Mean Square Asymptotic Analysis of Discretely Observed Hybrid Stochastic Systems by Feedback Control

Chao Wei, Xiaoyin Li, Xiang Zhang and Zhaoqian Liu

Abstract—**This paper is concerned with the mean square asymptotically boundedness control of hybrid stochastic systems with Markovian switching from discrete-time observations. Firstly, by using generalized Itô formula, stochastic analysis for martingale and Holder's inequality, the mean square asymptotically boundedness of the controlled system with common linear feedback control function is discussed. Secondly, by applying stochastic analysis for martingale and Cauchy-Schwarz inequality, the mean square asymptotically boundedness of the controlled system with the general form control function is studied. Finally, numerical examples are provided to show the usefulness of the proposed mean square asymptotically boundedness criterion.**

Index Terms—**Hybrid stochastic system, feedback control function, mean square asymptotically boundedness, discrete-time observations.**

I. Introduction

Hybrid stochastic systems, which describe the system may suffering abrupt changes in coefficients or structure, have been widely used in finance, biology, engineering, etc [1]–[7]. For the detailed introduction, we refer the readers to monographs [9], [10] and the references therein. The stabilization for such systems has been broadly studied as an important aspect of automatic control theory [11]–[17]. Recently, [18] introduced the state feedback stabilization theory for such systems based on discrete-time observations:

$$\begin{aligned} dx(t) &= [f(x(t), r(t), t) + u(x([t/\tau]\tau, r(t), t)]dt \\ &\quad + g(x(t), r(t), t)dB(t), \end{aligned} \tag{1}$$

where the discrete time gap τ is a positive constant and $[t/\tau]$ is the integer part of t/τ. It means the control function only require the states observations at $0, \tau, 2\tau, ...$ This kind of control mode is more realistic and economic obviously. After [18], Mao and his group have a series of studies [19]–[21] to improve the results or fix the relevant issues. We have to claim that there are a lot of studies, such as [22]–[24], have discussed the same idea for deterministic systems.

Meanwhile, sometime it is hard to stabilize the system or even impossible. In fact, one may only needs the controlled system to be bounded by a constant which is independent of the initial data. The boundedness control for a various of systems have been widely studied, such as [25]–[29]. There are some studies with different definitions that are similar to boundedness basically, like [30]. But to the best of our knowledge, there are few researches discuss the boundedness control for the stochastic system (1) based on discrete-time observations. Therefore, it is important to discuss the mean square asymptotically boundedness to such controlled system. In this paper, by using generalized Itô formula, stochastic analysis for martingale and Cauchy-Schwarz inequality and Holder's inequality, the mean square asymptotically boundedness of one controlled system with common linear feedback control function and the other one with the control function hasing a more general form are studied from discrete-time observations and numerical examples are provided to show the usefulness of the proposed mean square asymptotically boundedness criterion.

This paper is constructed in the following way. In Section II some mathematical preliminaries and basic assumptions are given. Section III discusses the common situation of a linear feedback control function. Section IV is devoted to more general situation with a more general result by the Lyapunov method. A brief example and numerical simulations are displayed in Section V.

II. Problem Formulation and Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions that it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. For a vector or matrix A, A^T denotes its transpose. For $x \in R^n$, $|x|$ denotes its Euclidean norm. $|A| = \sqrt{\text{trace}(A^T A)}$ and $\|A\| = \max\{|Ax| : |x| = 1\}$ denote the trace and operator norms of a matrix A. For a symmetric matrix A i.e. $A = A^T$, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote its smallest and largest eigenvalues respectively. We mean A is non-positive and negative definite by $A \leq 0$ and $A < 0$. Denote by $L^2_{\mathcal{F}_t}(R^n)$ the family of all $\mathcal{F}_t$-measurable R^n-valued random variables ξ such that $\mathbb{E}|\xi|^2 < \infty$, where $\mathbb{E}$ is the expectation with respect to the probability measure $\mathbb{P}$. For a non-negative real number a, let $[a]$ denote the integer part of a.

Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j

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if $i \neq j$ while

$$\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is known that $r(t)$ is a time-continuous and state-discrete Markov chain. Thus, for any finite subinterval of it when $t \in [0, \infty)$, $r(t)$ only have a finite number of jumps. And except these jumps times, almost all path of $r(t)$ are constant. We stress that almost all sample paths of $r(t)$ are right continuous.

Consider an n-dimensional controlled hybrid SDE

$$\begin{aligned} dx(t) &= [f(x(t), r(t), t) + u(x(\delta(t)), r(t), t)]dt \\ &\quad + g(x(t), r(t), t)dB(t), \end{aligned} \tag{2}$$

on $t \geq 0$, with initial data $x(0) = x_0 \in L^2_{\mathcal{F}_0}(R^n)$ and $r(0) = r_0 \in M_{\mathcal{F}_0}(S)$. Here $\tau > 0$ and

$$\delta(t) = [t/\tau]\tau, \tag{3}$$

in which $[t/\tau]$ is the integer part of t/τ. Where the original system, which is (2) with $u(x(t), r(t), t) = 0$, is unbounded.

