Non-equidistance Fractional Order Accumulation Grey Power Model NFGPM (1, 1) and Its Application

XiaoGao Yang，Deqion Ding, Youxin Luo

Abstract—**In scientific research and practical engineering technology applications, there are a large number of research objects with nonlinear fractional order characteristics. The correct establishment of analysis and prediction model is related to the ability to reveal the laws between parameters and the accuracy of prediction. In recent years, the gray system theory has been developed in small sample and poor information engineering practice. The present paper, a non-linear fractional accumulation gray power model NFGPM (1, 1) is proposed. Based on the multiple variables of the fractional accumulation order, the initial value correction of differential equation and the background value, the non-linear fractional cumulative gray power model was constructed, which expanded the theoretical research and practical application of fractional cumulative gray model. An example verification analysis shows that the results obtained by using the fractional cumulative gray power model are in good agreement with the references, and the analysis accuracy is high, At the same time, the model has the ability to fully discover the explicit and hidden information in the data in the case of little data and poor information, which has high engineering practical value.**

Index Terms—**fractional accumulation order, fractional order accumulation grey power model, nonlinear, analysis accuracy, background values**

I. Introduction

In the study of a large number of practical engineering problems, building the analytical models, estimating parameters, optimizing algorithm, etc., need to establish a relationship between variables or parameters through mathematical models [1-3]. Establishing the correct analytical model for the research object is the key link to solve the practical problem of the project. The fractional order is an analytical model other than the integer-order analytical model. It first appeared in pure mathematical theory analysis and was introduced into practical engineering applications this year [4-7]. Gray modeling can model prediction without requiring a large amount of sample data. Because the modeling process is simple and easy to operate, it has a unique advantage in short-term prediction of small sample sequences. The classical gray prediction model also has high engineering application value [8-10]. In recent years, the grey system prediction method has been widely used in modeling and forecasting non-equidistant sequence, especially in building deformation, material experiment, rock mechanics, resource exploration and other engineering fields [11]. Through the analysis of the literature, it can be known the grey model GM (1, 1) is the integer order derivative model, which cannot accurately expose the essential characteristics and behaviors of a large number of research objects with fractional order attributes in engineering practice. In order to solve the problems in engineering practice, scholars at home and abroad have analyzed the fractional order theory model. Wu and Liu etc. provided the fractional order accumulated grey model GM (1, 1) [12]. Fang and Wu etc. predict the weapon maintenance fee by adopting the fractional order accumulated grey model GM (1, 1) [13]. Wu, Liu and Yao established the discrete fractional order accumulated grey model GM (1, 1) [14]. Meng and Zeng summarized the fractional order accumulated grey model GM (1, 1) and discrete fractional order accumulated grey model GM (1, 1) [15]. Literature [16] established the new information priority fractional order direct grey model NIGM (1, 1). Literature [17] established fractional order discrete grey GM (1, 1) power model. Literature [18] established grey GM (1, 1) fractional order accumulation model which expanded the integral order grey model. The character and the application of these models were also analyzed. The order GM (1, 1) model is an important nonlinear gray model, which can find the power exponent matching with the actual data, so as to reflect the nonlinear characteristics of the actual data. Literature [19] firstly provided the solution of classic GM (1, 1) order model and researched the character of solution for this model. Literature [11] provided the unequal GM (1, 1) power model. In this paper, a fractional cumulative grey power model NFGPM (1, 1) is proposed to solve the problem of modeling non-equidistant sequences in engineering. In this paper, aiming at the problem that it is difficult to establish a correct analysis model or the analysis accuracy is not high for a large number of researches on non-equidistant sequences in engineering, the non-equidistant fractional order accumulation grey power model NFGPM (1, 1) is proposed. In the process of modeling, by minimizing the absolute value of the average relative error, a nonlinear optimization model is constructed to estimate the parameters of non-equidistant fractional order accumulation grey power model NFGPM (1,

