Turing Instabilities Analysis and Spatiotemporal Patterns Formation Near a Hopf Bifurcation

Lianchao Gu, Shihong Zhong, Peiliang Gong, Bo Wang, and Hongqing Wang

Abstract—**The Gierer-Meinhardt model is a prototypical and significant activator-inhibitor model of the reaction-diffusion system. This paper focuses on the Turing instabilities analysis and pattern formation in the general Gierer-Meinhardt model. Based on the analysis of eigenvalues of the eigenpolynomial, we derive the existence and stability of the positive equilibrium. By using center manifold theory, the critical value and type of Hopf bifurcation are obtained. The effects of diffusions on the stability of the equilibrium and the bifurcated limit cycle are studied by employing normal form and center manifold reduction. The results show that the equilibrium undergoes a supercritical Hopf bifurcation. If the diffusion coefficients of the two species are sufficiently different, the stable equilibrium and the limit cycle will occur Turing instability, respectively. Moreover, we perform the numerical simulations for the derived results, which states clearly that the Turing patterns are either spot or stripe patterns.**

Index Terms—**asymptotic stability, Hopf bifurcation, supercritical, Turing instability.**

I. Introduction

As early as 1952s, to understand the underlying mechanism for some patterns, Turing [1] proposed the coupled reaction-diffusion equation in his percussive paper. Reaction-diffusion systems are mathematical models that describe how the concentrations of substances distributed in space change under the influence of local chemical reactions and diffusion. As a fact, they play a vital role in explaining and revealing the spatial pattern formations in many fields, such as predator-prey model [2-3], forest fire model [4], models of infectious disease in medicine [5-7], and so on. Turing indicated that although diffusion has a smoothing and trivializing effect on a single compound, in the case of interaction of two or more compounds, different diffusion rates will make the uniform steady state of the corresponding reaction-diffusion systems becomes unstable, resulting in a nonhomogeneous distribution of such reactants. This phenomenon is now known as Turing instability.

In 1972, Gierer and Meinhardt [8] proposed a prototypical model of coupled reaction-diffusion equations through a typical experiment that describes the interaction between two substances, i.e., the activator and the inhibitor. This model can be widely used to model patterns in biology and chemistry, which has the following forms:

$$\begin{cases} \dfrac{\partial a}{\partial t} = \rho_a \dfrac{a^2}{(1+k_a a)\,h} - \mu_a a + \sigma_a + D_a \dfrac{\partial^2 a}{\partial x^2}, \\ \dfrac{\partial h}{\partial t} = \rho_h a^2 - \mu_h h + \sigma_h + D_h \dfrac{\partial^2 h}{\partial x^2}, \end{cases} \tag{1}$$

where $a\,(x,t)$ and $h\,(x,t)$ represent the density of the activator and inhibitor, respectively. D_a and D_h represent the constant diffusion coefficients of the a and h; μ_a, μ_h are the removal rates and ρ_a, ρ_h are the cross-reactions coefficients; σ_a, σ_h are basic production terms; k_a is a saturation constant.

The aforementioned system has experienced significant attention from many scholars. For instance, Ruan [9] has investigated the instability of the homogeneous equilibrium and periodic solution under different diffusion coefficients. Since the Turing criterion depends on the period of the limit cycle obtained from the Hopf bifurcation, it is not easy to verify and apply. As a consequence, many researchers [10-13] also studied the Turing instability of the Gierer-Meinhardt system by using some numerical analysis methods.

Moreover, Liu et al. [14] have considered a reaction-diffusion Gierer-Meinhardt model of morphogenesis subject to Dirichlet fixed boundary condition in the one-dimensional spatial domain. Also, they have conducted a detailed study of the model and obtained the results of the bifurcation. Mai [15] has investigated the spatial patterns of two-dimensional discrete systems with periodic boundary conditions and time continuity and compared them with continuous models. The reaction-diffusion Gierer-Meinhardt system with saturation in the activator production has been considered in reference [16]. Chen et al.[17] have discussed global and blow-up solutions for the general Gierer-Meinhardt system with zero Neumann boundary conditions and obtained some new sufficient conditions for global existence and finite time blow-up of solutions. Furthermore, the existence and asymptotic behaviors of solutions and their stability in terms of diffusion effects have been extensively investigated [18-25].