Assumption 2.1: Assume that both f and g are locally Lipschitz. Moreover, assume that there are positive constants a_1, a_2, b_1, b_2 such that both f and g satisfied the following linear growth condition

$$|f(x, i, t)|^2 \leq a_1|x|^2 + b_1, |g(x, i, t)|^2 \leq a_2|x|^2 + b_2, \tag{4}$$

for all $(x, i, t) \in R^n \times S \times R_+$.

Because $u(x, i, t)$ are human-designed function, therefore we have the following assumption.

Assumption 2.2: Assume that there are positive constants a_3, b_3, such that u satisfy the following globally Lipschitz condition and linear growth condition

$$|u(x, i, t) - u(y, i, t)|^2 \leq a_3|x - y|^2, \tag{5}$$

$$|u(x, i, t)|^2 \leq a_3|x|^2 + b_3, \tag{6}$$

for all $(x, y, i, t) \in R^n \times R^n \times S \times R_+$.

To get our main result, let us present a useful lemma.

Lemma 2.3: Let Assumptions 2.1 and 2.2 hold. For any initial data $(x_0, r_0, 0)$, write $x(t; x_0, r_0, 0) = x(t)$. If the time gap τ satisfy that $K_1(\tau) < 1$, then, for all $t \geq 0$, we have

$$\mathbb{E}|x(t) - x(\delta(t))|^2 \leq \frac{K_1(\tau)}{1 - K_1(\tau)}\mathbb{E}|x(t)|^2 + \frac{K_2(\tau)}{1 - K_1(\tau)} \tag{7}$$

where

$$K_1(\tau) = 6\tau(2\tau a_1 + 2a_2 + \tau a_3)e^{6\tau(\tau a_1 + a_2)}, \tag{8}$$

and

$$K_2(\tau) = 3\tau(\tau b_1 + b_2 + \tau b_3)e^{6\tau(\tau a_1 + a_2)}, \tag{9}$$

are both positive.

Proof: For any $t > 0$, there exists a integer k such that $t \in [k\tau, (k+1)\tau)$, which means $\delta(t) = k\tau$ as well. It is easy to see that

$$\begin{aligned} & x(t) - x(\delta(t)) \\ &= \int_{k\tau}^{t} [f(x(s), r(s), s) + u(x(\delta(s)), r(s), s)]ds \\ &+ \int_{k\tau}^{t} g(x(s), r(s), s)dB(s). \end{aligned}$$

By Assumption 2.1 and 2.2, we can derive

$$\begin{aligned} & \mathbb{E}|x(t) - x(\delta(t))|^2 \\ \leq\ & 3(\tau a_1 + a_2)\int_{k\tau}^{t} \mathbb{E}|x(s)|^2 ds + 3\tau^2 a_3 \mathbb{E}|x(k\tau)|^2 \\ & + 3\tau(\tau b_1 + b_2 + \tau b_3) \\ \leq\ & 6(\tau a_1 + a_2)\int_{k\tau}^{t} \mathbb{E}|x(s) - x(k\tau)|^2 ds + \\ & 3\tau(2\tau a_1 + 2a_2 + \tau a_3)\mathbb{E}|x(k\tau)|^2 \\ & + 3\tau(\tau b_1 + b_2 + \tau b_3) \\ \leq\ & [3\tau(2\tau a_1 + 2a_2 + \tau a_3)\mathbb{E}|x(k\tau)|^2 \\ & + 3\tau(\tau b_1 + b_2 + \tau b_3)]e^{6\tau(\tau a_1 + a_2)} \\ \leq\ & K_1(\tau)(\mathbb{E}|x(t) - x(\delta(t))|^2 + \mathbb{E}|x(t)|^2) + K_2(\tau) \\ \leq\ & \frac{K_1(\tau)}{1 - K_1(\tau)}\mathbb{E}|x(t)|^2 + \frac{K_2(\tau)}{1 - K_1(\tau)}. \end{aligned}$$

The proof is completed. ■

Remark 2.4: If the n-dimensional controlled hybrid SDE is changed as follows

$$\begin{aligned} dx(t) &= [f(x(t), r(t), t) + u(x(\delta(t)), r(t), t)]dt \\ &\quad + \int_Y g(x(t), r(t), t)N(dt, dy), \end{aligned} \tag{10}$$

where $t \geq 0$, with initial data $x(0) = x_0 \in L^2_{\mathcal{F}_0}(R^n)$ and $r(0) = r_0 \in M_{\mathcal{F}_0}(S)$, $N(t, y)$ is an l-dimensional $\mathscr{F}_t$-adapted Poisson random measure on $[0, +\infty) \times \mathbb{R}^l$ with compensator $\widetilde{N}(t, y)$ which satisfies $\widetilde{N}(dt, dy) = N(dt, dy) - \lambda\phi(dy)dt$, where λ is the probability density of Poisson process and ϕ is the probability distribution of y, the Lemma 2.3 is still correct.

Lemma 2.5: Let Assumptions 2.1 and 2.2 hold, together with

$$\int_Y |g(x, i, t, y)|^2 \nu(dy) \leq \lambda|x|^2 + \beta.$$

For any initial data $(x_0, r_0, 0)$, write $x(t; x_0, r_0, 0) = x(t)$. If the time gap τ satisfy that $M_1(\tau) < 1$, then, for all $t \geq 0$, we have

$$\mathbb{E}|x(t) - x(\delta(t))|^2 \leq \frac{M_1(\tau)}{1 - M_1(\tau)}\mathbb{E}|x(t)|^2 + \frac{M_2(\tau)}{1 - M_1(\tau)}$$

where

$$M_1(\tau) = 6\tau(2\tau a_1 + 2\lambda + \tau a_3)e^{6\tau(\tau a_1 + \lambda)},$$

and

$$M_2(\tau) = 3\tau(\tau b_1 + \beta + \tau b_3)e^{6\tau(\tau a_1 + \lambda)},$$

are both positive.

