Some Oscillation Results for a Class of Advanced Partial Difference Equations

Huili Ma, Jiaofeng Wang

Abstract—**In this article, the following advanced partial difference equation**

$$pu_{m+2,n}+qu_{m,n+2}-u_{m,n}+ru_{m+\sigma,n+\tau}=0$$

is considered, where p,q,r are real numbers satisfying $p^2+q^2+r^2\neq 0$, and m,n,σ,τ are non-negative integers. It derives some necessary and sufficient conditions by means of the envelope theory to ensure the oscillatory properties of solutions for the equation.

Index Terms—**advanced partial difference equation, oscillation, envelope, characteristic equation.**

I. Introduction

OSCILLATORY behavior is always one of the important study fields in the qualitative theory research. Meanwhile, partial difference equations have numerous applications as in population control, image processing, economic time series problem, material mechanics, etc[1-3]. In recent years, the study of oscillatory solution of partial difference has attracted considerable attention. In [4], by means of zero point theorem, B. G. Zhang and R. P. Agarwal have investigated the oscillatory property of the following partial difference equation

$$A_{m+1,n}+A_{m,n+1}-pA_{m,n}+\sum_{i=1}^{\mu}q_iA_{m-k_i,n-l_i}=0,$$

where p,q_i are real numbers, $k_i,l_i\in N_0(i=1,2,...,\mu)$, $N_t=\{t,t+1,...\}$, and μ is a positive integer.

There are many literature dealing with oscillatory studies for difference equations, we refer the reader to [5-10] for some references. However, most research handled ordinary or delay partial difference equations. For the oscillatory studies, there appeared few sample of work related to advanced partial difference equations, especially for the second order case. In this paper, we consider the following advanced partial difference equation

$$pu_{m+2,n}+qu_{m,n+2}-u_{m,n}+ru_{m+\sigma,n+\tau}=0, \qquad (1)$$

where p,q are real parameters satisfying $p^2+q^2+r^2\neq 0$, and m,n,σ,τ are nonnegative real numbers. We will apply the envelope theory to derive necessary and sufficient conditions for the advanced partial difference equation (1) to be oscillatory without the sign constrains for parameters p, q and r.

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Before stating the main results, we provide some definitions used in this paper.

Definition 1 If $\sigma\geq 0,\tau\geq 0$ and $\sigma^2+\tau^2\neq 0$. Then the equation(1) is called advanced equation.

Definition 2 A solution of (1) is a real double sequence $\{u_{m,n}\}$ which is defined for $m\geq -\sigma$,$n\geq -\tau$ and satisfies (1) for $m\geq 0$ and $n\geq 0$.

Definition 3 A solution $\{u_{m,n}\}$ of (1) is said to be eventually positive (or negative) if $u_{m,n}>0$ (or $u_{m,n}<0$) for large numbers m and n. It is said to be oscillatory if it is neither eventually positive nor eventually negative. (1) is called oscillatory if all of its nontrivial solutions are oscillatory.

II. Preliminaries

In this section, we give some lemmas that will be used in the proof of the main results in section 3.

Lemma 1 [11] The following statements are equivalent:

(i) Every solution of equation (1) is oscillatory.

(ii) The characteristic equation of equation (1)

$$p\lambda^2+q\mu^2-1+r\lambda^{\sigma}\mu^{\tau}=0$$

has no positive root.

Lemma 2 [12] Suppose that $f(x),g(x)$ and $v(x)$ are differentiable on $(-\infty,+\infty)$. Let Γ be the one-parameter family of lines defined by the equation

$$f(\lambda)x+g(\lambda)y=v(\lambda)$$

, where λ is a parameter. Let Σ be the envelope of the family Γ. Then the equation

$$f(\lambda)a+g(\lambda)b=v(\lambda)$$

has no real root if and only if there is no tangent line of Σ passing through the point (a,b) in xy-space.

Lemma 2' [12] Suppose that $f(x),g(x),h(x)$ and $v(x)$ are differentiable on $(-\infty,+\infty)$. Let Γ be the one-parameter family of planes defined by the equation

$$f(\lambda)x+g(\lambda)y+h(\lambda)z=v(\lambda)$$

, where λ is a parameter. Let Σ be the envelope of the family Γ. Then the equation

$$f(\lambda)a+g(\lambda)b+h(\lambda)c=v(\lambda)$$

has no real root if and only if there is no tangent plane of Σ passing through the point (a,b,c) in xyz-space.

Lemma 3 [12] Assume $f(x,y),g(x,y),h(x,y)$ and $v(x,y)$ are differentiable on $(-\infty,+\infty)\times(-\infty,+\infty)$. Let Γ be the

the two-parameter family of planes defined by the following equation

$$f(\lambda,\mu)x+g(\lambda,\mu)y+h(\lambda,\mu)z=v(\lambda,\mu),$$

where λ,μ are real numbers. Let Σ be the envelop of family of planes Γ. Then the equation

$$f(\lambda,\mu)a+g(\lambda,\mu)b+h(\lambda,\mu)c=v(\lambda,\mu)$$

has no real root if and only if there is no tangent plane of Σ passing through the point (a,b,c) in xyz-plane.

Lemma 3' [12] If the function $f(x,y),g(x,y),h(x,y)$ are differentiable on $(-\infty,+\infty)\times(-\infty,+\infty)$, and Γ is the the two-parameter family of lines defined by the following equation

$$f(\lambda,\mu)x+g(\lambda,\mu)y=h(\lambda,\mu),$$

where λ,μ are real numbers. Let Σ be the envelop of family of lines Γ. Then the equation

$$f(\lambda,\mu)a+g(\lambda,\mu)b=h(\lambda,\mu)$$

has no real root if and only if there is no tangent line of Σ passing through the point (a,b) in xy-plane.