In this paper, we mainly consider the stability of the positive equilibrium and the bifurcation limit cycle with diffusion. Through qualitative analysis, we know that the stable equilibrium and the stable limit cycle of this system will become unstable under diffusion conditions. The calculations are done explicitly for system (1) with the restriction $k_a = \sigma_h = 0$ (this greatly simplifies the calculations) [26]. To make the later exposition easier, we nondimensionalize

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system (1). Let

$$\bar{t}=\mu_a t, \bar{a}=\frac{\mu_a\rho_h}{\mu_h\rho_a}a, \bar{h}=\frac{\mu_a^2\rho_h}{\mu_h\rho_a^2}h,$$
$$d=\frac{D_a}{D_h}, \mu=\frac{\mu_h}{\mu_a}, \sigma=\frac{\sigma_a\rho_h}{\mu_h\rho_a},$$

we have

$$\begin{cases}\dfrac{\partial\bar{a}}{\partial\bar{t}}=\dfrac{\bar{a}^2}{\bar{h}}-\bar{a}+\sigma+d\dfrac{\partial^2\bar{a}}{\partial\bar{x}^2},\\ \dfrac{\partial\bar{h}}{\partial\bar{t}}=\mu\left(\bar{a}^2-\bar{h}\right)+\dfrac{\partial^2\bar{h}}{\partial\bar{x}^2}.\end{cases} \tag{2}$$

For convenience, we will use a, h, t, x instead of $\bar{a}, \bar{h}, \bar{t}, \bar{x}$. we write

$$\begin{cases}\dfrac{\partial a}{\partial t}=\dfrac{a^2}{h}-a+\sigma+d\dfrac{\partial^2 a}{\partial x^2},\\ \dfrac{\partial h}{\partial t}=\mu\left(a^2-h\right)+\dfrac{\partial^2 h}{\partial x^2}.\end{cases} \tag{3}$$

In addition, we will also assume the following Neumann boundary conditions

$$\begin{aligned}\frac{\partial a}{\partial x}(0,t)=\frac{\partial a}{\partial x}(\pi,t)=0,\\ \frac{\partial h}{\partial x}(0,t)=\frac{\partial h}{\partial x}(\pi,t)=0.\end{aligned} \tag{4}$$

The rest of the paper is organized as follows: In section II, we analyze the existence and stability of the positive equilibrium and the Hopf bifurcation. Section III studies the Turing instability of the equilibrium and limit cycles, and theoretically elaborates on the sufficient conditions and periodic solutions for the equilibrium and spatial homogeneous Turing instability. Finally, in section IV, we give examples to illustrate the analytic conditions and the numerical simulations are employed to verify the theoretical analysis.

II. Existence of Hopf bifurcation

In this section, the stability of system (5) without diffusion at positive equilibrium is investigated.

$$\begin{cases}\dfrac{da}{dt}=\dfrac{a^2}{h}-a+\sigma,\\ \dfrac{dh}{dt}=\mu\left(a^2-h\right).\end{cases} \tag{5}$$

Obviously, the aforementioned system has a unique positive equilibrium $(a^*,h^*)=\Big(1+\sigma,(1+\sigma)^2\Big)$. The Jacobian matrix of (5) at the equilibrium (a^*,h^*) is

$$J(\mu)=\begin{pmatrix}\dfrac{1-\sigma}{1+\sigma} & -\dfrac{1}{(1+\sigma)^2}\\ 2\mu(1+\sigma) & -\mu\end{pmatrix} \tag{6}$$

and the corresponding characteristic equation is

$$\lambda^2-Tr\lambda+Det=0,$$

where $Tr=-1-\mu+\dfrac{2}{1+\sigma}, Det=\mu$.

It is easy to solve that the characteristic equation has two eigenvalues

$$\lambda_{1,2}=-\frac{1}{2}\Big[\frac{\sigma-1}{1+\sigma}+\mu(1+\sigma)^2 \pm\frac{\sqrt{\Big(\mu(1+\sigma)^3+\sigma-1\Big)^2-4\mu(1+\sigma)^4}}{1+\sigma}\Big].$$

Let

$$\mu_0=\frac{1-\sigma}{1+\sigma},$$

and

$$\mu_1=\frac{(3+\sigma)-2\sqrt{2}\sqrt{(1+\sigma)}}{1-\sigma}\mu_0,$$
$$\mu_2=\frac{(3+\sigma)+2\sqrt{2}\sqrt{(1+\sigma)}}{1-\sigma}\mu_0.$$

From this, we have the following Theorem 1.