Proof: For any $t > 0$, there exists a integer k such that $t \in [k\tau, (k+1)\tau)$, which means $\delta(t) = k\tau$ as well. It is easy to see that

$$\begin{aligned} & x(t) - x(\delta(t)) \\ &= \int_{k\tau}^{t} [f(x(s), r(s), s) + u(x(\delta(s)), r(s), s)]ds \\ &+ \int_{k\tau}^{t}\int_Y g(x(s), r(s), s)N(ds, dy). \end{aligned}$$

Then, we can obtain that

$$
\begin{aligned}
& \mathbb{E}|x(t)-x(\delta(t))|^2 \\
\leq\ & 3(\tau a_1+\lambda)\int_{k\tau}^{t}\mathbb{E}|x(s)|^2 ds+3\tau^2 a_3\mathbb{E}|x(k\tau)|^2 \\
& +3\tau(\tau b_1+\beta+\tau b_3) \\
\leq\ & 6(\tau a_1+\lambda)\int_{k\tau}^{t}\mathbb{E}|x(s)-x(k\tau)|^2 ds+ \\
& 3\tau(2\tau a_1+2\lambda+\tau a_3)\mathbb{E}|x(k\tau)|^2 \\
& +3\tau(\tau b_1+\beta+\tau b_3) \\
\leq\ & [3\tau(2\tau a_1+2\lambda+\tau a_3)\mathbb{E}|x(k\tau)|^2 \\
& +3\tau(\tau b_1+\beta+\tau b_3)]e^{6\tau(\tau a_1+\lambda)} \\
\leq\ & M_1(\tau)(\mathbb{E}|x(t)-x(\delta(t))|^2+\mathbb{E}|x(t)|^2)+M_2(\tau) \\
\leq\ & \frac{M_1(\tau)}{1-M_1(\tau)}\mathbb{E}|x(t)|^2+\frac{M_2(\tau)}{1-M_1(\tau)}.
\end{aligned}
$$

The proof is completed. ■

Remark 2.6: If the n-dimensional controlled hybrid SDE is changed as follows

$$
\begin{aligned}
dx(t) &= [f(x(t),r(t),t)+u(x(\delta(t)),r(t),t)]dt \\
&\quad +g(x(t),r(t),t)dZ(t), \qquad (11)
\end{aligned}
$$

where $t \geq 0$, with initial data $x(0)=x_0 \in L^2_{\mathcal{F}_0}(R^n)$ and $r(0)=r_0 \in M_{\mathcal{F}_0}(S)$, $Z=\{Z_t, t\geq 0\}$ is a strictly symmetric α-stable Lévy motion.

A random variable η is said to have a stable distribution with index of stability $\alpha \in (0,2]$, scale parameter $\sigma \in (0,\infty)$, skewness parameter $\beta \in [-1,1]$ and location parameter $\mu \in (-\infty,\infty)$ if it has the following characteristic function:

$$
\phi_\eta(u)=\begin{cases}\exp\{-\sigma^\alpha|u|^\alpha(1-i\beta sgn(u)\tan\frac{\alpha\pi}{2})+i\mu u\}, \\ \exp\{-\sigma|u|(1+i\beta\frac{2}{\pi}sgn(u)\log|u|)+i\mu u\}.\end{cases}
$$

We denote $\eta \sim S_\alpha(\sigma,\beta,\mu)$. When $\mu=0$, we say η is strictly α-stable, if in addition $\beta=0$, we call η symmetrical α-stable. Throughout this paper, it is assumed that α-stable motion is strictly symmetrical and $\alpha \in (1,2)$.

Lemma 2.7: Let Assumptions 2.1 and 2.2 hold, together with

$$\sup_t \mathbb{E}|X_t|^2<\infty.$$

For any initial data $(x_0,r_0,0)$, write $x(t;x_0,r_0,0)=x(t)$. If the time gap τ satisfy that $H_1(\tau)<1$, then, for all $t\geq 0$, we have

$$\mathbb{E}|x(t)-x(\delta(t))|^2 \leq \frac{H_1(\tau)}{1-H_1(\tau)}\mathbb{E}|x(t)|^2+\frac{H_2(\tau)}{1-H_1(\tau)}$$

where

$$H_1(\tau)=6\tau(2\tau a_1+\tau a_3)e^{6\tau^2 a_1},$$

and

$$H_2(\tau)=(6\tau(\tau b_1+\tau b_3)+6(K\tau)^{\frac{2}{\alpha}})e^{6\tau^2 a_1},$$

are both positive.