Lemma 4 [12] Let $f(x)$ be continuously differentiable and not equivalent to zero on $(0,+\infty)$. If f satisfies $\lim_{x\to+\infty}f(x)>0$ or $\lim_{x\to0^+}f(x)>0$, then the equation

$$F(x,y)=y+f(x)=0$$

has no positive root on $(0,+\infty)\times(0,+\infty)$ if and only if the equation $f(x)=0$ has no positive root on $(0,+\infty)$.

Lemma 5 Let

$$f(\lambda,p,r)=r\lambda^k+p\lambda^2-1=0, \tag{2}$$

where $k\in\mathbb{Z}$ and $k\geq3$, p,r are real parameters. Then the equation $f(\lambda,p,r)=0$ has no positive root if and only if $p\leq0,r\leq0$ or $p>0,r<-2(k-2)^{\frac{k-2}{2}}p^{\frac{k}{2}}/k^{\frac{k}{2}}$.

Proof (I)If $r=0$, then $f(\lambda,p)=p\lambda^2-1=0$ has no positive root if and only if $p\leq0$.

(II)If $r\neq0$, we will consider (p,r) as a point in xy-plane, and try to search for the exact regions including points (p,r) in xy-plane such that (2) has no positive root. Actually, $f(\lambda,x,y)=0$ can be regarded as an equation describing a one-parameter family of lines in xy-plane where λ is the parameter. According to the envelop theory, the points of the envelope of the one-parameter family of lines defined by (2) satisfy the following equations

$$\begin{cases} f(\lambda,x,y)=\lambda^ky+\lambda^2x-1=0,\\ f_\lambda(\lambda,x,y)=k\lambda^{k-1}y+2\lambda x=0,\end{cases} \tag{3}$$

where $\lambda>0$. Eliminating the two parameters λ from (3), we get the equation of the envelope

$$y(x)=-\frac{2(k-2)^{\frac{k-2}{2}}}{k^{\frac{k}{2}}}x^{\frac{k}{2}},\qquad x>0. \tag{4}$$

From (4), we get

$$y'=-\frac{(k-2)^{\frac{k-2}{2}}}{k^{\frac{k-2}{2}}}x^{\frac{k-2}{2}},\quad y''=-\frac{(k-2)^{\frac{k}{2}}}{2k^{\frac{k-2}{2}}}x^{\frac{k-4}{2}},\quad x>0.$$

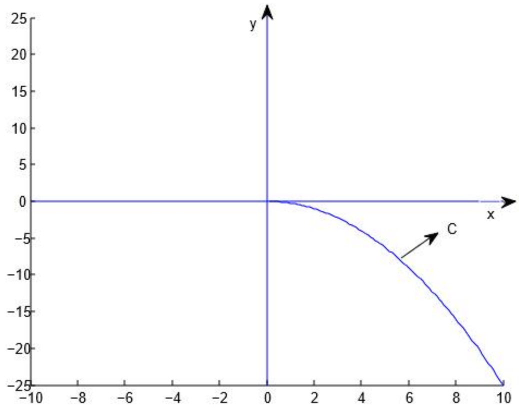

Fig. 1. The envelope curve for $k=4$

Then we have $y(x)<0,y'<0,y''<0$ on $(0,+\infty)$, which implies y is strictly decreasing and convex. Furthermore, $\lim_{x\to0^+}y(x)=\lim_{x\to0^+}\frac{-2(k-2)^{\frac{k-2}{2}}x^{\frac{k}{2}}}{k^{\frac{k}{2}}}=0,\lim_{x\to+\infty}y(x)=\lim_{x\to+\infty}\frac{-2(k-2)^{\frac{k-2}{2}}x^{\frac{k}{2}}}{k^{\frac{k}{2}}}=-\infty$. The envelope C defined by (4) is negative, strictly decreasing and convex as described in Figure 1. It is clearly seen that when the point (p,r) is under the envelope C, namely, $p>0$ and $r<-2(k-2)^{\frac{k-2}{2}}p^{\frac{k}{2}}/k^{\frac{k}{2}}$, there cannot be any tangent line of C which passes through the point (p,r). When the point (p,r) is in the third quadrant, observing that $p\leq0$ and $r<0$, there cannot be any tangent line of C which passes through the point (p,r).

According to lemma 2, (2) has no positive real root if and only if $p\leq0$ and $r\leq0$ or $p>0$ and $r<-2(k-2)^{\frac{k-2}{2}}p^{\frac{k}{2}}/k^{\frac{k}{2}}$.

III. Main Results

In this section, some necessary and sufficient conditions for oscillations of all solutions of equation (1) are established.

Theorem 1 Let $\sigma\geq2,\tau\geq1$ or $\sigma\geq1,\tau\geq2$. Then every solution of equation (1) oscillates if and only if $p\leq0,q\leq0,r\leq0$.

Proof When $\sigma\geq2,\tau\geq1$ or $\sigma\geq1,\tau\geq2$, the characteristic equation of equation (1) is

$$\phi(p,q,r,\lambda,\mu)=p\lambda^2+q\mu^2-1+r\lambda^\sigma\mu^\tau=0. \tag{5}$$