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1) with fractional order accumulation order, the correction of initial value of differential equation and the coefficient of background value as design variables., and MATLAB program was compiled. Through the analysis of the established model, the results show the established model extends the non-equidistance fractional-order cumulative gray model. The non-equidistance fractional order accumulation grey model NFGPM(1,1) and non-equidistance fractional order accumulation grey Verhulst model are both the special cases of non-equidistance fractional order accumulation power grey model NFGPM(1,1). The engineering example verified the reliability and practicality of on-equidistance fractional order accumulation power grey model NFGPM (1, 1). It is deserved to apply into the engineering fields.

II. THE UNEQUAL FRACTIONAL ORDER ACCUMULATION GREY POWER MODEL NFGPM (1, 1)

In order to derive the model, the following definition of sequence is made.

Definition 1: The nonnegative sequence is given as follows, $X^{(0)}=[x^{(0)}(t_1),\cdots,x^{(0)}(t_m)]$, $\Delta t_i=t_i-t_{i-1}$ is not, so $X^{(0)}$ is the non-negative sequence.

Definition 2: Given sequence $X^{(1)}=[x^{(1)}(t_1),x^{(1)}(t_2),\cdots,x^{(1)}(t_m)]$, if $x^{(1)}(t_1)=x^{(0)}(t_1)$, $x^{(1)}(t_{k+1})=x^{(1)}(t_k)+x^{(0)}(t_{k+1})\cdot\Delta t_{k+1}$, $k=1,\cdots,m-1$, so $X^{(1)}$ is the one order accumulated operation sequence of $X^{(0)}$.

Definition 3: Given the non-negative sequence as $X^{(0)}=[x^{(0)}(t_1),\cdots,x^{(0)}(t_m)]$, if $\Delta t_i=t_i-t_{i-1}\neq constant$, $i=2,\cdots,m$, $X^{(r)}=[x^{(r)}(t_1)\quad x^{(r)}(t_2)\quad\cdots\quad x^{(r)}(t_m)]$ is the r-th accumulated operation, where

$$x^{(r)}(t_k)=\sum_{i=1}^{k}x^{(r-1)}(t_i)\Delta t_i \tag{1}$$

According to the matrix operation principle, $x^{(r)}=Ax^{(r-1)}=AAx^{(r-2)}=\cdots=A^r x^{(0)}$, where the first accumulated operation matrix A is:

$$A=\begin{vmatrix}1&0&0&\cdots&0\\1&1&0&\cdots&0\\1&1&1&\cdots&0\\\cdots&\cdots&\cdots&\cdots&\cdots\\1&1&1&\cdots&1\end{vmatrix},$$

A^r is the r accumulated operation matrix, $A^r=(a^r_{i_1j_1})_{m\times m}$,

where, $(a^r_{i_1j_1})_{m\times m}=\begin{cases}\dfrac{(i_1-j_1+r-1)!}{(r-1)!(i-j)!} & i_1\geq j_1\\ 0 & i_1<j_1\end{cases}$. By combining the expansion of combinatorial number and expanding r from integer to fraction, the fractional order cumulative operation matrix can be expressed as.

$$A^r=(a^r_{i_1j_1})_{m\times m}=\begin{vmatrix}1&0&0&\cdots&0\\r&&&\cdots&0\\\dfrac{r(r+1)}{2!}&r&&\cdots&0\\\vdots&\vdots&\vdots&\ddots&0\\\dfrac{r(r+1)\cdots(r+m-2)}{(m-1)!}&\dfrac{r(r+1)\cdots(r+m-3)}{(m-2)!}&\dfrac{r(r+1)\cdots(r+m-4)}{(m-3)!}&\cdots&1\end{vmatrix}$$