Proof: For any $t>0$, there exists a integer k such that $t\in[k\tau,(k+1)\tau)$, which means $\delta(t)=k\tau$ as well. It is easy to see that

$$
\begin{aligned}
& x(t)-x(\delta(t)) \\
& =\int_{k\tau}^{t}[f(x(s),r(s),s)+u(x(\delta(s)),r(s),s)]ds \\
& +\int_{k\tau}^{t} g(x(s),r(s),s)dZ(s).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
& \mathbb{E}|x(t)-x(\delta(t))|^2 \\
\leq\ & 3\tau a_1\int_{k\tau}^{t}\mathbb{E}|x(s)|^2 ds+3\tau^2 a_3\mathbb{E}|x(k\tau)|^2 \\
& +3\tau^2 b_1+3\mathbb{E}|\int_{k\tau}^{t} g(x(s),r(s),s)dZ(s)|^2 \\
\leq\ & 6\tau a_1\int_{k\tau}^{t}\mathbb{E}|x(s)-x(k\tau)|^2 ds+ \\
& 3\tau(2\tau a_1+\tau a_3)\mathbb{E}|x(k\tau)|^2+3\tau(\tau b_1+\tau b_3) \\
& +3(K\tau)^{\frac{2}{\alpha}} \\
\leq\ & [3\tau(2\tau a_1+\tau a_3)\mathbb{E}|x(k\tau)|^2 \\
& +3\tau(\tau b_1+\tau b_3)+3(K\tau)^{\frac{2}{\alpha}}]e^{6\tau^2 a_1} \\
\leq\ & H_1(\tau)(\mathbb{E}|x(t)-x(\delta(t))|^2+\mathbb{E}|x(t)|^2)+H_2(\tau) \\
\leq\ & \frac{H_1(\tau)}{1-H_1(\tau)}\mathbb{E}|x(t)|^2+\frac{H_2(\tau)}{1-H_1(\tau)}.
\end{aligned}
$$

The proof is completed. ■

Remark 2.8: If the n-dimensional controlled hybrid SDE is changed as follows

$$
\begin{aligned}
dx(t) &= [f(x(t),r(t),t)+u(x(\delta(t)),r(t),t)]dt \\
&\quad +g(x(t),r(t),t)dB(t)+dL_t, \qquad (12)
\end{aligned}
$$

where $t\geq 0$, with initial data $x(0)=x_0\in L^2_{\mathcal{F}_0}(R^n)$ and $r(0)=r_0\in M_{\mathcal{F}_0}(S)$, L_t is Lévy noises and $B(t)$ is Brownian motion.

Lemma 2.9: Let Assumptions 2.1 and 2.2 hold, together with

$$\sup_t \mathbb{E}|X_t|^2<\infty.$$

For any initial data $(x_0,r_0,0)$, write $x(t;x_0,r_0,0)=x(t)$. If the time gap τ satisfy that $F_1(\tau)<1$, then, for all $t\geq 0$, we have

$$\mathbb{E}|x(t)-x(\delta(t))|^2 \leq \frac{F_1(\tau)}{1-F_1(\tau)}\mathbb{E}|x(t)|^2+\frac{F_2(\tau)}{1-F_1(\tau)}$$

where

$$F_1(\tau)=6\tau(2\tau a_1+2+\tau a_3)e^{6\tau(\tau a_1+1)},$$

and

$$F_2(\tau)=3\tau(\tau b_1+1+\tau b_3)e^{6\tau(\tau a_1+1)},$$

are both positive.

Proof: For any $t>0$, there exists a integer k such that $t\in[k\tau,(k+1)\tau)$, which means $\delta(t)=k\tau$ as well. It is easy to see that

$$
\begin{aligned}
& x(t)-x(\delta(t)) \\
& =\int_{k\tau}^{t}[f(x(s),r(s),s)+u(x(\delta(s)),r(s),s)]ds \\
& +\int_{k\tau}^{t} g(x(s),r(s),s)dB(t)+\int_{k\tau}^{t} dL_t.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
& \mathbb{E}|x(t)-x(\delta(t))|^2 \\
\leq\ & 3\tau a_1\int_{k\tau}^{t}\mathbb{E}|x(s)|^2ds+3\tau^2a_3\mathbb{E}|x(k\tau)|^2 \\
& +3\tau^2b_1+3\mathbb{E}|\int_{k\tau}^{t}g(x(s),r(s),s)dB(s)|^2 \\
& +3\mathbb{E}|\int_{k\tau}^{t}dL(s)|^2 \\
\leq\ & 6\tau a_1\int_{k\tau}^{t}\mathbb{E}|x(s)-x(k\tau)|^2ds+ \\
& 3\tau(2\tau a_1+\tau a_3)\mathbb{E}|x(k\tau)|^2+3\tau(\tau b_1+\tau b_3) \\
& +3(K\tau)^{\frac{2}{\alpha}} \\
\leq\ & [3\tau(2\tau a_1+1+\tau a_3)\mathbb{E}|x(k\tau)|^2 \\
& +3\tau(\tau b_1+2+\tau b_3)+3(K\tau)^{\frac{2}{\alpha}}]e^{6\tau^2a_1} \\
\leq\ & F_1(\tau)(\mathbb{E}|x(t)-x(\delta(t))|^2+\mathbb{E}|x(t)|^2)+F_2(\tau) \\
\leq\ & \frac{F_1(\tau)}{1-F_1(\tau)}\mathbb{E}|x(t)|^2+\frac{F_2(\tau)}{1-F_1(\tau)}.
\end{aligned}
$$

The proof is completed. ■

III. LINEAR FEEDBACK CONTROL

In this section, let us first consider a linear control function $u(x,i,t)=D(i)x(t)$ at first, such that the controlled system is

$$
\begin{aligned}
dx(t) &= [f(x(t),r(t),t)+D(r(t))x(\delta(t))]dt \\
&\quad +g(x(t),r(t),t)dB(t).
\end{aligned} \tag{13}
$$

It is easy to see that system (13) fulfills Lemma 2.3 with $a_3=\eta_D$ and $b_3=0$, where $\eta_D=\max_{i\in S}\|D_i\|^2$. Now let us state our main result.