According to lemma 1, we only need to consider the positive solution of (5), that is $\lambda>0,\mu>0$. We will consider (p,q,r) as a point in xyz-space and search for the exact regions including points (p,q,r) such that (5) has no positive root. Actually, $\phi(x,y,z,\lambda,\mu)=0$ can be considered as an equation describing a two-parameter family of planes in xyz-space, where λ,μ are two real parameters. According to the envelop theory, the points of the envelope of the two-parameter family of planes defined by (5) satisfy the following equations

$$\begin{cases} \phi(x,y,z,\lambda,\mu)=0,\\ \phi_\lambda(x,y,z,\lambda,\mu)=2\lambda x+\sigma\lambda^{\sigma-1}\mu^\tau z=0,\\ \phi_\mu(x,y,z,\lambda,\mu)=2\mu y+\tau\lambda^\sigma\mu^{\tau-1}z=0,\end{cases} \tag{6}$$

where $\lambda > 0, \mu > 0$. Eliminating the parameter λ, μ from (6), we obtain the equation of the envelope

$$z(x,y) = -\frac{2(\sigma+\tau-2)^{\frac{\sigma+\tau-2}{2}}}{\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau}{2}}} x^{\frac{\sigma}{2}} y^{\frac{\tau}{2}}, \tag{7}$$

where $x > 0, y > 0$. Consequently we have

$$\begin{aligned}
\frac{\partial z}{\partial x} &= -\frac{(\sigma+\tau-2)^{\frac{\sigma+\tau-2}{2}}}{\sigma^{\frac{\sigma-2}{2}}\tau^{\frac{\tau}{2}}} x^{\frac{\sigma-2}{2}} y^{\frac{\tau}{2}},\\
\frac{\partial z}{\partial y} &= -\frac{(\sigma+\tau-2)^{\frac{\sigma+\tau-2}{2}}}{\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau-2}{2}}} x^{\frac{\sigma}{2}} y^{\frac{\tau-2}{2}},\\
\frac{\partial^2 z}{\partial x^2} &= -\frac{(\sigma-2)(\sigma+\tau-2)^{\frac{\sigma+\tau-2}{2}}}{2\sigma^{\frac{\sigma-2}{2}}\tau^{\frac{\tau}{2}}} x^{\frac{\sigma-4}{2}} y^{\frac{\tau}{2}},\\
\frac{\partial^2 z}{\partial y^2} &= -\frac{(\tau-2)(\sigma+\tau-2)^{\frac{\sigma+\tau-2}{2}}}{2\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau-2}{2}}} x^{\frac{\sigma}{2}} y^{\frac{\tau-4}{2}},\\
\frac{\partial^2 z}{\partial x \partial y} &= -\frac{2(\sigma+\tau-2)^{\frac{\sigma+\tau-2}{2}}}{2\sigma^{\frac{\sigma-2}{2}}\tau^{\frac{\tau-2}{2}}} x^{\frac{\sigma-2}{2}} y^{\frac{\tau-2}{2}}.
\end{aligned}$$

Then $\partial^2 z/\partial x^2 < 0, \partial^2 z/\partial y^2 < 0, \partial^2 z/\partial x^2 \cdot \partial^2 z/\partial y^2 -$

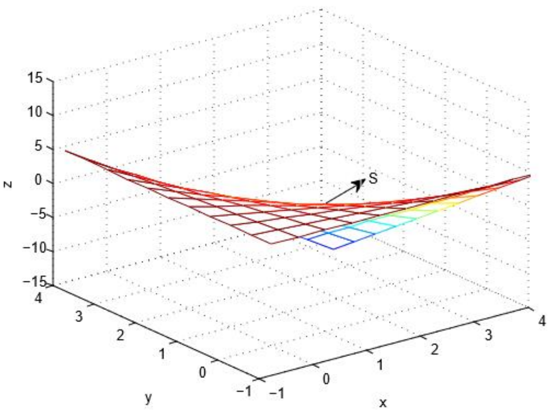

Fig. 2. The envelope surface for $\sigma = 2$, $\tau = 2$

$(\partial^2 z/\partial x \partial y)^2 < 0$, so the function $z(x,y)$ is neither convex nor concave on $(0,+\infty)\times(0,+\infty)$. Furthermore, when $x > 0, y > 0$, $z(x,y) < 0$, $\lim_{(x,y)\to(+\infty,+\infty)} z(x,y) = -\infty$ and $\lim_{(x,y)\to(0^+,0^+)} = 0$. The envelope S defined by (7) is in the fifth quadrant as described in Figure 2. It is clearly seen that when the point (p,q,r) in the seventh quadrant and $p \le 0, q \le 0$ and $r \le 0$, there cannot be any tangent plane of the envelope S which passes through the point (p,q,r). According to lemma 3, the characteristic equation of equation (5) has no positive real root if and only if $p \le 0, q \le 0$ and $r \le 0$. Associated with lemma 1, the proof is accomplished.

Theorem 2 Let $\sigma \ge 3, \tau = 0$. Then every solution of equation (1) oscillates if and only if $p \le 0, q \le 0, r \le 0$ or $p > 0, q \le 0, r < -2(\sigma-2)^{\frac{\sigma-2}{2}} p^{\frac{\sigma}{2}}/\sigma^{\frac{\sigma}{2}}$.

Proof When $\sigma \ge 3, \tau = 0$ the characteristic equation of equation (1) is

$$\phi(p,q,r,\lambda,\mu) = p\lambda^2 + q\mu^2 - 1 + r\lambda^\sigma = 0. \tag{8}$$

If $q > 0$, it is clear that (8) has positive root. So we only need to consider the case $q \le 0$.

(I) For $q = 0$, (8) can be rewritten as

$$\phi(p,r,\lambda) = r\lambda^\sigma - 1 + p\lambda^2 = 0. \tag{9}$$

According to lemma 5, (9) has no positive root if and only if $p \le 0, r \le 0$ or $p > 0$, $r < -\frac{2(\sigma-2)^{\frac{\sigma-2}{2}}}{\sigma^{\frac{\sigma}{2}}} p^{\frac{\sigma}{2}}$.