The coefficient of $x^{(0)}(t_i)$ in the expanded formula of $x^{(r)}(t_k)$ is:

$$\begin{aligned}a_k&=\frac{(k-i+1)(k-i+2)\cdots(k-i+r-1)}{(r-1)!}\\&=\frac{(r+k-i-1)!}{(k-i)!(r-1)!}=\frac{\Gamma(r+k-1)}{\Gamma(k-i+1)\Gamma(r)}\qquad(i=1,2,\cdots,m)\end{aligned}$$

So $x^{(r)}(t_k)$ can be expressed as

$$x^{(r)}(t_k)=\sum_{i=1}^{k}\frac{\Gamma(r+k-1)}{\Gamma(k-i+1)\Gamma(r)}(x^{(0)}(t_i)\Delta t_i),\ k=1,2,\cdots,m \tag{2}$$

where, Γ is Gamma function.

Equation (2) is the general form in which r is expanded from integer to fraction. So the Equation (2) is adopted to calculate the fractional order of $x^{(r)}(t_k)$.

Definition 4

If $k=2,3,\cdots,m$, $x^{(r-1)}(t_k)=x^{(r)}(t_k)-x^{(r)}(t_{k-1})$ in the equation (2), the unequal fractional accumulated grey power model GM (1, 1), which labelling as NFGPM (1, 1), can expressed as

$$x^{(r-1)}(t_k)+az^{(r)}(t_k)=b[z^{(r)}(t_k)]^{\gamma} \tag{3}$$

where, the coefficient *a* and *b* are the developed coefficient and the actuating quantity in the model, respectively.

The background values is expressed as

$$z^{(r)}=[z^{(r)}(t_2)\quad z^{(r)}(t_3)\quad\cdots\quad z^{(r)}(t_m)] \tag{4}$$

where, the mean values is operated as shown in the following formula

$$z^{(r)}(t_k)=0.5x^{(r)}(t_{k-1})+0.5x^{(r)}(t_k) \tag{5}$$

Especially, as the exponent γ, on the right-hand side of Definition 4, is equal to 2, equation (3) is called the unequal grey fractional order *Verhulst* model. When γ is equal to 0, the equation (3) is called the unequal grey fractional order GM (1,1) model. In summary, it is not difficult to find that the model extends the range of nonlinear fractional-order models. By the least square method, the parameters of the unequal fractional order accumulated grey power model $x^{(r-1)}(t_k)+az^{(r)}(t_k)=b[z^{(r)}(t_k)]^{\gamma}$ can be got

$$P=\begin{pmatrix} a \\ b \end{pmatrix}=(B^T B)^T B^T Y \tag{6}$$

where,

$$\mathbf{Y}=\begin{vmatrix} x^{(r-1)}(t_2) \\ x^{(r-1)}(t_3) \\ \cdots \\ x^{(r-1)}(t_m) \end{vmatrix}, \mathbf{B}=\begin{vmatrix} -z^r(t_2) & [z^r(t_2)]^{\gamma} \\ -z^r(t_3) & [z^r(t_3)]^{\gamma} \\ \cdots & \cdots \\ -z^r(t_m) & [z^r(t_m)]^{\gamma} \end{vmatrix},$$

Definition 5: $\frac{dx^r}{dt}+ax^{(r)}=b[x^{(r)}]^{\gamma}$ is the white equation of the fractional order GM (1, 1) model and the general solution of this equation is

$$x^{(r)}=[\frac{b}{a}+ce^{(\gamma-1)at}]^{\frac{1}{1-\gamma}} \tag{7}$$

where, the coefficient c is the constant.

