Robust Control for a Class of Nonlinear Switched Systems with Mixed Delays

Yongzhao Wang, Wenqiong Hou, Jiangrui Ding

Abstract—In this paper, we consider the problem of exponential \( H_{\infty} \) control for a class of nonlinear switched systems subject to exogenous disturbance and mixed time-varying delays. Specifically, the mixed time-varying delays with upper bound are considered to describe the delay-dependent multi-Lyapunov-Krasovskii functional, and the nonlinear perturbation satisfies the Lipschitz condition. Then, based on Lyapunov stability theory and average dwell time method, new delay-dependent sufficient results are obtained such that switched stability theory and average dwell time method, new delay-dependent sufficient results are obtained such that switched stability theory and average dwell time method, new delay-dependend sufficient results are obtained such that switched stability theory and average dwell time method, new delay-dependend sufficient results are obtained. Then, based on Lyapunov-Krasovskii functional, and the nonlinear perturbation satisfies the Lipschitz condition. Then, based on Lyapunov stability theory and average dwell time method, new delay-dependent sufficient results are obtained such that switched stability theory and average dwell time method, new delay-dependent sufficient results are obtained such that switched stability theory and average dwell time method, new delay-dependent sufficient results are obtained such that switched stability theory and average dwell time method, new delay-dependent sufficient results are obtained such that switched stability theory and average dwell time method, new delay-dependent sufficient results are obtained such that switched stability theory and average dwell time method, new delay-dependent sufficient results are obtained such that switched stability theory and average dwell time method, new delay-dependent sufficient results are obtained such that switched stability theory and average dwell time method, new delay-dependent sufficient results are obtained such that switched stability theory and average dwell time method, new delay-dependent sufficient results are obtained. Finally, a numerical example and a practical example of river pollution control are used to show the effectiveness of the proposed method.

Index Terms—\( H_{\infty} \) control, Switched systems, Average dwell time, Mixed delays, Multi-Lyapunov-Krasovskii functional.

I. INTRODUCTION

As special classes of hybrid systems, switched systems are widely used in engineering applications, which are composed of a set of subsystems and a special switching law[1-4]. In accordance with switching among the subsystems, switching technique ensures the stability of systems as well as specified performance. With the deepening of research, scholars have found that many real-world models can be represented by switched systems, for instance, tracking control systems[5], networked control systems[6], power electronic and robot control device[7], hot metal processing system[8]. In addition, some significant achievements of switched systems get more and more attention in theory as well as practical applications due to their advantages of increased system flexibility and high reliability. For example, the problems of exponentially mean-square stable and \( H_{\infty} \) filter of switched stochastic systems are addressed in [9]. Moreover, the conservatism of systems can not be guaranteed among the existing results. In view of this, we will consider the issue of mixed time-varying delays with upper bound, and use inequality technique to reduce the conservatism. In addition, the existence of external disturbances and nonlinear perturbations often cause undesirable performance of dynamical systems such as unstable and performance degradation. It is necessary to pay attention to the impact of these factors in most practical systems. During the last two decades, \( H_{\infty} \) control strategies are widely applied to deal with the stability analysis and control synthesis of switched systems subject to exogenous disturbance. The concept of \( H_{\infty} \) finite-time boundedness is first introduced in [16], and based on the average dwell time method, some new delay-dependent criteria are obtained to ensure the \( H_{\infty} \) finite-time boundedness of discrete-time switched delay systems. The issue of finite-time \( L_{\infty} \) filter design for networked Markov switched singular systems is addressed and a satisfied \( H_{\infty} \) performance is achieved by designing a unified method in [17]. The maximum and minimum dwell time scheme and multi-Lyapunov-Krasovskii functional technique are considered in [18]. By applying these combinations, the exponential stabilization and non-weighted \( H_{\infty} \) control performance of switched control systems are addressed and criteria on external stability are also obtained, respectively.
The $H_\infty$ control for two-dimensional switched systems with time-delayed are investigated in [19] by using the discrete Jensen inequality and the Lyapunov method. To the best of our knowledge, the issue of $H_\infty$ control of switched systems with mixed time-varying delays and exogenous disturbance has not been yet completely solved and results are relatively infrequent.

Inspired by the results mentioned above, the objective of this paper is to study $H_\infty$ control of nonlinear switched systems in the presence of exogenous disturbance and mixed time-varying delays. Our contributions to the literature are four-fold:

- Time delay and nonlinear are often coexisting in practical implementations, which are often give rise to instability and oscillation. In this paper, the mixed time-varying delays with upper bound and the nonlinear perturbation satisfies the Lipschitz condition are considered.
- Based on Lyapunov stability theory and average dwell time scheme, a novel delay-dependent multi-Lyapunov-Krasovskii functional is constructed that utilizes the complete available information about the upper bound of mixed time-varying delays.
- New delay-dependent exponentially stable criteria and a prescribed $H_\infty$ control performance index of switched systems with mixed time-varying delays are obtained by resorting to Jensen’s inequality. Moreover, the design of the proposed state feedback controllers of switched systems subject to exogenous disturbance and mixed time-varying delays is given through deformation of Linear Matrix Inequalities.
- On the basis of the proposed schemes, a practical example of river pollution control is carried out to confirm the validity and potential of the developed results.

The remainder of the paper is organized as follows. Section 2 presents the problem description and preliminaries. Section 3 derives the results on exponential stabilization and predefined $H_\infty$ control performance of switched systems in the presence of exogenous disturbance and mixed time-varying delays. In Section 4, two examples are carried out to illustrate the effectiveness of the proposed approach. Concluding remarks and directions for future research are proposed in Section 5.