(II) For $q < 0$, the characteristic equation (8) can be rewritten as

$$\phi(p,q,r,\lambda,\mu) = q(\mu^2 + \frac{p}{q}\lambda^2 - \frac{1}{q} + \frac{r}{q}\lambda^\sigma) = 0. \tag{10}$$

Set

$$F(\frac{1}{q}, \frac{p}{q}, \frac{r}{q}, \lambda) = \frac{p}{q}\lambda^2 - \frac{1}{q} + \frac{r}{q}\lambda^\sigma. \tag{11}$$

Since $q < 0, \lim_{\lambda\to 0^+} F(1/q, p/q, r/q, \lambda) = -1/q > 0$ and $F(1/q,p/q,r/q,\lambda)$ is differentiable on $(0,+\infty)$ with regard to λ. According to lemma 1, we only need to consider the positive solution of (11), that is $\lambda > 0$. Now consider $(1/q, p/q, r/q)$ as a point in xyz-space and search for the exact regions including points $(1/q, p/q, r/q)$ such that (11) has no positive root. Actually, $F(x,y,z,\lambda) = 0$ can be regarded as an equation describing a one-parameter family of planes in xyz-space , where λ is the parameter. According to the envelop theory, the points of the envelope of the one-parameter family of planes defined by (11) satisfy the following equations

$$\begin{cases} F(x,y,z,\lambda) = -x + \lambda^2 y + \lambda^\sigma z = 0, \\ F_\lambda(x,y,z,\lambda) = 2\lambda y + \sigma\lambda^{\sigma-1} z = 0, \end{cases} \tag{12}$$

where $\lambda > 0$. Eliminating the parameter λ from (12), we obtain the equation of the envelope

$$z(x,y) = -\frac{2(\sigma-2)^{\frac{\sigma-2}{2}} y^{\frac{\sigma}{2}}}{\sigma^{\frac{\sigma}{2}} x^{\frac{\sigma-2}{2}}}, \tag{13}$$

where $x < 0, y < 0$. Then

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{(\sigma-2)^{\frac{\sigma}{2}} y^{\frac{\sigma}{2}}}{\sigma^{\frac{\sigma}{2}} x^{\frac{\sigma}{2}}}, \qquad \frac{\partial z}{\partial y} = -\frac{(\sigma-2)^{\frac{\sigma-2}{2}} y^{\frac{\sigma-2}{2}}}{\sigma^{\frac{\sigma-2}{2}} x^{\frac{\sigma-2}{2}}},\\
\frac{\partial^2 z}{\partial x^2} &= -\frac{(\sigma-2)^{\frac{\sigma}{2}} y^{\frac{\sigma}{2}}}{2\sigma^{\frac{\sigma-2}{2}} x^{\frac{\sigma+2}{2}}}, \quad \frac{\partial^2 z}{\partial y^2} = -\frac{(\sigma-2)^{\frac{\sigma}{2}} y^{\frac{\sigma-4}{2}}}{2\sigma^{\frac{\sigma-2}{2}} x^{\frac{\sigma-2}{2}}},\\
\frac{\partial^2 z}{\partial x \partial y} &= \frac{(\sigma-2)^{\frac{\sigma}{2}} y^{\frac{\sigma-2}{2}}}{2\sigma^{\frac{\sigma-2}{2}} x^{\frac{\sigma}{2}}},
\end{aligned}$$

where $x < 0, y < 0$. Thus $z(x,y) > 0, \partial^2 z/\partial x^2 > 0, \partial^2 z/\partial y^2 > 0$ and $\partial^2 z/\partial x^2 \cdot \partial^2 z/\partial y^2 - (\partial^2 z/\partial x \partial y)^2 = 0$, and so $z(x,y)$ is convex on $(-\infty,0)\times(-\infty,0)$. The envelope defined by (13) is a convex surface S in the third quadrant, described in Figure 3. It is clearly seen that in the first place, when the point $(1/q,p/q,r/q)$ is directly over the envelop S, that is, when $1/q < 0, p/q < 0, r/q > -2(\sigma-2)^{\frac{\sigma-2}{2}}(p/q)^{\frac{\sigma}{2}}/\sigma^{\frac{\sigma}{2}}(1/q)^{\frac{\sigma-2}{2}}$, which can be simplified as $p > 0, q < 0$ and $r < -2(\sigma-2)^{\frac{\sigma-2}{2}} p^{\frac{\sigma}{2}}/\sigma^{\frac{\sigma}{2}}$, there cannot be any tangent plane of the envelope S which passes through the point $(1/q,p/q,r/q)$. According to lemma 2', the characteristic equation of equation (8) has no positive real root if and only if $p > 0, q < 0$ and $r < -2(\sigma-2)^{\frac{\sigma-2}{2}} p^{\frac{\sigma}{2}}/\sigma^{\frac{\sigma}{2}}$. In the second place, when the point $(1/q,p/q,r/q)$ is in the second quadrant, which means $q < 0, p/q \ge 0$ and $r/q \ge 0$, and simplified as $p \le 0, q < 0$ and $r \le 0$, similarly there cannot be any tangent plane of the envelope S which passes through the point $(1/q,p/q,r/q)$. Since (8) and (11) has the same positive roots, based on lemma 2', the characteristic equation (8) has no positive real root if and only if $p \le 0, q < 0$ and $r \le 0$ or $p > 0, q < 0$ and $r < -\frac{2(\sigma-2)^{\frac{\sigma-2}{2}}}{\sigma^{\frac{\sigma}{2}}} p^{\frac{\sigma}{2}}$.

Combined with (I) and (II), (8) has no positive real root if and only if $p \le 0, q \le 0$ and $r \le 0$ or $p > 0, q \le 0$ and $r < -2(\sigma-2)^{\frac{\sigma-2}{2}} p^{\frac{\sigma}{2}}/\sigma^{\frac{\sigma}{2}}$. Associated with lemma 1, the proof is accomplished.