Theorem 3.1: Assume that for all $(x,i,t)\in R^n\times S\times R_+$, there exist a pair of symmetric matrices Q_i and $\hat{Q}_i$, such that

$$2x^TQ_if(x,i,t)+g^T(x,i,t)Q_ig(x,i,t)\leq x^T\hat{Q}_ix+\beta, \tag{14}$$

where Q_i is positive-definite and $\beta\geq 0$. And there exists solutions D_i to the following LMIs

$$P_i=\hat{Q}_i+2Q_iD_i+\sum_{j=1}^{N}\gamma_{ij}Q_j<0. \tag{15}$$

Set $0<\lambda_m=\min_{i\in S}\lambda_{min}(Q_i)$, $0<\lambda_M=\max_{i\in S}\lambda_{max}(Q_i)$, $0>-\alpha=\max_{i\in S}\lambda_{max}(P_i)$ and $\eta_{QD}=\max_{i\in S}\|Q_iD_i\|^2$. If the time gap τ satisfy that

$$K_1(\tau)<\frac{\alpha^2}{4\eta_{QD}+\alpha^2}. \tag{16}$$

Then the solution of system (13) satisfies

$$\lim_{t\to\infty}\mathbb{E}|x_t|^2\leq\frac{1}{\theta\lambda_m}[\beta+\frac{\eta_{QD}K_2(\tau)}{\alpha_\tau[1-K_1(\tau)]}], \tag{17}$$

where $0<\theta=\frac{\alpha-2\alpha_\tau}{\lambda_M}$, $\alpha_\tau=\sqrt{\frac{\eta_{QD}K_1(\tau)}{1-K_1(\tau)}}$ and K_1,K_2 are both defined in Lemma 2.3, which means the system (13) is mean square asymptotically bounded.

Proof: By the generalized Itô formula, we have

$$
\begin{aligned}
& d[x_t^TQ(r_t)x_t] \\
=\ & [2x_t^TQ(r_t)f(x_t,r_t,t)+2x_t^TQ(r_t)D(r_t)x_t \\
& +g(x_t,r_t,t)^TQ(r_t)g(x_t,r_t,t) \\
& +\sum_{j=1}^{N}\gamma_{r_tj}x_t^TQ_jx_t-2x_t^TQ(r_t)D(r_t)(x_t-x_{\delta_t})]dt \\
& +dM_1(t),
\end{aligned} \tag{18}
$$

where $M_1(t)$ is a martingale with $M_1(0)=0$.

Then using the Itô formula to $e^{\theta t}x_t^TQ(r_t)x_t$, we have

$$
\begin{aligned}
& \lambda_me^{\theta t}\mathbb{E}|x_t|^2 \\
\leq\ & \lambda_M\mathbb{E}|x_0|^2+\int_0^te^{\theta s}\theta\lambda_M\mathbb{E}|x_s|^2ds \\
& +\mathbb{E}\int_0^te^{\theta s}x_s^TP(r_s)x_sds+\int_0^te^{\theta s}\beta ds \\
& +\mathbb{E}\int_0^t2e^{\theta s}[x_s^TQ(r_s)D(r_s)(x_s-x_{\delta_s})]ds \\
\leq\ & \lambda_M\mathbb{E}|x_0|^2+\int_0^te^{\theta s}(\theta\lambda_M-\alpha)\mathbb{E}|x_s|^2ds \\
& +\int_0^te^{\theta s}\beta ds \\
& +\int_0^t2\eta_{QD}^{\frac{1}{2}}e^{\theta s}\mathbb{E}(|x_s||x_s-x_{\delta_s}|)ds.
\end{aligned} \tag{19}
$$

By the definition of α_τ, it is easy to see from (16) that $\alpha_\tau>0$ and

$$
\begin{aligned}
& 2\eta_{QD}^{\frac{1}{2}}\mathbb{E}(|x_s||x_s-x_{\delta_s}|) \\
\leq\ & \alpha_\tau\mathbb{E}|x_s|^2+\frac{\eta_{QD}}{\alpha_\tau}\mathbb{E}|x_s-x_{\delta_s}|^2 \\
\leq\ & \alpha_\tau\mathbb{E}|x_s|^2+\frac{\eta_{QD}}{\alpha_\tau}\frac{K_1(\tau)}{1-K_1(\tau)}\mathbb{E}|x(t)|^2 \\
& +\frac{\eta_{QD}K_2(\tau)}{\alpha_\tau[1-K_1(\tau)]} \\
\leq\ & 2\alpha_\tau\mathbb{E}|x_s|^2+\frac{\eta_{QD}K_2(\tau)}{\alpha_\tau[1-K_1(\tau)]}.
\end{aligned} \tag{20}
$$