II. PROBLEM FORMULATION AND PRELIMINARIES

In this section, we consider nonlinear switched systems in the presence of exogenous disturbance and mixed time-varying delays as follows:

$$
\dot{x}(t) = A_{1i}(t)x(t) + A_{2i}(t)x(t - d(t)) + B_{\sigma(t)}u(t) + C_{\sigma(t)}\int_{t-\tau(t)}^{t} x(s)ds + g(t, x(t-\tau(t))) + D_{\sigma(t)}\nu(t),
$$

where $x(t) \in \mathbb{R}^n$, $\varphi(s) \in \mathbb{R}^n$, $\nu(t) \in \mathbb{R}^m$ and $u(t) \in \mathbb{R}^m$ denote the state vector, initial condition, exogenous disturbance, and the control input, respectively. $z(t) \in \mathbb{R}^n$ is the measured output, the switching signal $\sigma(t) : [0, \infty] \to M = \{1, 2, \ldots, n\}$ is a piecewise continuous (from the right) function, where $n$ is the number of subsystems. Specifically, we denote $\Sigma : \{ (t_0, \sigma(t)), \ldots, (t_k, \sigma(t)), \ldots, k = 0, 1, 2, \ldots \}$, where $t_0$ is the initial switching instant and $t_k$ denotes the $k$th switching instant.

For $t \in [t_k, t_{k+1})$, we assume that the $i$th subsystem is activated. $A_{1j}, A_{2j}, B_j, C_i, D_i, E_i, F_i$ are constant matrices. $d(t)$ and $\tau(t)$ denote the time-varying delay satisfying

$$
0 \leq d(t) \leq d_M, \quad 0 \leq \tau(t) \leq \tau_M, \quad \dot{x}(t) \leq \tau(t).
$$

Remark 1. External disturbances and nonlinear perturbations often cause undesirable performance such as instability and performance degradation in many control applications. In this paper, we consider the mixed time-varying delays with upper bound and the nonlinear perturbation satisfying the Lipschitz condition. Compared with the existing results, the switched model in this paper is more comprehensive and practical in engineering.

For switched systems (1), the state feedback is described as:

$$
u(t) = K_{\sigma(t)}x(t).
$$

For convenience of calculation and analysis, we recorded as $\tilde{A}_{1i} = A_{1i} + B_i K_i$ such that the resulting closed-loop (1) by following:

$$
\dot{x}(t) = \tilde{A}_{1i}x(t) + A_{2i}x(t - d(t)) + C_{\sigma(t)}\int_{t-\tau(t)}^{t} x(s)ds + g(t, x(t-\tau(t))) + D_{\sigma(t)}\nu(t)
$$

Before proving theorem, the following definitions and lemmas are crucial for the development of our main results.

Definition 1.([20]) $N_{\sigma}(t, T)$ is the switching number of $\sigma(t)$ on an interval $(t, T)$. For any $T > t \geq 0$, if

$$
N_{\sigma}(t, T) \leq N_0 + (T-t)/\tau_{\alpha},
$$

holds for given $N_0 \geq 0$ and $\tau_{\alpha} \geq 0$, the constant $\tau_{\alpha}$ is called the average dwell time. In this paper, $N_0 = 0$.

Remark 2. Average dwell time method is an available scheme to obtain a satisfied performance by designating the maximum switching numbers over a operating interval. Moreover, with the deepening of research, the concept of maximum and minimum dwell time scheme is also considered in [18]. It is worth noting that the weighted term $e^{-\alpha t}$ can be canceled in [18] when applying maximum and minimum dwell time method and the results obtained also increase conservatism. In order to obtain the lower bound for the dwell time (i.e., $\tau_{\alpha} > \tau_{\alpha} = \frac{\ln N_0}{\epsilon}$), the average dwell time method will be applied in this paper.

Definition 2.([21]) The equilibrium $x^* = 0$ of the system (1) with $\nu(t) = 0$ is said to be exponentially stable under switching signal $\sigma(t)$ if the solution $x(t)$ satisfies

$$
\|x(t)\|^2 \leq re^{-\alpha(t-t_0)}\|x_{t_0}\|, \quad \forall t \geq t_0
$$


for constants \( r \geq 1, \alpha > 0 \), where \( \| x_{t_o} \|_e = \sup_{-\max(dM, \tau_M) \leq \theta \leq 0} \| x(t_0 + \theta) \|, \| \dot{x}(t_0 + \theta) \| \).

**Definition 3.**([22]) For a given performance level \( \gamma > 0 \), switched system (1) is said to be exponential stabilization with an \( H_{\infty} \) disturbance attenuation level \( \gamma \), if the following conditions hold:

1. Switched system (1) is exponentially stabilizable when the exogenous disturbance \( \nu(t) = 0 \).
2. Under the zero initial condition, for any nonzero exogenous disturbance, there is
   \[
   \int_0^\infty e^{-\lambda t} z(t) dt \leq e^{2 \int_0^\infty \nu^T(t) \nu(t) dt}
   \]
   where \( \lambda \geq 0, \gamma > 0 \).

**Lemma 1.**([23]) For a given matrix \( S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \) with \( S_{11} = S_{11}^T, S_{22} = S_{22}^T \), then the following conditions are equivalent:

1. \( S < 0 \),
2. \( S_{11} < 0, S_{22} - S_{12} S_{11}^{-1} S_{12} < 0 \),
3. \( S_{22} < 0, S_{11} - S_{12} S_{22}^{-1} S_{12} < 0 \).

**Lemma 2.**([17]) For any positive definite constant matrix \( M \), scalars \( a \) and \( b \) : \( b < a \), vector function \( \gamma : [b, a] \rightarrow \mathbb{R}^d \) such that the integration constraint are well defined, then:

\[
(a-b) \int_b^a \gamma(s)M \gamma(s) ds \geq \left( \int_b^a \gamma(s) ds \right)^2 \int_b^a \gamma(s) ds.
\]

**III. MAIN RESULTS**

The exponential stable and \( H_{\infty} \) control of nonlinear switched systems (1) in the presence of mixed time-varying delays and exogenous disturbance are investigated by resorting to the average dwell time scheme and Jensen’s Inequality technique in this section. In view of this, we firstly consider that the resulting closed-loop system (5) is exponential stable when \( \nu(t) = 0 \).