Given the initial condition $x^{(r)}(t)|_{t=t_1}=x^{(r)}(t_1)$, the time response sequence of the unequal fractional order GM (1, 1) power model is

$$\hat{x}^{(r)}(t_k)=\{\frac{b}{a}+[x^{(r)}(t_1)^{1-\gamma}-\frac{b}{a}]e^{(\gamma-1)a(t_k-t_1)}\}^{\frac{1}{1-\gamma}} \tag{8}$$

where, k=1,2⋯m

For A^r is satisfied $(A^r)^{-1}=A^{-r}, A^r A^{-r}=I$, $\hat{x}^{(0)}=A^{-r}A^r\hat{x}^{(0)}=A^{-r}x^{(r)}$. Defining A^{-r} is the r-*th* inverse accumulated operation matrix. So the original $\hat{x}^{(0)}$ can be restored from $\hat{x}^{(r)}$ and the model can be verified.

$$\hat{x}^{(0)}(t_k)=\sum_{i=0}^{k-1}(-1)^i\frac{\Gamma(r+1)}{\Gamma(i+1)\Gamma(r-i+1)}\hat{x}^{(r)}(t_{k-i})/\Delta t_k \tag{9}$$

where, $(k=1,2,\cdots,m)$

According to the above method the unequal fractional order accumulated grey power model NFGPM (1, 1) can be established. Giving the initial value $x^{(r)}(t)|_{t=t_1}=x^{(r)}(t_1)$ in the Equation (8), the values of $x^{(r)}(t_1)$ can be replaced with $x^{(r)}(t_1)+\beta$, so

$$\hat{x}^{(r)}(t_k)=\{\frac{b}{a}+[(x^{(r)}(t_1)+\beta)^{1-\gamma}-\frac{b}{a}]e^{(\gamma-1)a(t_k-t_1)}\}^{\frac{1}{1-\gamma}} \tag{10}$$

where, $k=1,2,\cdots,m$

In the formula (4) the background value are got by the mean values operation and if it is revised as

$$z_i^{(r)}(t_k)=\lambda x_i^{(r)}(t_k)+(1-\lambda)x_i^{(r-1)}(t_{k-1}) \tag{11}$$

$$\text{so Y}=\begin{vmatrix} x^{(r-1)}(t_2) \\ x^{(r-1)}(t_3) \\ \cdots \\ x^{(r-1)}(t_m) \end{vmatrix}, \text{B}=\begin{vmatrix} -z^r(t_2) & [z^r(t_2)]^{\gamma} \\ -z^r(t_3) & [z^r(t_3)]^{\gamma} \\ \cdots & \cdots \\ -z^r(t_m) & [z^r(t_m)]^{\gamma} \end{vmatrix},$$

$$z^{(r)}(t_k)=\lambda x^{(r)}(t_{k-1})+(1-\lambda)x^{(r)}(t_k) \tag{12}$$

where $\lambda\in[0,1]$.

The absolute error of the fitting data can be expressed as

$$q(t_k)=\hat{x}^{(0)}(t_k)-x^{(0)}(t_k) \tag{13}$$

Similarly, the relative error of the fitting data is shown below

$$e(t_k)=\frac{\hat{x}^{(0)}(t_k)-x^{(0)}(t_k)}{x^{(0)}(t_k)}\times 100\% \tag{14}$$

The mean value of relative error of the fitting data is

$$f=\frac{1}{m}\sum_{k=1}^{m}\left|e(t_k)\right| \tag{15}$$

Taking the minimum mean absolute relative error as the objective, the non-equidistant fractional accumulation grey power model NFGPM (1, 1) is expressed as the following optimization model with the fractional accumulation order, the initial value of differential equation and the coefficient of background value as the design variables.

$$\min \quad f(r,\gamma,\beta)=\frac{1}{m}\sum_{k=1}^{m}\left|\frac{x^{(0)}(t_k)-\hat{x}^{(0)}(t_k)}{x^{(0)}(t_k)}\right| \tag{16}$$

III. APPLICATIONS

In the section, some applications were presented on fatigue strength for the results established above. Fatigue strength refers to the maximum stress of material under infinite multiple alternating loads. Fatigue failure is one of the main causes of mechanical parts failure. According to statistics, more than 80% of the failed mechanical parts belong to fatigue failure, and there is no obvious deformation before fatigue failure, and fatigue failure often causes major accidents. Therefore, the fatigue strength research of key components such as shaft, gear, bearing, blade and spring is related to the service performance and life of the equipment. Because of the nonlinear fractional characteristic of fatigue strength at different temperatures, it is difficult to simulate and analyze the fatigue strength by modeling. The previous research on fatigue strength is mainly in the form of experiment. The experimental data of fatigue strength of titanium alloy with temperature, researched by P.G. Fore, was shown in Table 1 [11].