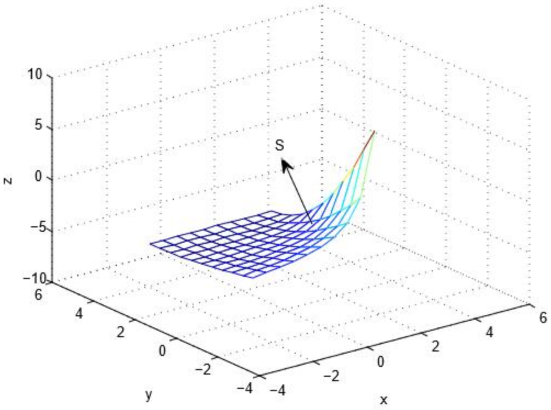

Fig. 3. The envelope surface for $\sigma = 4$, $\tau = 0$

Theorem 3 Let $\sigma = 0, \tau \ge 3$. Then every solution of equation (1) oscillates if and only if $p \le 0, q \le 0, r \le 0$ or $p \le 0, q > 0, r < -2(\tau-2)^{\frac{\tau-2}{2}} q^{\frac{\tau}{2}}/\tau^{\frac{\tau}{2}}$.

Proof The proof is similar to that of theorem 2 since the equation has symmetric property.

Theorem 4 Let $\sigma = 0, \tau = 2$. Then every solution of equation (1) oscillates if and only if $q + r \le 0$.

Proof When $\sigma = 0, \tau = 2$, the characteristic equation of equation (1) is

$$\phi(p,q,r,\lambda,\mu) = p\lambda^2 + (q+r)\mu^2 - 1 = 0. \quad (14)$$

We discuss it in three cases.

(I) When $p = 0$, the characteristic equation is

$$\phi(q,r,\mu) = (q+r)\mu^2 - 1 = 0. \quad (15)$$

It is obvious that the equation (15) has no positive real root if and only if $q + r \le 0$.

(II) When $p < 0$, (14) can be rewritten as

$$\phi(p,q,r,\lambda,\mu) = p(\lambda^2 + \frac{q+r}{p}\mu^2 - \frac{1}{p}) = 0. \quad (16)$$

Set

$$F(\frac{1}{p}, \frac{q+r}{p}, \mu) = \frac{q+r}{p}\mu^2 - \frac{1}{p} = 0. \quad (17)$$

It is obvious that the equation (17) has no positive real root if and only if $q + r \le 0$. Since $\lim_{\mu\to 0^+} F(\frac{1}{p}, \frac{q+r}{p}, \mu) = -\frac{1}{p} > 0$ and $F(\frac{1}{p}, \frac{q+r}{p}, \mu)$ is continuously differentiable with respect to $\mu \in (0, +\infty)$. Based on lemma 4, (14) and (17) has the same positive roots, which implies that the characteristic equation (14) has no positive real root if and only if $q+r \le 0$.

(III)For $p > 0$, we have

$$(q+r)\mu^2 - 1 = -p\lambda^2 \le 0, \quad (18)$$

which means $\mu^2 \le \frac{1}{q+r}$. Thus (18) has no positive root if and only if $q + r < 0$.

Combined with the above three cases, equation (14) has no positive root if and only if $q + r \le 0$. Associated with lemma 1, the proof is accomplished.

Similarly, the following theorem holds true symmetrically.

Theorem 5 Let $\sigma = 2, \tau = 0$. Then every solution of equation (1) oscillates if and only if $p + r \le 0$.

Theorem 6 Let $\sigma = 0, \tau = 1$. Then every solution of equation (1) oscillates if and only if $p \le 0, q < 0, r < \sqrt{-4q}$ or $p > 0, q < 0, r > -\sqrt{-4q}$ or $p > 0, q = 0, r < 0$.

Proof When $\sigma = 0, \tau = 1$, the characteristic equation of equation (1) is

$$\phi(p,q,r,\lambda,\mu) = p\lambda^2 + q\mu^2 - 1 + r\mu = 0. \quad (19)$$

we discuss it in three cases.

(I) $p = 0$. In this case, the characteristic equation has the form

$$\phi(q,r,\mu) = q\mu^2 - 1 + r\mu = 0. \quad (20)$$

According to lemma 1, we only need to consider the positive solution of (20), that is $\mu > 0$. We will consider (q, r) as a point in xy-space and search for the exact regions including points (q, r) such that (20) has no positive root. Actually, $\phi(x, y, \mu) = 0$ can be considered as an equation describing a one-parameter family of planes in xy-space , where μ is the parameter. According to the envelop theory, the points of the envelope of the one-parameter family of lines defined by (20) satisfy the following equations

$$\begin{cases} \phi(x,y,\mu) = \mu^2 x + \mu y - 1 = 0, \\ \phi_\mu(x,y,\mu) = 2\mu x + y = 0, \end{cases} \quad (21)$$

where $\mu > 0$. Eliminating the parameter μ from (21), we obtain the equation of the envelope

$$y(x) = \sqrt{-4x}, \quad (22)$$

where $x < 0$. Consequently,

$$y'(x) = -2 \cdot (-4x)^{-\frac{1}{2}}, \quad y''(x) = -4 \cdot (-4x)^{-\frac{3}{2}}, \quad x < 0.$$

Therefore, from $y(x) > 0, y'(x) < 0, y''(x) < 0, x \in (-\infty, 0)$, we conclude that $y(x)$ is a positive strictly convex on $(-\infty, 0)$ and $\lim_{x\to-\infty} y(x) = \lim_{x\to-\infty} \sqrt{-4x} = +\infty, \lim_{x\to 0^-} y(x) = \lim_{x\to 0^-} \sqrt{-4x} = 0$. The envelope curve defined by (21) is a convex curve C in the second quadrant, described in Figure 4. It is clearly seen that when the point (q, r) is directly below the envelop C, that is $q < 0, r < \sqrt{-4q}$, there cannot be any tangent line of the envelop C pass through the point (q, r). According to lemma 2, equation (20) has no positive root if and only if $q < 0, r < \sqrt{-4q}$.

