fractional Nizhnik-Novikov-Veselov System and the time fractional seventh-order Kaup-Kupershmidt equation. With the aid of the mathematical software, some new exact solutions with general forms have been successfully found for them.

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