Theorem 1. *The unique equilibrium of system (5) is asymptotically stable if either condition $(H1)$ or condition $(H2)$ holds and unstable if condition $(H3)$ holds.*

$(H1)$ $\sigma\geq 1$;
$(H2)$ $\mu>\mu_0, 0<\sigma<1$;
$(H3)$ $\mu<\mu_0, 0<\sigma<1$.

Furthermore, we analyze the equilibrium in detail and obtain the following properties. Firstly, if $\sigma\geq 1$, then $\mu_1<\mu_2$, we have $\Big(\mu(1+\sigma)^3+\sigma-1\Big)^2-4\mu(1+\sigma)^4 < 0\,(\geq 0)$ if $\mu_1<\mu<\mu_2\,(0<\mu\leq\mu_1 \text{ or } \mu\geq\mu_2)$. For $0<\sigma<1$, it is easy to note that $\mu_1<\mu_0<\mu_2$, and then $\lambda_{1,2}$ are negative real numbers and $\Big(\mu(1+\sigma)^3+\sigma-1\Big)^2-4\mu(1+\sigma)^4 \geq 0$ if $\mu\geq\mu_2$. $\lambda_{1,2}$ are a pair of conjugate complex numbers with negative real parts if $\mu_0<\mu<\mu_2$ and we have $\Big(\mu(1+\sigma)^3+\sigma-1\Big)^2-4\mu(1+\sigma)^4 < 0$; $\lambda_{1,2}$ are positive real numbers if $\mu_0<\mu<\mu_1$ and we have $\Big(\mu(1+\sigma)^3+\sigma-1\Big)^2-4\mu(1+\sigma)^4 \geq 0$; $\lambda_{1,2}$ are a pair of conjugate complex numbers with positive real parts if $\mu_1<\mu<\mu_0$ and we have $\Big(\mu(1+\sigma)^3+\sigma-1\Big)^2-4\mu(1+\sigma)^4<0$.

Hence, we know that (i) the equilibrium (a^*,h^*) is a stable node if one of the following conditions holds: (ia) $\sigma\geq 1, 0<\mu\leq\mu_1$, (ib) $\sigma\geq 1, \mu\geq\mu_2$, (ic) $0<\sigma<1, \mu\geq\mu_2$; (ii) the equilibrium (a^*,h^*) is a stable foucs if one of the following conditions holds: (iia) $\sigma\geq 1, \mu_1<\mu<\mu_2$, (iib) $0<\sigma<1, \mu_0<\mu<\mu_2$; (iii) the equilibrium (a^*,h^*) is an unstable node if $0<\sigma<1$ and $0<\mu\leq\mu_1$; (iv) the equilibrium (a^*,h^*) is an unstable foucs if $0<\sigma<1$ and $\mu_1<\mu<\mu_0$.

From the analyses made above, we may come to the conclusion that the Jacobian matrix (6) has a pair of pure conjugate imaginary eigenvalues. As a result, the system (5) may undergo Hopf bifurcation at $\mu=\mu_0$ under condition $0<\sigma<1$.

Let $\hat{x}=a-a^*$, $\hat{y}=h-h^*$, and the system (5) at equilibrium (a^*,h^*) becomes

$$\frac{d}{dt}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}=J(\mu)\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}+\begin{pmatrix}f_1(\hat{x},\hat{y},\mu)\\ g_1(\hat{x},\hat{y},\mu)\end{pmatrix}, \tag{7}$$

where

$$\begin{aligned} f_1(\hat{x},\hat{y},\mu) &= \frac{\hat{x}^2}{(1+\sigma)^2} - \frac{2\hat{x}\hat{y}}{(1+\sigma)^3} + \frac{\hat{y}^2}{(1+\sigma)^4} - \frac{\hat{y}^3}{(1+\sigma)^6} \\ &\quad + \frac{2\hat{x}\hat{y}^2}{(1+\sigma)^5} - \frac{\hat{x}^2\hat{y}}{(1+\sigma)^4} + O(4), \\ g_1(\hat{x},\hat{y},\mu) &= \mu\hat{x}^2 + O(4), \end{aligned}$$

and $O(4)$ represents the remaining terms with order greater than or equal to 4.