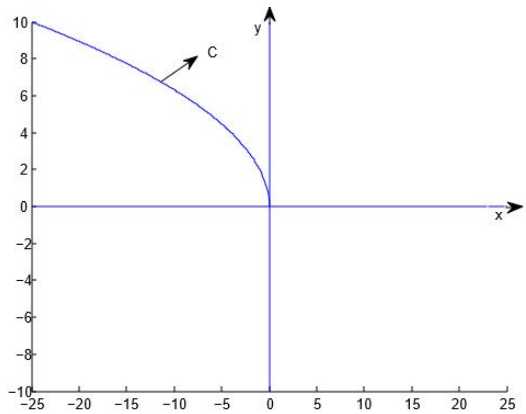

Fig. 4. The envelope curve $y = \sqrt{-4x}$

(II) For $p < 0$, (19) can be rewritten as

$$\phi(p,q,r,\lambda,\mu) = p(\lambda^2 + \frac{q}{p}\mu^2 - \frac{1}{p} + \frac{r}{p}\mu) = 0. \quad (23)$$

Set

$$F(\frac{1}{p},\frac{q}{p},\frac{r}{p},\mu) = \frac{q}{p}\mu^2 - \frac{1}{p} + \frac{r}{p}\mu = 0. \quad (24)$$

We have $\lim_{\mu\to 0^+} F(\frac{1}{p},\frac{q}{p},\frac{r}{p},\mu) = -\frac{1}{p} > 0$ and $F(\frac{1}{p},\frac{q}{p},\frac{r}{p},\mu)$ is continuously differentiable with respect to $\mu \in (0,+\infty)$. According to lemma 4, (23) and (24) has the same positive roots. Then by lemma 1, we only need to study the positive solution of (24), that is $\mu > 0$. Consider $(1/p, q/p, r/p)$ as a point in xyz-space and search for the exact regions including points $(1/p, q/p, r/p)$ such that (20) has no positive root. Actually, $F(x,y,z,\mu) = 0$ can be considered as an equation describing a one-parameter family of plane in xyz-space , where μ is the parameter. According to the envelop theory, the points of the envelope of the one-parameter family of planes defined by (24) satisfy the following equations

$$\begin{cases} F(x,y,z,\mu) = -x + \mu^2 y + \mu z = 0, \\ F_\mu(x,y,z,\mu) = 2\mu y + z = 0, \end{cases} \quad (25)$$

where $\mu > 0$. Eliminating the parameter μ from (25), we obtain the equation of the envelope

$$z(x,y) = -\sqrt{-4xy}, \quad (26)$$

where $x < 0, y > 0$. Thus we have

$$\frac{\partial z}{\partial x} = 2y \cdot (-4xy)^{-\frac{1}{2}}, \qquad \frac{\partial z}{\partial y} = 2x \cdot (-4xy)^{-\frac{1}{2}},$$
$$\frac{\partial^2 z}{\partial x^2} = 4y^2 \cdot (-4xy)^{-\frac{3}{2}}, \qquad \frac{\partial^2 z}{\partial y^2} = 4x^2 \cdot (-4xy)^{-\frac{3}{2}},$$
$$\frac{\partial^2 z}{\partial x \partial y} = (-4xy)^{-\frac{1}{2}}, \quad x < 0, y > 0.$$

Therefore, for $x < 0, y > 0, z(x,y) < 0, \partial^2 z/\partial x^2 > 0, \partial^2 z/\partial y^2 > 0$ and $\partial^2 z/\partial x^2 \cdot \partial^2 z/\partial y^2 - (\partial^2 z/\partial x \partial y)^2 = 0$, which imply that $z(x,y)$ is convex on $(-\infty, 0) \times (0, +\infty)$. The envelope plane defined by (19) is a convex surface S in the sixth quadrant , described in Figure 5. It is clearly seen that when the point $(1/p, q/p, r/p)$ is directly over S, that is $1/p < 0, q/p > 0, r/p > -\sqrt{-4 \cdot \frac{1}{p} \cdot \frac{q}{p}}$, which can be simplified as $p < 0, q < 0$ and $r < \sqrt{-4q}$, there cannot be any tangent plane of the envelope S which passes through the point. For (23) and (24) has the same positive root, based on lemma 4 the characteristic equation has no positive root if and only if $p < 0, q < 0$ and $r < \sqrt{-4q}$.

(III) For $p > 0$, we have $q\mu^2 + r\mu - 1 = -p\lambda^2 \le 0$. Set

$$f(q,r,\mu) = q\mu^2 + r\mu - 1. \quad (27)$$

(i) When $q = 0$, then the equation $f(q,r,\mu) = 0$ can be rewritten as $r\mu - 1 = 0$, which implies $\mu = \frac{1}{r} > 0$ if $r > 0$. Therefore, when $p > 0, q = 0, r < 0$, the equation $f(q,r,\mu) = 0$ has no positive solution.

(ii) When $q \ne 0$, the equation $f(q,r,\mu) = 0$ can be regarded as a function of μ and the root μ_1, μ_2 satisfy

$$\begin{cases} \mu_1 + \mu_2 = -\frac{r}{q}, \\ \mu_1 \cdot \mu_2 = -\frac{1}{q}. \end{cases} \quad (28)$$

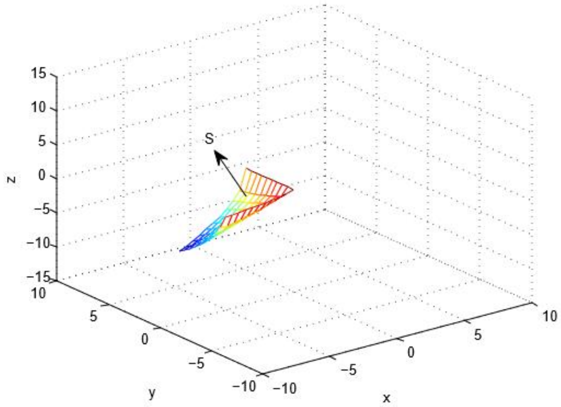

Fig. 5. The envelope surface $z = -\sqrt{-4xy}$

We can see that

(a) if $q > 0$, $r^2 + 4q > 0$ and $\mu_1\mu_2 < 0$, consequently, $f(q,r,\mu) = 0$ has one positive solution.