By (16) we see that $2\alpha_\tau<\alpha$ which means $\theta>0$, then substituting (20) into (19), we have

$$
\begin{aligned}
& \lambda_me^{\theta t}\mathbb{E}|x_t|^2 \\
\leq\ & \lambda_M\mathbb{E}|x_0|^2+\int_0^te^{\theta s}(\theta\lambda_M-\alpha+2\alpha_\tau)\mathbb{E}|x_s|^2ds \\
& +\int_0^te^{\theta s}[\beta+\frac{\eta_{QD}K_2(\tau)}{\alpha_\tau[1-K_1(\tau)]}]ds \\
\leq\ & \lambda_M\mathbb{E}|x_0|^2+\frac{1}{\theta}[\beta+\frac{\eta_{QD}K_2(\tau)}{\alpha_\tau[1-K_1(\tau)]}](e^{\theta t}-1).
\end{aligned} \tag{21}
$$

Thus it is easy to get

$$
\begin{aligned}
& \mathbb{E}|x_t|^2\leq\frac{\lambda_M}{\lambda_m}\mathbb{E}|x_0|^2e^{-\theta t} \\
& +\frac{1}{\theta\lambda_m}[\beta+\frac{\eta_{QD}K_2(\tau)}{\alpha_\tau[1-K_1(\tau)]}](1-e^{-\theta t}).
\end{aligned} \tag{22}
$$

Letting $t\to\infty$, and we see

$$\lim_{t\to\infty}\mathbb{E}|x_t|^2\leq\frac{1}{\theta\lambda_m}[\beta+\frac{\eta_{QD}K_2(\tau)}{\alpha_\tau[1-K_1(\tau)]}]. \tag{23}$$

The proof is completed. ■

IV. MORE GENERAL SITUATION

In the previous section, we have discussed the common linear feedback control function $D(r(t))x(\delta(t))$ based on the discrete-time observations. While in this section, let us consider the controlled systems(2), which the control function has a more general form $u(x(\delta(t)), r(t), t)$ that fulfill the Assumption 2.2.

It has to be point out that with a more general form $u(x(\delta(t)), r(t), t)$, the control function may not only contains more functions of x, but can also deal with more control problems in different situations. For example, in many situations, the Markov chain in the system (2) is an implicit variable that can not been observed. It is actually a special case of the general form $u(x(\delta(t)), r(t), t)$ that one can design a feedback control function $u(x(\delta(t)), t)$ independent of $r(t)$ to control the system. In addition, the general form $u(x(\delta(t)), r(t), t)$ contains the situations that the control function is a time-varying function which means a better control effect or less economic cost.

Except a more general form of the control function, we will use the Lyapunov function method to get a more general result. Let us give some very general definition of Lyapunov function. For any open subset G of R^n, let $C^{2,1}(G \times S \times R_+; R_+)$ denote the family of all non-negative functions $V(x, i, t)$ on $G \times S \times R_+$ which are continuously twice differentiable in x and once in t. For $V \in C^{2,1}(G \times S \times R_+; R_+)$, let us define an operator $\mathcal{L}V : R^n \times S \times R_+ \to R$ by

$$\begin{aligned}\mathcal{L}V(x,i,t) = V_t(x,i,t) + V_x(x,i,t)[f(x,i,t) + u(x,i,t)] \\ + \frac{1}{2}\text{trace}[g^T(x,i,t)V_{xx}(x,i,t)g(x,i,t)] \\ + \sum_{j=1}^{N} \gamma_{ij} V(x,j,t), \end{aligned} \quad (24)$$

where

$$V_t(x,i,t) = \frac{\partial V(x,i,t)}{\partial t},$$

$$V_x(x,i,t) = \Big(\frac{\partial V(x,i,t)}{\partial x_1}, \cdots, \frac{\partial V(x,i,t)}{\partial x_n}\Big),$$

and

$$V_{xx}(x,i,t) = \Big(\frac{\partial^2 V(x,i,t)}{\partial x_i \partial x_j}\Big)_{n\times n}.$$

Now let us give the general theorem.

Theorem 4.1: Assume that there exist functions $V \in C^{2,1}(G \times S \times R_+; R_+)$ and two groups of positive numbers c_1, c_2 and $\kappa_1, \kappa_2, \kappa_3$ such that

$$c_1|x|^2 \le V(x,i,t) \le c_2|x|^2 \quad (25)$$

and

$$\mathcal{L}V(x,i,t) + \kappa_1|V_x(x,i,t)|^2 \le -\kappa_2|x|^2 + \kappa_3 \quad (26)$$

for all $(x, i, t) \in R^n \times S \times R_+$. If the time gap τ satisfy that

$$K_1(\tau) < \frac{4\kappa_1\kappa_2}{4\kappa_1\kappa_2 + a_3} \quad (27)$$

Then the solution of system (2) satisfies

$$\lim_{t\to\infty} \mathbb{E}|x_t|^2 \le \frac{1}{c_1\gamma}[\kappa_3 + \frac{a_3K_2(\tau)}{4\kappa_1(1-K_1(\tau))}] \quad (28)$$

where $0 < \gamma = \frac{\kappa_2}{c_2} - \frac{a_3K_1(\tau)}{4c_2\kappa_1(1-K_1(\tau))}$ and K_1, K_2 are both defined in Lemma 2.3, which means the system (2) is mean square asymptotically bounded.