**A. Stability analysis**

**Theorem 1.** For given positive constants \( \alpha, d_M, \tau_M, \) and \( \mu \geq 1 \), if there exist positive constant \( \epsilon_i \) and symmetric and positive definite matrices \( P_i, S_{1i}, S_{2i}, T_{1i}, T_{2i} \) such that the following matrix inequalities hold for all \( i, j \in M, i \neq j \),

\[
P_i \leq \mu P_j, S_{1i} \leq \mu S_{1j}, S_{2i} \leq \mu S_{2j},
T_{1i} \leq \mu T_{1j}, T_{2i} \leq \mu T_{2j}, \quad (9)
\]

\[
\Xi_1 = \begin{pmatrix} \phi_{11}^0 & \phi_{12}^0 & 0 & 0 & \phi_{16}^0 \\
\phi_{22}^0 & 0 & 0 & 0 & 0 \\
\phi_{33}^0 & 0 & 0 & 0 \\
\phi_{44}^0 & 0 & 0 & 0 \\
\phi_{55}^0 & 0 & 0 & 0 \\
\phi_{66}^0 & 0 & 0 & 0
\end{pmatrix} < 0, \quad (10)
\]

where \( \phi_{11}^0 = P_i A_{1i} + \tilde{A}_{1i}^T P_i + S_{1i} + S_{2i} + d_M^2 T_{1i} + \tau_M T_{2i} + \alpha P_i, \phi_{12}^0 = P_i A_{2i}, \phi_{16}^0 = P_i C_i, \phi_{16}^0 = -(1-d) e^{-\alpha t_M} S_{1i}, \phi_{22}^0 = \phi_{44}^0 = -I, \phi_{55}^0 = -e^{-\alpha t_M} T_{1i}, \phi_{66}^0 = -e^{-\alpha t_M} T_{2i} \).

Then, the resulting closed-loop system (5) with \( \omega(t) = 0 \) is exponentially stabilizable for any switching signal with the average dwell time satisfying

\[
\tau_a > \tau_a^* = \frac{\ln \mu}{\alpha} \quad (11)
\]

**Proof:** When \( t \in [t_k, t_{k+1}] \) and \( \omega(t) = 0 \), we assume that the \( i \)th subsystem is activated, the delay-dependent Lyapunov-Krasovskii functional is constructed as follows:

\[
V(t) = V_i(t) = \sum_{s=1}^5 V_{si}(t),
\]

where

\[
V_{1i}(t) = 2x^T(t) P_i x(t), \quad V_{2i}(t) = \int_{t-d(t)}^t e^{\alpha(s-t)} x^T(s) S_{1i} x(s) ds,
\]

\[
V_{3i}(t) = \int_{t-\tau(t)}^t e^{\alpha(s-t)} x^T(s) S_{2i} x(s) ds, \quad V_{4i}(t) = d_M \int_{t-d_M}^t e^{\alpha(s-t)} x^T(s) T_{1i} x(s) ds d\theta,
\]

\[
V_{5i}(t) = \tau_M \int_{t-\tau_M}^t e^{\alpha(s-t)} x^T(s) T_{2i} x(s) ds d\theta.
\]

The time derivative of (12) yields

\[
\dot{V}_{ix}(t) = 2x^T(t) P_i \dot{x}(t), \quad \dot{V}_{2i}(t) = -(1-d(t)) e^{-\alpha d(t)} x(t-d(t)) S_{1i} x(t-d(t)) + x(t) S_{1i} x(t) - \alpha V_{2i}(t),
\]

\[
\dot{V}_{3i}(t) = -(1-\tau(t)) e^{-\alpha \tau(t)} x(t-d\tau(t)) S_{2i} x(t-\tau(t)) + x(t) S_{2i} x(t) - \alpha V_{3i}(t),
\]

\[
\dot{V}_{4i}(t) = -\tau_M e^{-\alpha \tau_M} x(t) \dot{V}_{4i}(t) + x(t) S_{2i} x(t) - \alpha V_{4i}(t), \quad \dot{V}_{5i}(t) = -\tau_M e^{-\alpha \tau_M} x(t) \dot{V}_{5i}(t) + x(t) S_{2i} x(t) - \alpha V_{5i}(t).
\]

According (3), we have

\[
\delta^2 x^T(t-\tau(t)) x(t-\tau(t)) - g^T(t, x(t-\tau(t))) g(t, x(t-\tau(t))) \geq 0.
\]
Form Jensens inequality, we get
\[
\tau_M \int_{t-\tau_M}^{t} e^{-\alpha \tau_M} x_T(s) T_{2i} x(s) ds \\
\leq -e^{-\alpha \tau_M} \left( \int_{t-\tau_M}^{t} x_T(s) ds \right) T_{2i} \left( \int_{t-\tau_M}^{t} x(s) ds \right) \\
\leq -e^{-\alpha \tau_M} \left( \int_{t-\tau(t)}^{t} x_T(s) ds \right) T_{2i} \left( \int_{t-\tau(t)}^{t} x(s) ds \right); \\
d_M \int_{t-d_M}^{t} e^{-\alpha d_M} x_T(s) T_{1i} x(s) ds \\
\leq -e^{-\alpha d_M} \left( \int_{t-d_M}^{t} x_T(s) ds \right) T_{1i} \left( \int_{t-d_M}^{t} x(s) ds \right).
\]
(15)