(b) if $q < 0$, if $r^2 + 4q < 0$, $f(q,r,\mu) = 0$ has no positive solution; if $r^2 + 4q \ge 0$ and $r < 0$, $f(q,r,\mu) = 0$ has just one positive solution; if $r^2 + 4q \ge 0$ and $r > 0$, $f(q,r,\mu) = 0$ has two positive solutions.

Combined with the above cases, equation (19) has no positive root if and only if $p \le 0, q < 0, r < \sqrt{-4q}$ or $p > 0, q < 0, r > -\sqrt{-4q}$ or $p > 0, q = 0, r < 0$. Associated with lemma 1, the proof is accomplished.

Similarly, using the symmetric property, we have the following theorem.

Theorem 7 If $\sigma = 1, \tau = 0$. Then every solution of equation (1) oscillates if and only if $p < 0, q \le 0, r < \sqrt{-4p}$ or $p < 0, q > 0, r > -\sqrt{-4p}$ or $p = 0, q > 0, r < 0$.

Theorem 8 Let $\sigma = 1, \tau = 1$. If $p \le \frac{r^2}{4q}, q < 0, r > 0$ or $p < 0, q = 0, r < 0$, then every solution of equation (1) oscillates.

Proof Set $\mu = c\lambda (c > 0)$, the characteristic equation is

$$\phi(p,q,r,\lambda) = p\lambda^2 + q\mu^2 - 1 + r\lambda\mu = (p + c^2 q + cr)\lambda^2 - 1 = 0. \quad (29)$$

To prove (29) has no positive root, we only need to prove $qc^2 + rc + p \le 0$. Set

$$f(p,q,r,c) = qc^2 + rc + p. \quad (30)$$

Since $c > 0$, the symmetric axis of (30) should be positive, that is, $\frac{-r}{2q} > 0$. We consider the following three cases.

(I) When $q < 0$, we have $r > 0$. If $p \le \frac{r^2}{4q}$, we have $\max_{c>0} f(p,q,r,c) = \frac{4pq - r^2}{4q} \le 0$. Consequently, $f(p,q,r,c) \le 0$, which implies that (29) has no positive root.

(II) When $q > 0$, we have $r < 0$. Since (30) is an upwards parabola, f can not be always negative, which means (29) could have positive root.

(III) When $q = 0$, $f(p,q,r,c) = rc + p$. If $r < 0, p < 0$, with the condition $c > 0$, we have $f(p,q,r,c) < 0$. Then (29) has no positive root.

Combined with lemma 1, we conclude by the above three cases that the theorem holds.

Theorem 9 Let $\sigma = 0, \tau = 0$. If $p \ge 0, q \ge 0, r > 1$ or $p \le 0, q \le 0, r < 1$ or $p^2 + q^2 \ne 0, r = 1$, then every solution of equation (1) oscillates.

Proof Set $\mu = c\lambda (c > 0)$, the characteristic equation is

$$\phi(p, q, r, \lambda) = p\lambda^2 + q\mu^2 - 1 + r = (p + c^2 q)\lambda^2 - 1 + r = 0. \quad (31)$$

We consider the following three cases

(I) When $r = 1$, (31) has no positive root as long as $p^2 + q^2 \neq 0$.

(II) When $r > 1$, if $p \geq 0, q \geq 0$, then (31) has no positive root.

(III) When $r < 1$, if $p \leq 0, q \leq 0$, then (31) has no positive root.

Combined with lemma 1, we conclude by the above three cases that the theorem holds.

IV. Illustrative examples

In this section, we give some examples to illustrate the results obtained in Section 3.

Example 1 Consider the partial difference equation

$$-0.2u_{m+2,n} - 0.1u_{m,n+2} - u_{m,n} - u_{m+2,n+2} = 0, \quad (32)$$

where $\sigma = 2, \tau = 2$. Since $p = -0.2 < 0$, $q = -0.1 < 0$ and $r = -1 < 0$, by theorem 1, every solution of equation (32) is oscillatory. The oscillatory behavior of equation (32) is demonstrated by Figure 6.

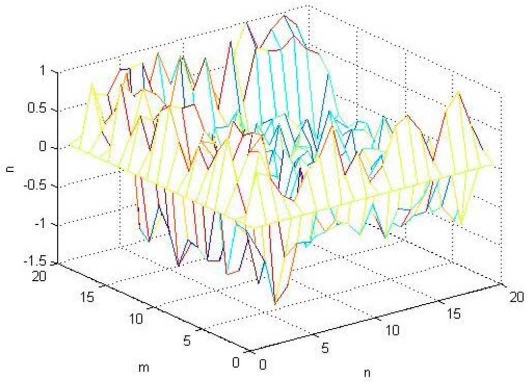

Fig. 6. Oscillatory behavior of (32)

Example 2 Consider the partial difference equation

$$0.2u_{m+2,n} - u_{m,n+2} - u_{m,n} - 0.02u_{m+4,n} = 0, \quad (33)$$

where $\sigma = 4, \tau = 0$. Since $p = 0.2 > 0$, $q = -1 < 0$, $r = -0.02 < -0.01 = -2(\sigma-2)^{\frac{\sigma-2}{2}} p^{\frac{\sigma}{2}} / \sigma^{\frac{\sigma}{2}}$, by theorem 2, every solution of equation (33) is oscillatory. The oscillatory behavior of equation (33) is demonstrated by Figure 7.