TABLE I
EXPERIMENTAL DATA OF FATIGUE STRENGTH OF Ti ALLOY ON DIFFERENT TEMPERATURE (MPa)

Ordinal	Temperature(t_k / °C)	fatigue strength of titanium alloy(σ_{-1})
1	100	560.00
2	130	557.54
3	170	536.10
4	210	516.10
5	240	505.60
6	270	486.10
7	310	467.40
8	340	453.80
9	380	436.40

It can be seen from Tab.1 that the experimental data is an unequally spaced sequence. The sample size of test data

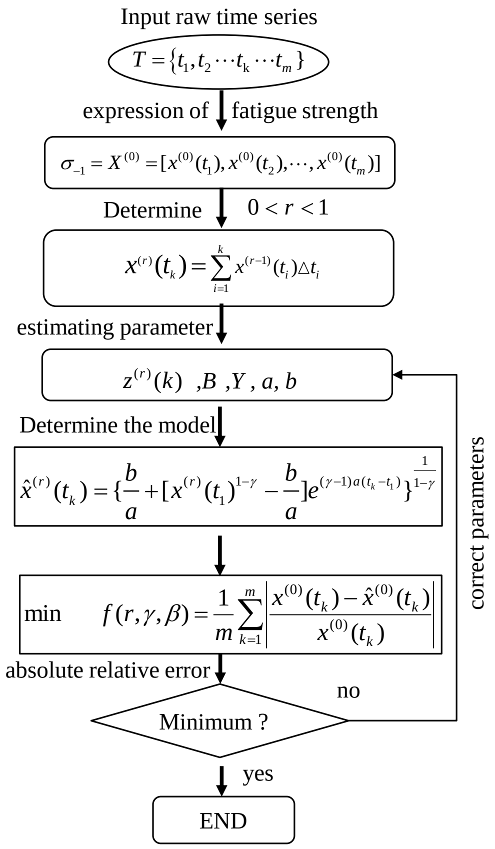

Fig.1 The flowchart of the solution

between fatigue strength and temperature of titanium alloy is small and the degree of information is low. The effect of the temperature on the fatigue strength in the condition of the cyclic symmetry of the long life has nonlinear fractional order characteristics.

The analysis and prediction model of effect of the temperature on the fatigue strength of the titanium alloy was established in the section by means of the non-equidistance fractional cumulative grey power mode. Then the analysis data will be compared with the reference value to verify the correctness of the model and the accuracy of the analysis. The analysis was performed with deliberate default data (σ_{-1}(170 ℃), σ_{-1}(310 ℃)), which purpose is to verify the ability and accuracy of model mining data. The solution flow is shown in Figure 1.

TABLE II
CASE ANALYSIS OF TI ALLOY FATIGUE STRENGTH ALONG WITH TEMPERATURE (MPA)

t_k / °C	Raw Data	Fitting Data	Fitting Error: Absolute Error	Fitting Error: Relative Error
100	560	560.0851	-0.0851	-0.015194
130	557.54	555.5778	1.9622	0.35193
170	536.10	538.0955	-1.9955	-0.37223
210	516.10	518.075	-1.9750	-0.38267
240	505.60	502.6442	2.9558	0.58461
270	486.1	487.0975	-0.9975	-0.20521
310	467.4	467.7774	-0.3774	-0.080735
340	453.8	454.3012	-0.5012	-0.11045
380	436.4	435.4366	0.9634	0.22076

Suppose t_k is the element of the temperature sequence and that $x^{(0)}(t_k)$ is the element of the fatigue strength change sequence of Ti alloy, the values of fatigue strength can be expressed

$$\sigma_{-1}=[x^{(0)}(t_1),x^{(0)}(t_2),\cdots,x^{(0)}(t_k),\cdots,x^{(0)}(t_m)]$$

where the value of t_k is $t=[t_1,t_2,\cdots,t_k,\cdots,t_m]$.