Considering (13), (14), and (15)
\[
\dot{V}_i(t) + \alpha V_i(t) \\
\leq x_T(t) \left [ P_i \tilde{A}_{i1} + \tilde{A}_{i1}^T P_i + S_{i1} + S_{2i} + d_M^2 T_{1i} + \alpha P_i \right. \\
\left. + \tau_M^2 T_{2i} x_T(t) - g^T(t, x(t-\tau(t))) g(t, x(t-\tau(t))) \right] \\
+ x_T(t-\tau(t)) \left [ 2 \sigma^2 1 - (1 - \tau) e^{-\alpha \tau_M} S_{i1} x(t-\tau(t)) \right] \\
- (1 - d_i) e^{-\alpha d_M} x_T(t-d(t)) S_{i1} x(t-d(t)) \\
+ x_T(t-d(t)) A_{2i} P_i x(t) + x_T(t) P_i C_i \left( \int_{t-d(t)}^{t} x(s) ds \right) \\
+ x_T(t) P_i g(t, x(t-\tau(t))) + g^T(t, x(t-\tau(t))) P_i x(t) \\
+ x_T(t) P_i A_{2i} x(t-d(t)) + \int_{t-d(t)}^{t} x_T(s) ds C_i P_i x(t) \\
- e^{-\alpha \tau_M} \left( \int_{t-\tau(t)}^{t} x_T(s) ds \right) T_{2i} \left( \int_{t-\tau(t)}^{t} x(s) ds \right) \\
- e^{-\alpha d_M} \left( \int_{t-d_M}^{t} x_T(s) ds \right) T_{1i} \left( \int_{t-d_M}^{t} x(s) ds \right)
\]
(16)

Then,
\[
\dot{V}_i(t) + \alpha V_i(t) \leq \psi^T(t) \Xi_i \psi(t),
\]
(17)

where
\[
\psi(t) = \begin{pmatrix} x_T(t) \\ x_T(t-d(t)) \\ g^T(t, x(t-\tau(t))) \\ \int_{t-d_M}^{t} x_T(s) ds \end{pmatrix} \\
\psi^T(t) = \begin{pmatrix} x_T(t-d(t)) \\ x_T(t-\tau(t)) \end{pmatrix}.
\]

According (10), we have
\[
\dot{V}_i(t) - \alpha V_i(t) \leq 0.
\]
(18)

When \( t \in [t_k, t_{k+1}) \), simultaneously integrating from \( t_k \) to \( t \) on both sides of the above formula, we can get
\[
V(t) = V_{\sigma(t)}(t) \leq e^{-\alpha(t-t_k)} V_{\sigma(t_k)}(t_k), \quad t \leq t < t_{k+1}.
\]
(19)

Form (9), (19) and \( k = N_{\sigma}(t, t_0) \leq (t - t_0)/\tau_a \), we have
\[
V(t) \leq e^{-\alpha(t-t_k)} \mu V_{\sigma(t-\tau_a)}(t_k) \\
\leq \cdots \leq e^{-\alpha(t-t_k)} \mu^k V_{\sigma(t-\tau_a)}(t_0^-) \\
\leq e^{-(\alpha t - \alpha m_1)/\tau_a} V_{\sigma(t_0)}(t_0).
\]
(20)

According Lyapunov-Krasovskii functional (12), we can get
\[
V(t) \geq a \| x(t) \|^2, \\
V(t_0) \leq b \| x_{t_0} \|,
\]
where
\[
a = \min_{i \in \mathbb{N}} \lambda_{\min}(P_i), \\
b = \max_{i \in \mathbb{N}} \lambda_{\max}(P_i) + \| M \| \max_{i \in \mathbb{N}} \lambda_{\max}(Q_{i1}) \\
+ \tau_M \max_{i \in \mathbb{N}} \lambda_{\max}(Q_{i2}) + \frac{1}{2} \| M \| \max_{i \in \mathbb{N}} \lambda_{\max}(R_{i1}) \\
+ \tau_M^2 \max_{i \in \mathbb{N}} \lambda_{\max}(R_{i2}).
\]

Hence,
\[
\| x(t_0) \| \leq \sup_{\tau = \max(h_M, d_M)} \| x(t_0 + \tau) \|, \| \dot{x}(t_0 + \tau) \|.
\]
(21)

By Definition 2, system (1) is exponentially stability. The proof is completed.

Remark 3. It should be pointed out that if \( \mu = 1 \), switching signals of switched systems degenerates to arbitrary switching, which implies that \( \tau_n = 0 \). However, in the mathematical model of the actual system, this phenomenon is often undesirable, because in practice it may cause the entire switched systems to be unstable due to fast switching (see [29] for more details).

Remark 4. If the matrix \( C_i \) is a 0 matrix, the term of \( \int_{t-\tau(t)}^{t} x(s) ds \) will not exist in (1). In this case, we get a special nonlinear switched time-delay system. Thus, the term of \( \int_{t-d(t)}^{t} x(s) ds \) not only makes the switched system more general than the model created in [11, 15]. In addition, the mixed time-delays with upper bound are considered in (12) to describe the delay-dependent multi-Lyapunov-Krasovskii functional and free-weighting matrices and Jensens inequality are also introduced in the process of calculation. Compared with [30], the results obtained in this paper are less conservative by the proposed approach.

B. \( H_\infty \) control

In the following subsection, we considered the issue of \( H_\infty \) control for the resulting system (5) with respect to the exogenous disturbance input \( \nu(t) \neq 0 \), and some new delay-dependent exponentially stable criteria with a prescribed weighted attenuation performance index \( \gamma \) are provided.

Theorem 2. For given positive constants \( \alpha, \gamma, d_M, \) and \( \tau_M \), if there exist symmetric and positive definite matrices \( P_i, Q_{i1}, Q_{i2}, R_{i1}, R_{i2}, \) and \( K_i \), such that the following matrix inequalities hold for all \( i, j \in \mathbb{M} \),
\[
P_i \leq \mu P_j, \quad S_{i1} \leq \mu S_{j1}, \quad S_{2i} \leq \mu S_{2j}, \\
T_{i1} \leq \mu T_{j1}, \quad T_{2i} \leq \mu T_{2j},
\]
(22)

\[
\Xi_i = \begin{pmatrix} \phi_{i1}^{11} & \phi_{i1}^{12} & 0 & P_i & 0 & \phi_{i1}^{16} & \phi_{i1}^{17} \\
& \phi_{i2}^{11} & 0 & 0 & 0 & 0 & 0 \\
& * & * & \phi_{i3}^{11} & 0 & 0 & 0 \\
& * & * & * & \phi_{i4}^{11} & 0 & 0 \\
& * & * & * & * & \phi_{i5}^{11} & 0 \\
& * & * & * & * & * & \phi_{i6}^{11} \\
& * & * & * & * & * & \phi_{i7}^{11} \end{pmatrix} < 0
\]
(23)

where
\[
\phi_{i1}^{11} = P_i \tilde{A}_{i1} + \tilde{A}_{i1}^T P_i + S_{i1} + S_{2i} + d_M^2 T_{i1} + \tau_M^2 T_{2i} + \alpha P_i, \\
\phi_{i1}^{17} = F_i^T F_i - \gamma^2 I, \quad \phi_{i7}^{11} = P_i D_i + E_i^T F_i.
\]
Then, the resulting system (5) is exponentially stabilizable and has weighted attenuation performance \( \gamma \) for any switching signal with the average dwell time satisfies (11).