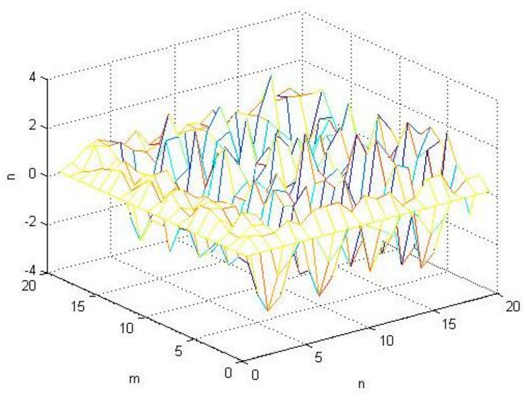

Fig. 7. Oscillatory behavior of (33)

Example 3 Consider the partial difference equation

$$-0.2u_{m+2,n} - 0.1u_{m,n+2} - u_{m,n} - 0.1u_{m+2,n} = 0, \quad (34)$$

where $\sigma = 2, \tau = 0$. Since $q = -0.1 < 0$ and $r = -0.1$ imply that $p + r = -0.3 < 0$, by theorem 5, every solution of equation (34) is oscillatory. The oscillatory behavior of equation (34) is demonstrated by Figure 8.

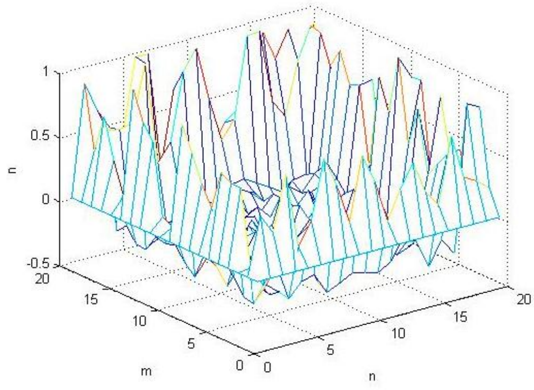

Fig. 8. Oscillatory behavior of (34)

Example 4 Consider the partial difference equation

$$-0.01u_{m+2,n} - 0.5u_{m,n+2} - u_{m,n} + 0.03u_{m+1,n} = 0, \quad (35)$$

where $\sigma = 1, \tau = 0$. Since $p = -0.01 < 0$, $q = -0.5 < 0$ and $r = 0.03 < 0.2 = \sqrt{-4p}$ imply that $p + r = -0.3 < 0$, by theorem 7, every solution of equation (35) is oscillatory. The oscillatory behavior of equation (35) is demonstrated by Figure 9.

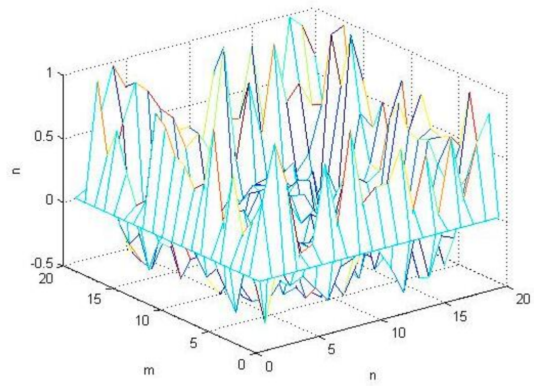

Fig. 9. Oscillatory behavior of (35)

Example 5 Consider the partial difference equation

$$-0.6u_{m+2,n} - 0.5u_{m,n+2} - u_{m,n} + 0.2u_{m+1,n+1} = 0, \quad (36)$$

where $\sigma = 1, \tau = 1$. Since $p = -0.6 < \frac{r^2}{4q}$, $q = -0.5 < 0$, $r = 0.2 > 1$, by theorem 8, every solution of equation (36) is oscillatory. The oscillatory behavior of equation (36) is demonstrated by Figure 10.

Example 6 Consider the partial difference equation

$$-0.24u_{m+2,n} - 0.18u_{m,n+2} - u_{m,n} + 0.12u_{m,n} = 0, \quad (37)$$

where $\sigma = 0, \tau = 0$. Since $p = -0.24 < 0$, $q = -0.18 < 0$, $r = 0.12 < 1$, by theorem 9, every solution of equation (37) is oscillatory. The oscillatory behavior of equation (37) is demonstrated by Figure 11.

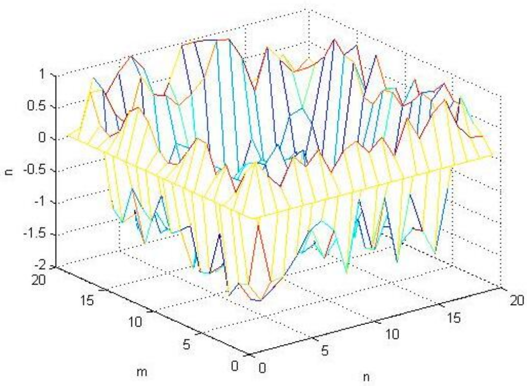

Fig. 10. Oscillatory behavior of (36)

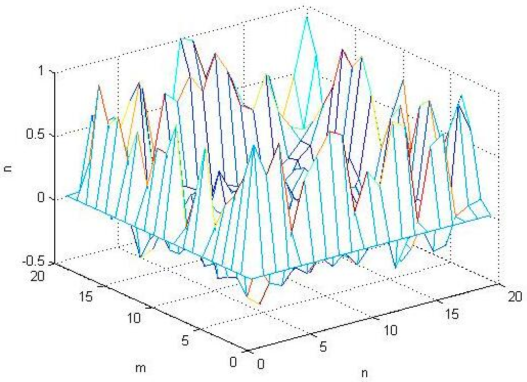

Fig. 11. Oscillatory behavior of (37)

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