For $\mu = \mu_0$, we can easily obtain $\lambda_{1,2}(\mu_0) = \pm i\omega_0$ and $\omega_0 = \sqrt{\frac{1-\sigma}{1+\sigma}} > 0$. Then, we know

$$J(\mu_0) = \begin{pmatrix} \dfrac{1-\sigma}{1+\sigma} & -\dfrac{1}{(1+\sigma)^2} \\ 2(1-\sigma) & -\dfrac{1-\sigma}{1+\sigma} \end{pmatrix}.$$

Besides, the eigenvector corresponding to the eigenvalue $i\omega_0$ of $J(\mu_0)$ is $\xi = \left(\frac{1}{2(1+\sigma)} + \frac{\omega_0}{2(1-\sigma)}i, 1\right)^T$ and nonzero transversally condition $\frac{d}{d\mu}Re(\lambda_{1,2})|_{\mu=\mu_0} = \frac{-2\sigma}{1+\sigma} < 0$.

Setting

$$P = \begin{pmatrix} \dfrac{\omega_0}{2(1-\sigma)} & \dfrac{1}{2(1+\sigma)} \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = P \begin{pmatrix} u \\ v \end{pmatrix},$$

then (7) becomes

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_2(u,v,\mu_0) \\ g_2(u,v,\mu_0) \end{pmatrix}, \tag{8}$$

where

$$\begin{aligned} f_2(u,v,\mu_0) &= \frac{\omega_0}{4(1-\sigma^2)}u^2 - \frac{3-\sigma}{2(1+\sigma)^3}uv \\ &\quad + \frac{\omega_0}{4(1+\sigma)^2}v^2 - \frac{\omega_0}{2(1+\sigma)^4(1-\sigma)}u^2v \\ &\quad + \frac{1}{(1+\sigma)^5}uv^2 - \frac{\omega_0}{2(1+\sigma)^5}v^3 + O(4), \\ g_2(u,v,\mu_0) &= \frac{1}{4(1+\sigma)^2}u^2 + \frac{\omega_0}{2(1+\sigma)^2}uv \\ &\quad - \frac{\sigma-1}{4(1+\sigma)^3}v^2 + O(4). \end{aligned}$$

According to the references [27-28], the type of the Hopf bifurcation at equilibrium (a^*, h^*) is determined by the following symbols

$$\begin{aligned} \sigma_0 &= \frac{1}{16}(f_{2uuu} + g_{2uuv} + f_{2uvv} + g_{2vvv}) \\ &\quad + \frac{1}{16\omega_0}[f_{2uv}(f_{2uu} + f_{2vv}) - g_{2uv}(g_{2uu} + g_{2vv}) \\ &\quad - f_{2uu}g_{2uu} + f_{2vv}g_{2vv}]. \end{aligned}$$

where

$$\begin{aligned} & f_{2uu} = \frac{\omega_0}{2(1-\sigma^2)}, g_{2uu} = \frac{1}{2(1+\sigma)^2}, \\ & f_{2vv} = \frac{\omega_0}{2(1+\sigma)^2}, g_{2vv} = -\frac{\sigma-1}{2(1+\sigma)^3}, \\ & f_{2uv} = -\frac{3-\sigma}{2(1+\sigma)^3}, g_{2uv} = \frac{\omega_0}{2(1+\sigma)^2}, \\ & f_{2uuu} = 0, g_{2uuv} = 0, \\ & f_{2uvv} = \frac{2}{(1+\sigma)^5}, g_{2vvv} = 0. \end{aligned}$$

Thus

$$\sigma_0 = \frac{-\sigma}{8(1-\sigma)(1+\sigma)^5} < 0.$$

Based on the above discussion, it is easy for us to draw the Theorem 2.

Theorem 2. *System (5) undergoes a Hopf bifurcation at the equilibrium (a^*, h^*) for $\mu = \mu_0$ if $0 < \sigma < 1$. The Hopf bifurcation is supercritical because of $\sigma_0 < 0$ and the bifurcated limit cycle is stable.*

III. Turing Instability of the Equilibrium and Limit Cycles

In this section, we deduce and obtain the results on Turing instability in equilibrium and limit cycles under the influence of diffusion.

A. Turing Instability of the Equilibrium

We first assume that condition $(H2)$ holds so that the equilibrium (a^*, h^*) is stable for system (5).

With Neumann boundary conditions (4), we consider the diffusion system (3) in the Banach space $\mathrm{H}^2((0,\pi)) \times \mathrm{H}^2((0,\pi))$, where

$$\mathrm{H}^2((0,\pi)) = \left\{ w(\cdot,t) | \frac{\partial^i w}{\partial x^i}(\cdot,t) \in L^2((0,\pi)), i = 0,1,2 \right\}.$$

Obviously, the equilibrium (a^*, h^*) is a stable solution of (3)-(4). The equilibrium (a^*, h^*) is nonlinearly unstable for (3)-(4), if it is unstable in $\mathrm{H}^2((0,\pi)) \times \mathrm{H}^2((0,\pi))$.