Proof: For \( t \in [t_k, t_{k+1}) \), we choose Lyapunov-Krasovskii functional as (12), thus, we can get

\[
\begin{align*}
\dot{V}(t) + \alpha V(t) + z^T(t)z(t) - \gamma^2 \nu T(t)\nu(t) &
\leq \dot{V}(t)\{P_A z(t) + \hat{A}^T P_A + S_{1i} + S_{2i} + d^2_{2i} T_i \}
\\& + \alpha P_i + \hat{D}_i^T D_i x(t) + \nu^T(t)\{P_i \hat{D}_i + E_i^T E_i\} x(t) \\
&- (1 - \tau)z^T(t - \tau(t))e^{-\alpha \tau T} S_{2i} z(t - \tau(t)) \\
\leq &
\dot{V}(t)P_i g(t, x(t - \tau(t))) + \int_{t-\tau(t)}^t x^T(s)ds C_i^T P_i x(t) \\
&+ \dot{V}(t)P_i A_{2i} x(t - d(t)) + \nu^T(t)(F_i^T F_i - \gamma^2 I)\nu(t) \\
&+ g^T(t, x(t - \tau(t)))P_i x(t) + x^T(t)\{P_i \hat{D}_i + E_i^T E_i\} x(t) \\
& + [\delta^2 I - (1 - d)]e^{-\alpha \tau T} x(t - d(t))S_{1i} x(t - d(t)) \\
&+ \dot{V}(t)P_i C_i \int_{t-\tau(t)}^t x(s)ds + x^T(t - d(t)) A_{2i} P_i x(t) \\
&- e^{-\alpha \tau T} \int_{t-d}^t x^T(s)ds T_{1i} \left( \int_{t-d}^t x(s)ds \right) \\
&- e^{-\tau T} \int_{t-\tau(t)}^t x^T(s)ds T_{2i} \left( \int_{t-\tau(t)}^t x(s)ds \right) \\
&- g^T(t, x(t - \tau(t)))g(t, x(t - \tau(t))).
\end{align*}
\]

\[
\psi(t) = |x^T(t) x(t - d(t)) x^T(t - \tau(t))| \begin{pmatrix} \int_{t-d}^t x(s)ds \end{pmatrix}^T \\
\nu^T(t) T.
\]

So

\[
\dot{V}(t) + \alpha V(t) + z^T(t)z(t) - \gamma^2 \nu T(t)\nu(t) \leq \dot{\psi}(t) \bar{\Xi}_1 \psi(t).
\]

We have

\[
\dot{V}(t) + \alpha V(t) + z^T(t)z(t) - \gamma^2 \nu T(t)\nu(t) \leq 0. \tag{24}
\]

Integrating from \( t_k \) to \( t \) on both sides of (24), we have

\[
V(t) \leq e^{-\alpha(t-t_k)}V(t_k) - \int_{t_k}^t e^{-\alpha(s-t_k)} \Gamma(s)ds. \tag{25}
\]

where \( \Gamma(t) = z^T(t)z(t) - \gamma^2 \nu T(t)\nu(t) \).

Combining (9) and (25), we obtain

\[
V(t) \leq e^{-\alpha(t-t_k)}V(t_k) - \int_{t_k}^t e^{-\alpha(s-t_k)} \Gamma(s)ds \\
\leq \mu^k V(t_0)e^{-\alpha t} - \mu^k \int_0^1 e^{-\alpha(s-t)} \Gamma(s)ds \\
- \mu^{k-1} \int_1^2 e^{-\alpha(s-t)} \Gamma(s)ds \\
- \cdots - \mu^{k-1} \int_{t_k-1}^{t_k} e^{-\alpha(s-t)} \Gamma(s)ds \\
\leq e^{-\alpha t + N_\alpha(t_0)} \ln \| u \| V(0) - \int_0^t e^{-\alpha t + N_\alpha(s)} \ln \| u \| \Gamma(s)ds.
\]

Then,

\[
0 \leq - \int_0^t e^{-\alpha(s-t) + N_\alpha(s)} \ln \| u \| \Gamma(s)ds. \tag{26}
\]

Using \( e^{-N_\alpha(t_0)} \ln \| u \| \) to pre-multiply and post-multiply the left term of (26), we have

\[
\int_0^t e^{-\alpha(s-t) - N_\alpha(t_0)} \ln \| u \| \Gamma(s)ds \leq 0.
\]

C. Controller design

The design of the proposed state feedback controllers of switched systems subject to exogenous disturbance and mixed time-varying delays is given through deformation of Linear Matrix Inequalities in this section.