Applying the non-equidistant fractional cumulative gray power model NFGPM (1, 1) proposed in this paper, the model parameters are as follows:

r =0.99327, γ=0, λ=0.49539, β=0.085084, a=b=564.9215

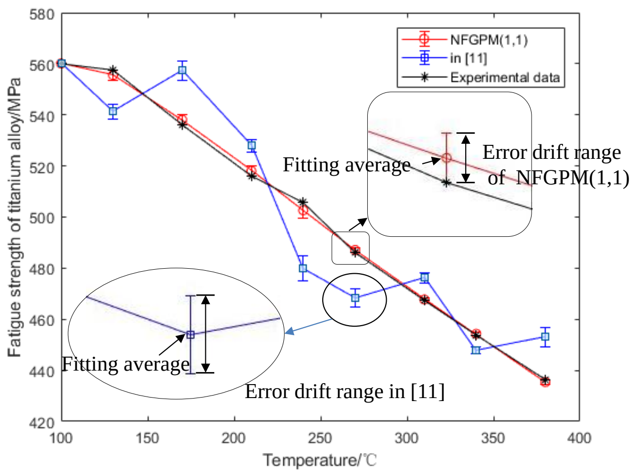

Fig.2 Comparative analysis error bar figure of NFGPM with [11]

After the optimization parameter model was obtained, we increased the temperature t_k (t_k =170℃, t_k =310℃) and still obtained a more accurate value σ_{-1}, respectively, is 538.0955 and 467.7774.

The fitting values of the original data, absolute error and relative error have been shown in Table II

The average relative error of the fitting data is 0.2582% and mean square error MSE=2.5326.

After optimization we get γ=0. The non-equidistance fractional order accumulation power grey model NFGPM (1,

1) has transformed to the non-equidistance fractional order accumulation grey model NFGPM (1, 1). The average relative error is 0.94% in the literature. The comparative analysis between NFGPM and [11] was shown in the figure2. It can be clearly see that the fitting value of references deviates from the experimental value and error drift range of the reference both much more than the NFGPM (1, 1) as shown in the figure.

The case verification analysis show the proposed model has higher parameter estimation accuracy than that of reference. So we can see the method in the paper is adaptive and scientific.

IV. CONCLUSIONS

In the present paper, the present paper, a non-linear fractional accumulation gray power model NFGPM (1, 1) a non-equidistance fractional cumulative grey power model is proposed to solve the problem of modeling non-equal spacing sequences in engineering. The establishment of this model makes the application range of grey theory analysis model expand from integer order to rational number, and promotes the application and research of grey theory. The case verification analysis, in which the effect of the temperature on the fatigue strength in the condition of the cyclic symmetry of the long life has studied by P.G. Fore's, shows the proposed model, and has higher parameter estimation accuracy than that of reference. At the same time, the model has the ability to fully discover the explicit and hidden information in the data in the case of little data and poor information.

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REFERENCES

[1] Jun Song, Yihan Xu, and Guanghao Li, "An Accurate Parameter Estimator for LFM Signals Based on Zoom Modified Discrete Chirp Fourier Transform," IAENG International Journal of Computer Science, vol. 46, no.3, pp467-474, 2019

[2] Hongbiao Li, Ruiying Liu, Jingdong Wang, and Qilong Wu, "An Enhanced and Efficient Clustering Algorithm for Large Data Using MapReduce," IAENG International Journal of Computer Science, vol. 46, no.1, pp61-67, 2019