Proof: By the definition of (24), we can get

$$\begin{aligned} &dV(x_t, r_t, t) \\ &= [\mathcal{L}V(x_t,r_t,t) - V_x(x_t,r_t,t) \\ &(u(x_t,r_t,t) - u(x_{\delta_t}, r_t, t))]dt + dM_2(t), \end{aligned} \quad (29)$$

where $M_2(t)$ is a martingale with $M_2(0) = 0$. By using the same technic in Theorem 3.1, we see

$$\begin{aligned} &\mathbb{E}\big[e^{\gamma t}V(x_t,r_t,t)\big] \\ = \; &V(x_0,r_0,0) + \mathbb{E}\int_0^t e^{\gamma s}[\gamma V(x_s,r_s,s) \\ &+LV(x_s,r_s,s) \\ &-V_x(x_s,r_s,s)(u(x_s,r_s,s) - u(x_{\delta_s},r_s,s))]ds \end{aligned} \quad (30)$$

From the basic inequality and Assumption 2.2, we know that

$$\begin{aligned} &-V_x(x_s,r_s,s)(u(x_s,r_s,s) - u(x_{\delta_s},r_s,s))] \\ &\le \kappa_1|V_x(x_s,r_s,s)|^2 + \frac{a_3}{4\kappa_1}|x_s - x_{\delta_s}|^2. \end{aligned} \quad (31)$$

Substituting this into (30) we have

$$\begin{aligned} &\mathbb{E}\big[e^{\gamma t}V(x_t,r_t,t)\big] \\ \le \; &V(x_0,r_0,0) + \int_0^t e^{\gamma s}[\gamma\mathbb{E}V(x_s,r_s,s) \\ &+\mathbb{E}(LV(x_s,r_s,s) + \kappa_1|V_x(x_s,r_s,s)|^2) \\ &+\frac{a_3}{4\kappa_1}\mathbb{E}|x_s - x_{\delta_s}|^2]ds \end{aligned} \quad (32)$$

By (25), (26) and Lemma 2.3, we see

$$\begin{aligned} &c_1e^{\gamma t}\mathbb{E}|x_t|^2 \\ \le \; &V(x_0,r_0,0) + \int_0^t e^{\gamma s}[\gamma c_2\mathbb{E}|x_s|^2 - \kappa_2\mathbb{E}|x_s|^2 + \kappa_3 \\ &+\frac{a_3}{4\kappa_1}(\frac{K_1(\tau)}{1-K_1(\tau)}\mathbb{E}|x_s|^2 + \frac{K_2(\tau)}{1-K_1(\tau)})]ds \\ \le \; &V(x_0,r_0,0) + \int_0^t e^{\gamma s}[(\gamma c_2 - \kappa_2 \\ &+\frac{a_3K_1(\tau)}{4\kappa_1(1-K_1(\tau))}) \\ &\mathbb{E}|x_s|^2 + \kappa_3 + \frac{a_3K_2(\tau)}{4\kappa_1(1-K_1(\tau))}]ds \\ \le \; &V(x_0,r_0,0) + \frac{1}{\gamma}[\kappa_3 + \frac{a_3K_2(\tau)}{4\kappa_1(1-K_1(\tau))}](e^{\gamma t} - 1). \end{aligned}$$

Thus, we have

$$\begin{aligned} \mathbb{E}|x_t|^2 \le &\frac{V(x_0,r_0,0)}{c_1}e^{-\gamma t} \\ &+\frac{1}{c_1\gamma}[\kappa_3 + \frac{a_3K_2(\tau)}{4\kappa_1(1-K_1(\tau))}](1-e^{-\gamma t}) \end{aligned}$$

Letting $t \to \infty$, and we see

$$\lim_{t\to\infty} \mathbb{E}|x_t|^2 \le \frac{1}{c_1\gamma}[\kappa_3 + \frac{a_3K_2(\tau)}{4\kappa_1(1-K_1(\tau))}].$$

The proof is completed. ■

V. Example

Example 5.1: Let us first consider an 2-dimensional hybrid system,

$$dx(t) = [A(r(t))x(t) + sin(t)(2,2)^T]dt$$
$$+[B(r(t))x(t) + cos(t)(1,1)^T]d\omega(t)$$

on $t \geq 0$. Here $\omega(t)$ is a scalar Brownian motion; $r(t)$ is a Markov chain on the state space $S = 1, 2$ with the generator

$$\Gamma = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix};$$

and the system matrices are

$$A_1 = \begin{bmatrix} 1 & 3 \\ 4 & -5 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -3 & 4 \\ 5 & 2 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The computer simulation (Figure 1) shows that this hybrid SDE is not mean-square bounded.