**Theorem 3.** For given positive constants \( \alpha, \gamma, \delta_t, \delta_M \), if there exist symmetric and positive definite matrices \( X_i, G_{1i}, G_{2i}, O_{1i}, O_{2i}, \) and \( K_i \), such that the following matrix inequalities hold for all \( i, j \in M \),

\[
\begin{align*}
N_i &\leq M N_j, \quad G_{1i} \leq M G_{1j}, \quad G_{2i} \leq M G_{2j}, \\
O_{1i} &\leq M O_{1j}, \quad O_{2i} \leq M O_{2j}.
\end{align*}
\]

Then, the system (1) is exponentially stabilizable and has weighted attenuation performance \( \gamma \) for any switching signal with the average dwell time satisfies (11). Furthermore, the controller can be designed by the following formula:

\[
K_i = Y_i N_i^{-1}
\]

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Proof: From $G_{11} > 0, G_{21} > 0, O_{11} > 0$, and $O_{21} > 0$, we can get

$$
(G_{11} - N_i)G_{11}^{-1}(G_{11} - N_i) \geq 0,
$$

$$
(G_{21} - N_i)G_{21}^{-1}(G_{21} - N_i) \geq 0,
$$

$$
(O_{11} - N_i)O_{11}^{-1}(O_{11} - N_i) \geq 0,
$$

$$
(O_{21} - N_i)O_{21}^{-1}(O_{21} - N_i) \geq 0.
$$

Then the following inequality can be obtained:

$$
G_{11} - 2N_i \geq -N_iG_{11}^{-1}N_i,
$$

$$
G_{21} - 2N_i \geq -N_iG_{21}^{-1}N_i,
$$

$$
O_{11} - 2N_i \geq -N_iO_{11}^{-1}N_i,
$$

$$
O_{21} - 2N_i \geq -N_iO_{21}^{-1}N_i.
$$

Substituting (33) into (31) and multiplying both sides of (31) by

$$\text{diag}(N_i^{-1}, N_i^{-1}, N_i^{-1}, I, N_i^{-1}, N_i^{-1}, I, I, I, I, I, I)$$

The following inequality is obtained,

$$
\left( \tilde{\Delta}_{11}^i \ast \tilde{\Delta}_{12}^i \right) < 0, \quad (34)
$$

where

$$
\tilde{\Delta}_{11}^i = \begin{pmatrix}
\phi_{i1} & \phi_{i2} & 0 & I & 0 & \phi_{i10} \\
\ast & \phi_{i22} & 0 & 0 & 0 & 0 \\
\ast & \ast & \phi_{i33} & 0 & 0 & 0 \\
\ast & \ast & \ast & \phi_{i44} & 0 & 0 \\
\ast & \ast & \ast & \ast & \phi_{i55} & 0 \\
\ast & \ast & \ast & \ast & \ast & \phi_{i66}
\end{pmatrix},
$$

$$
\tilde{\Delta}_{12}^i = \begin{pmatrix}
\phi_{i17} & 0 & d_M I & \tau_M I & I & I \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \delta I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$\phi_{i1} = N_i^{-1}A_{1i} + N_i^{-1}B_1Y_1N_i^{-1} + (N_i^{-1}A_{1i} + N_i^{-1}B_1Y_1N_i^{-1})^T + \alpha N_i^{-1}, \quad \phi_{i2} = N_i^{-1}A_{2i}, \quad \phi_{i10} = N_i^{-1}C_i,
$$

$$\phi_{i22} = -(1-d)e^{-\alpha M}S_{1i}^{-1}, \quad \phi_{i44} = -I, \quad \phi_{i55} = e^{-\alpha d M}T_{1i}^{-1},
$$

$$\phi_{i33} = -(1-\tau)e^{-\alpha M}S_{2i}^{-1}, \quad \phi_{i66} = e^{-\alpha M}T_{2i}^{-1}.
$$

Then setting

$$
Y_i = K_iN_i, \quad N_i^{-1} = P_i, \quad G_{11}^{-1} = S_{1i}, \quad G_{21}^{-1} = S_{2i}, \quad O_{11}^{-1} = T_{1i}, \quad O_{21}^{-1} = T_{2i},
$$

and using Lemma 2.2 in (34), it can be concluded that (24) holds. This means that (31) implies (24). Moreover, the controller gains are given by (32). The proof is completed.

Remark 5. In fact, $H_\infty$ control of switched systems is an important research issue in many engineering systems when switched systems are constantly disturbed by the external behavior, e.g., tracking process, networked control systems, circuit systems. Specifically, the issue of $H_\infty$ is considered such that the switched systems is exponential stabilization and satisfies a prescribed $H_\infty$ control performance level in this paper. Moreover, on the basis of Jensen’s Inequality, the average dwell time method and free weighting matrix technique, some new delay-dependent sufficient results are obtained. Compared with the existing of $H_\infty$ control, it is noted that the results in [5,22] can be further improved by our method.

Remark 6. A problem of finite-time $H_\infty$ control for uncertain switched neural networks is addressed in [14]. However, to the best of our knowledge, the finite-time $H_\infty$ control behavior can be termed as short-time stability or finite-time stability (see [16] for more details). However, in practical engineering systems, we need the system to be asymptotic stable in most cases. Motivated by this, we consider the $H_\infty$ control problem of the switched systems in $[0, \infty)$ instead of the interval of $[0, T]$.

Remark 7. In [31], the issue of dynamic output feedback controller was addressed for a linear switched systems. Moreover, as the switched systems state $x(t)$ is not always available and exogenous disturbance often exist in many practical systems. For this reason, $H_\infty$ control scheme is proposed in Theorem 2. Specifically, when we do not consider the effects of exogenous disturbance and nonlinear perturbation, the model of switched systems considered in [31] is a special case of this paper. It is shown that our results have a larger application range.

IV. NUMERICAL EXAMPLES

In this section, on the basis of the proposed schemes, we present a numerical example and a practical example of river pollution control to demonstrate the effectiveness of the proposed approach.