[3] Gonggui Chen, Jie Qian, Zhizhong Zhang, and Zhi Sun, "Multi-objective Improved Bat Algorithm for Optimizing Fuel Cost, Emission and Active Power Loss in Power System," IAENG International Journal of Computer Science, vol. 46, no.1, pp118-133, 2019

[4] Bin Zheng, "Some New Fractional Integral Inequalities in the Sense of Conformable Fractional Derivative," Engineering Letters, vol. 27, no.2, pp287-294, 2019

[5] Qinghua Feng, "Compact Difference Schemes for a Class of Space-time Fractional Differential Equations," Engineering Letters, vol. 27, no.2, pp269-277, 2019

[6] Tingting Xue, Xiaolin Fan, and Jiabo Xu, "Existence of Positive Solutions for a Kind of Fractional Multi-point Boundary Value Problems at Resonance," IAENG International Journal of Applied Mathematics, vol. 49, no.3, pp281-288, 2019

[7] Wei Gao, and Ce Shi, "Sufficient Conditions for The Existence of Fractional Factors in Different Settings," IAENG International Journal of Applied Mathematics, vol. 49, no.2, pp145-154, 2019

[8] Liu, S. F., Lin, Y. "Grey systems: theory and applications". Springer-Verlag London Ltd., London., 2010.

[9] Zheng, B. H., Luo, L.X. "Non-equidistance grey model GRM(1,1) and empirical research of city construction land demand prediction". Electronic Journal of Geotechnical Engineering, vol.20, no.14, pp. 6065-6071, 2015.

[10] Zheng, B. H., Luo, L.X. "Non-equidistance optimum grey model GM(1, 1) of requirement analysis of urban construction land and its empirical study". Electronic Journal of Geotechnical Engineering, vol. 20, no. 22, pp. 12227-12232,2015.

[11] Wang, Z. X., Dang, Y. G., Liu, S. F. "Non-equidistant GM(1,1) power model and its application in engineering". Chinese Journal of Management Science, vol. 14, no.7, pp. 98-102, 2012.

[12] Wu L. F., Liu S. F., Yao L. G., et al.. "Grey system model with the fractional order accumulation". Communications in Nonlinear Science and Numerical Simulation, vol. 18, no. 7, pp. 1775-1785, 2013.

[13] Fang, S., Wu, L., Fang, Z., Guo, X. "Using fractional GM(1,1) model to predict the maintenance cost of weapon system". Journal of Grey System, vol. 25, no.3, pp. 9-15, 2013.

[14] Wu, L. F., Liu, S. F., Yao, L. G. "Discrete grey model based on fractional order accumulate". Systems Engineering - Theory & Practice, vol. 34, no. 7, pp. 1822-1827, 2014.

[15] Meng, W., Zeng, B. "The fractional order operator and the research on grey forecasting model". Science Press, Beijing, 2015.

[16] Wu, L. F., Liu, S. F., Yao, L. G. "Grey model with Caputo fractional order derivative". Systems Engineering - Theory & Practice, vol. 35, no. 5, pp. 1311-1316, 2015.

[17] Yang, B. H., Zhao, J. S. "Fractional order discrete grey GM(1,1) power model and its application". Control and Decision, vol. 30, no.7, pp.1264-1268, 2015.

[18] Wu, L. F., Liu, S. F., Liu, J. "GM(1,1) model based on fractional order accumulating method and its stability". Control and Decision, vol. 29, no. 5, pp. 919-923, 2014.

[19] Wang, Z. X., Dang, Y. G., Liu, S. F. "Solution of GM (1,1) power model and its properties". Systems Engineering and Electronics. 31(10): 2380-2383. 2009.

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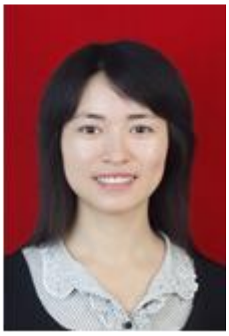

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