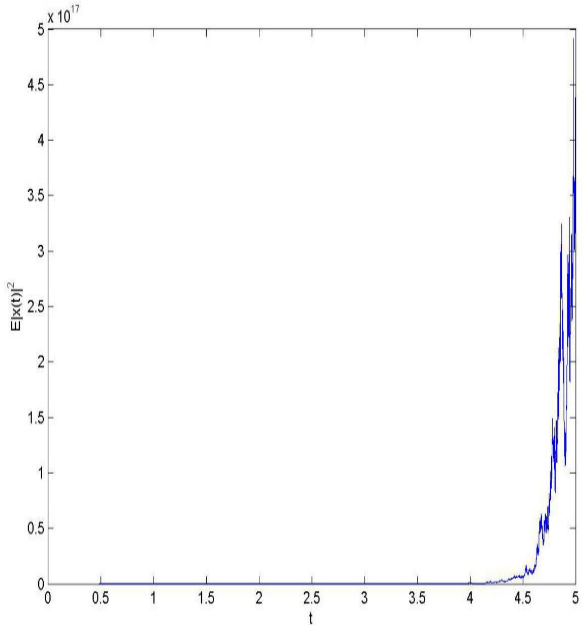

Fig. 1. Computer simulation of $\mathbb{E}|x_t|^2$ for the hybrid SDE (33) by using 10^4 paths of the Euler-Maruyama method with step size 10^{-5} and initial values $r(0) = 1$, $x_1(0) = -2$ and $x_2(0) = 2$.

Now let us design a discrete-time state feedback control to make the system to be mean square bounded with sampling gap τ. Assume that the controlled system has the following form

$$\begin{aligned} dx(t) &= [A(r(t))x(t) + sin(t)(2,2)^T \\ &\quad +D(r(t))x(\delta(t))]dt \\ &\quad +[B(r(t))x(t) + cos(t)(1,1)^T]d\omega(t). \quad (33) \end{aligned}$$

It is easy to calculate that $a_1 = 89.0132$, $a_2 = 10.472$, $a_3 = 185$, $b_1 = 8$, $b_2 = 2$, $b_3 = 0$. Our aim is to seek for D_1 and D_2 in $R^{2\times1}$ and then find the condition τ fitted so that the controlled system to be mean-square bounded. According to Theorem 3.1, we find that $Q_1 = Q_2 = I$ (the 2×2 identity matrix) and

$$D_1 = \begin{bmatrix} -7.5 & -7.5 \\ -3.5 & -3.5 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -4.5 & -4.5 \\ -8.5 & -8.5 \end{bmatrix},$$

and for these solutions we have

$$\bar{P}_1 = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix}, \quad \bar{P}_2 = \begin{bmatrix} -9 & 0 \\ 0 & -7 \end{bmatrix}.$$

It is easy to compute the parameters, we have

$$\lambda_m = \lambda_M = 1, \quad \alpha = 7, \quad \eta_{QD} = \eta_D = 185, \quad \beta = 4.$$

By (16), we have the boundedness of system (33) whenever $K_1(\tau) < 0.0621$, which means $\tau < 0.000475$. If we set $\tau = 0.0004$, then we have $K_1(\tau) = 0.0519$, $K_2(\tau) = 0.0025$, $\alpha_\tau = 3.1823$ and $\theta = 0.6354$. By this, we see that

$$\lim_{t\to\infty} \mathbb{E}|x_t|^2 \leq \frac{1}{\theta\lambda_m}[\beta + \frac{\eta_{QD}K_2(\tau)}{\alpha_\tau[1 - K_1(\tau)]}] = 6.5365 \quad (34)$$

The computer simulation (Figure 2) supports this result clearly.

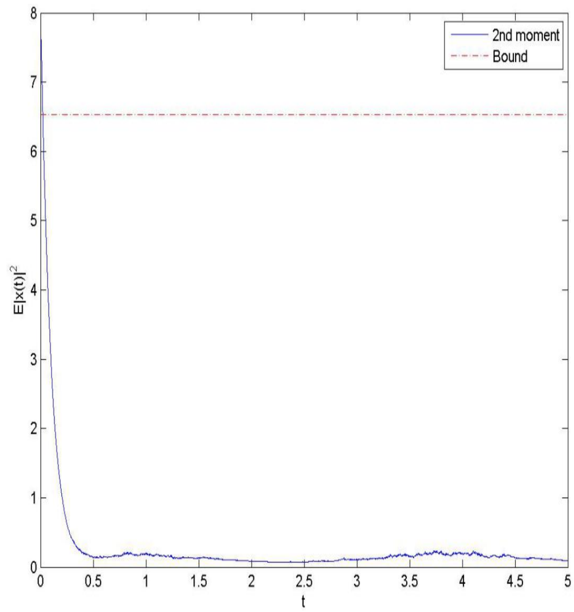

Fig. 2. Computer simulation of $\mathbb{E}|x_t|^2$ for the hybrid SDDE (33) with $\tau = 0.0004$ by using 10^4 paths of the Euler-Maruyama method with step size 10^{-5} and initial values $r(0) = 1$, $x_1(0) = -2$ and $x_2(0) = 2$.

VI. Conclusion

In this paper, the boundedness control problem of hybrid stochastic systems has been studied based on discrete-time observations. The mean square asymptotically boundedness of one controlled system with common linear feedback control function has been discussed through the generalized Itô formula, Itô formula, stochastic analysis for martingale and Holder's inequality. Moreover, by using stochastic analysis for martingale and Cauchy-Schwarz inequality, the mean square asymptotically boundedness of one controlled system with the control function hasing a more general form has been studied. Numerical examples have been provided to show the usefulness of the proposed mean square asymptotically boundedness criterion. Further research topics will include the boundedness control problem of hybrid stochastic systems with Lévy noises.

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