Example 1. Consider switched system (1) composed of two subsystems with the following parameters:

$$A_{11} = \begin{pmatrix} -1.8 & 0.2 \\ 0.3 & -1.5 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} -0.9 & 0.1 \\ 0.1 & -1.4 \end{pmatrix},
$$

$$A_{12} = \begin{pmatrix} -2.1 & 0.3 \\ 0.1 & -1.7 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} -0.6 & 0.2 \\ 0.1 & -1.7 \end{pmatrix},
$$

$$B_1 = \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}, \quad C_1 = \begin{pmatrix} -0.2 & 0.3 \\ 0.1 & -0.3 \end{pmatrix},
$$

$$B_2 = \begin{pmatrix} 0.1 \\ 0.8 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0.2 & -0.1 \\ 0.2 & -0.1 \end{pmatrix},
$$

$$D_1 = \begin{pmatrix} -0.9 & 0.2 \\ 0.2 & -1.6 \end{pmatrix}, \quad D_2 = \begin{pmatrix} -1.1 & 0.1 \\ 0.1 & -1.8 \end{pmatrix},
$$

$$E_1 = \begin{pmatrix} -0.7 & 0.1 \\ 0.1 & -0.8 \end{pmatrix}, \quad E_2 = \begin{pmatrix} -1.2 & 0.1 \\ 0.2 & -1.3 \end{pmatrix},
$$

$$F_1 = \begin{pmatrix} -1.1 & 0.2 \\ 0.1 & -1.9 \end{pmatrix}, \quad F_2 = \begin{pmatrix} -1.1 & 0.2 \\ 0.1 & -1.2 \end{pmatrix}.
$$

Let $\alpha = 0.8, d_M = 0.6, \tau_M = 0.9, d = 0.3, \tau = 0.5, \mu = 1.7, \gamma = 0.6, d(t) = 0.3 + 0.3 \sin(t), \tau(t) = 0.4 + 0.5 \sin(t)$, and $\nu(t) = (0.1e^{-2t} 0.2e^{-3t})^T$. By calculating, we can get the average dwell time is $\tau_D > 0.6633$. Moreover, we consider $g(t, x(t - \tau(t))) = ((0.3 \sin(x(t))) 0.2 \cos(x(t - \tau(t))))^T$. 

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By solving (23) and (24), we can get
\[
P_1 = \begin{bmatrix} 0.9762 & -0.0175 \\ -0.0175 & 1.0073 \end{bmatrix},
\]
\[
P_2 = \begin{bmatrix} 0.8256 & -0.2334 \\ -0.2334 & 0.9011 \end{bmatrix},
\]
\[
S_{11} = \begin{bmatrix} 1.0122 & -0.1325 \\ -0.1325 & 1.2117 \end{bmatrix},
\]
\[
S_{21} = \begin{bmatrix} 1.0231 & -0.1332 \\ -0.1332 & 1.1239 \end{bmatrix},
\]
\[
S_{12} = \begin{bmatrix} 0.9263 & -0.1286 \\ -0.1286 & 1.0092 \end{bmatrix},
\]
\[
S_{22} = \begin{bmatrix} 1.0128 & -0.2176 \\ -0.2176 & 1.1254 \end{bmatrix},
\]
\[
T_{11} = \begin{bmatrix} 0.9729 & -0.2825 \\ -0.2825 & 1.0726 \end{bmatrix},
\]
\[
T_{21} = \begin{bmatrix} 1.0127 & -0.2903 \\ -0.2903 & 1.2527 \end{bmatrix},
\]
\[
T_{12} = \begin{bmatrix} 1.7822 & -0.3037 \\ -0.3037 & 1.6735 \end{bmatrix},
\]
\[
T_{22} = \begin{bmatrix} 0.9923 & -0.3826 \\ -0.3826 & 1.0771 \end{bmatrix},
\]
\[
Y_1 = \begin{bmatrix} 1.2733 & 1.8682 \end{bmatrix},
\]
\[
Y_2 = \begin{bmatrix} 0.9678 & 1.0245 \end{bmatrix}.
\]

Then, the controller gains are
\[
K_1 = \begin{bmatrix} 1.3380 & 1.8779 \end{bmatrix},
\]
\[
K_2 = \begin{bmatrix} 1.6117 & -1.5544 \end{bmatrix}.
\]

The trajectories of the closed-loop switched system (5) with \( x(0) = (2, -2)^T \) the initial state and switching signal \( \sigma(t) \) are shown in Fig.1 and Fig.2, respectively. Fig.2 shows that the trajectory of subsystem 1 and subsystem 2 gradually tend to equilibrium position under the designed switching signal. As a consequence, The implication of this numerical example is to illustrate the effectiveness of the approach proposed.

**Example 2.** It is worth mentioning that the safe water resources are important for social development. In fact, the current problem of water pollution is also an urgent issue for every country. Next, on the basis of the proposed \( H_{\infty} \) control schemes in this paper, we present a practical example of river pollution control to confirm the effectiveness of the approach proposed.

According to [27,28], the concentrations per unit volume of biochemical oxygen demand (BOD) and dissolved oxygen (DO) are denoted as \( m(t) \) and \( l(t) \), respectively. \( m^* \) and \( l^* \) denote the desired steady values of \( m(t) \) and \( l(t) \), respectively. Let
\[
x_1(t) = m(t) - m^*, \quad x_2(t) = l(t) - l^*,
\]
\[
x(t) = [x_1^T(t) x_2^T(t)]^T.
\]

Subsequently, the dynamic equation for \( x(t) \) is given by
\[
\dot{x}(t) = Ax(t) + \dot{A}x(t) - d(t) + Bu(t) + \nu(t)
\]
(36)
where
\[
A = \begin{bmatrix} -\xi_1 - \xi_2 - \xi_2 & 0 \\ -\xi_1 - \xi_2 & -\xi_2 - \xi_1 - \xi_2 \end{bmatrix},
\]
\[
\dot{A} = \begin{bmatrix} \xi_2 & 0 \\ 0 & \xi_2 \end{bmatrix}, \quad B = \begin{bmatrix} \xi_1 \\ 1 \end{bmatrix},
\]
\( x(t) \in \mathbb{R}^2, u(t) \in \mathbb{R}^2, \nu(t) \in \mathbb{R}^2 \) are the state vector, control input and exogenous disturbance of water pollution control system, respectively. In addition, \( \xi_i (i = 1, 2, 3) \) and \( \xi_j (j = 1, 2) \) are known constants and the physical meaning of parameters can be found in [27,28]. Moreover, in order to match the actual situation, we considered nonlinear term and conservativeness during the modeling process. Simultaneously, we assume that there are two actuators, which correspond to no failures occur and failures occur situations, respectively. Therefore, water pollution control system can be described as the following switched system:

\[
\begin{cases}
\dot{x}(t) = A_1 x(t) + \dot{A}_2 x(t - d(t)) + Bu(t) + g(t, x(t - \tau(t))) + C \int_{t-\tau(t)}^{t} x(s) ds + D\nu(t), \quad \text{no failures occur;} \\
\dot{x}(t) = A_3 x(t) + \dot{A}_4 x(t - d(t)) + Bu(t) + g(t, x(t - \tau(t))) + C \int_{t-\tau(t)}^{t} x(s) ds + D\nu(t), \quad \text{failures occur.}
\end{cases}
\]
(37)

Next, we choose \( \xi_1 = 1.5, \xi_2 = 0.6, \xi_3 = 1.4, \xi_1 = 0.4, \xi_2 = 0.6 \), and get that
\[
A = \begin{bmatrix} -2.5 & 0 \\ -1.4 & -0.6 \end{bmatrix}, \quad \dot{A} = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 \\ 1 \end{bmatrix}.
\]

Let \( A_1 = A, \dot{A}_2 = \dot{A}, \)
\[
A_3 = A + \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \quad \dot{A}_4 = \dot{A} + \begin{bmatrix} -0.2 & 0 \\ 0 & -0.2 \end{bmatrix},
\]
\[
C = D = \begin{bmatrix} 0.11 & 0 \\ 0 & 0.11 \end{bmatrix}.
\]
In addition, we choose \( \alpha = 0.45, d_M = 0.8, \tau_M = 0.7, \bar{d} = 0.4, \bar{\tau} = 0.6, \mu = 1.85, \gamma = 0.7, d(t) = 0.4 + 0.4 \sin(t), \tau(t) = 0.1 + 0.6 \sin(t), \nu(t) = (0.02e^{-2t} \frac{0.01}{1+\tau^{20}})^T \),

\[
g(t, x(t-\tau(t))) = \begin{pmatrix} 0.01 \sin(x_1(t-\tau(t))) \\ 0.02 \sin(x_2(t-\tau(t))) \end{pmatrix}.
\]

By solving (11), (23) and (24), we can get average dwell time \( \tau_a > 1.3671 \) and

\[
P_1 = \begin{bmatrix} 1.1552 & -1.4957 \\ -1.4957 & 2.3321 \end{bmatrix},
P_2 = \begin{bmatrix} 7.3874 & -3.7818 \\ -3.7818 & 4.1623 \end{bmatrix},
S_{11} = \begin{bmatrix} -0.3013 & 0.5532 \\ 0.3192 & -1.3771 \end{bmatrix},
S_{21} = \begin{bmatrix} -0.1248 & 0.3584 \\ 0.8228 & -1.626 \end{bmatrix},
S_{12} = \begin{bmatrix} -0.1626 & 0.6926 \\ 0.5098 & -0.0753 \end{bmatrix},
S_{22} = \begin{bmatrix} -0.0753 & 0.4517 \\ 0.5442 & -0.5975 \end{bmatrix},
T_{11} = \begin{bmatrix} 0.5442 & -0.9310 \\ -0.6156 & 0.6971 \end{bmatrix},
T_{21} = \begin{bmatrix} 1.9692 & -0.7040 \\ -0.7040 & 1.4063 \end{bmatrix},
T_{12} = \begin{bmatrix} 2.7299 & -0.1392 \\ -0.8392 & 1.5794 \end{bmatrix},
Y_1 = \begin{bmatrix} 0.0648 & 0.9883 \\ 0.5828 & 0.4235 \end{bmatrix}.
\]

Then, the controller gains are

\[
K_1 = \begin{bmatrix} -1.4034 & 2.2080 \\ 2.7037 & -0.4413 \end{bmatrix},
K_2 = \begin{bmatrix} -2.0797 & 0.8432 \\ 2.6456 & -1.2342 \end{bmatrix}.
\]

Fig. 3 and Fig. 4 correspond to the states response of subsystem 1 and subsystem 2 with the initial condition \( x(0) = (1.5, -1.5)^T \), respectively. Moreover, we can also observe from Fig. 4 that subsystem 2 is unstable when the actuator occur failures. To overcome this obstacle, we design switching signal and use the proposed \( H_\infty \) control schemes to achieve the system (37) with the initial condition \( x(0) = (1.5, -1.5)^T \) is exponentially stabilizable. Form Fig.5 and Fig.6, we can obviously see that the developed method is correct in the control of water pollution process.

Remark 8. Compared with some existing the stability results of nonlinear switched systems, the proposed method has a wider application range from the physical point of view by example of water pollution control. Specifically, it should be pointed out our method can also handle the case that the switched systems contains unstable subsystems. Therefore, our results have improved in terms of conservatism.

V. CONCLUSIONS

This paper mainly studies the issue of exponential \( H_\infty \) control for a class of nonlinear switched systems subject to exogenous disturbance and mixed time-varying delays. Considering the mixed time-varying delays with upper bound and the nonlinear perturbation satisfies the Lipschitz condition, a novel delay-dependent multi-Lyapunov-Krasovskii functional is constructed that utilizes the complete available information about the upper bound of mixed time-varying delays. Subsequently, based on Lyapunov stability theory and average dwell time method, some delay-dependent exponentially stable criteria and a prescribed \( H_\infty \) control performance of switched systems with fixed time-varying delays are obtained by resorting to Jensen’s inequality. Specifically, compared with the existing literature, our conservatism is guaranteed by simulation results, and the proposed control scheme have wide practical range by example of water pollution control. Finally, the state feedback controllers gains are also derived by solving some linear matrix inequalities.

In addition, control system applications are becoming widespread [24-26], as we all know that random delay is caused by network constraints in some microprocessors control systems. It is important to deal with the effects of
random delay while maintaining the control performance. Therefore, the time-varying random delay is my next research plan.

REFERENCES