Let $u_1 = a - a^*, u_2 = h - h^*$, the linearized system of (3) at the equilibrium (a^*, h^*) is

$$\begin{aligned} \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} &= \begin{pmatrix} \dfrac{1-\sigma}{1+\sigma} - dk^2 & -\dfrac{1}{(1+\sigma)^2} \\ 2\mu(1+\sigma) & -\mu - k^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ &\doteq L \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \end{aligned} \tag{9}$$

with Neumann boundary conditions

$$u_{1x}(0,t) = u_{2x}(0,t) = u_{1x}(\pi,t) = u_{2x}(\pi,t) = 0. \tag{10}$$

The solution (u_1, u_2) of (9)-(10) in $\mathrm{H}^2((0,\pi)) \times \mathrm{H}^2((0,\pi))$ has the following form

$$\begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} A_k \\ H_k \end{pmatrix} e^{\lambda t} cos(kx), \tag{11}$$

where $\lambda \in C$ is the temporal spectrum, k is the wave number and A_k, H_k are complex numbers for $k = 0, 1, 2, \cdots, n, \cdots$. Substituting (11) into (9), we have

$$\begin{aligned} &\sum_{k=0}^{\infty} \lambda \begin{pmatrix} A_k \\ H_k \end{pmatrix} e^{\lambda t} cos(kx) \\ &\quad = \sum_{k=0}^{\infty} \begin{pmatrix} \dfrac{1-\sigma}{1+\sigma} - dk^2 & -\dfrac{1}{(1+\sigma)^2} \\ 2\mu(1+\sigma) & -\mu - k^2 \end{pmatrix} \\ &\quad \times \begin{pmatrix} A_k \\ H_k \end{pmatrix} e^{\lambda t} cos(kx). \end{aligned}$$

We collect items equal to the power of k and in that way we have

$$(\lambda I - J_k(\mu)) \begin{pmatrix} A_k \\ H_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ k = 0, 1, 2, \cdots, n, \cdots \tag{12}$$

where

$$J_k(\mu) = \begin{pmatrix} \dfrac{1-\sigma}{1+\sigma} - dk^2 & -\dfrac{1}{(1+\sigma)^2} \\ 2\mu(1+\sigma) & -\mu - k^2 \end{pmatrix}. \tag{13}$$

Equation (12) has a nonzero solution if and only if

$$det(\lambda I - J_k(\mu)) = 0. \tag{14}$$

Nextly, we rewrite (14) into equation

$$\lambda^2 - Tr(k)\lambda + Det(k) = 0, \ k = 0, 1, 2, \cdots, n, \cdots \tag{15}$$

where

$$\begin{aligned} Tr(k) &= -(1+d)k^2 - 1 - \mu + \frac{2}{1+\sigma}, \\ Det(k) &= \left(dk^2 - \frac{1-\sigma}{1+\sigma}\right)k^2 + d\mu k^2 + \mu. \end{aligned}$$

If condition $(H2)$ holds, $Tr(k) < 0$ for all $k = 0, 1, 2, \cdots, n, \cdots$ and $Det(0) > 0$. Note that if $d > \frac{1-\sigma}{1+\sigma}$, then $Det(k) > 0$, thus, we have the equilibrium (a^*, h^*) is still stable for (3). If $m^2 < \frac{1-\sigma}{d(1+\sigma)} \leq (m+1)^2$ and $d < D_m$, where $D_m = \min\limits_{1 \leq k \leq m} \frac{k^2(1-\sigma)-\mu(1+\sigma)}{k^2(k^2+\mu)(1+\sigma)}$, we have the equilibrium (a^*, h^*) is still stable for (3). If $m^2 < \frac{1-\sigma}{d(1+\sigma)} \leq (m+1)^2$ and $d > D_m$, then there exists at least one negative in $Det(1), Det(2), \cdots, Det(m)$. It implies that the equilibrium (a^*, h^*) will lose its stability and occur Turing pattern for (3).

Now, after the aforementioned analysis, we summarize the Theorem 3 as follows.

Theorem 3. *Assume condition $(H2)$ is satisfied, let*

$$D_m = \min_{1 \leq k \leq m} \frac{k^2(1-\sigma) - \mu(1+\sigma)}{k^2(k^2+\mu)(1+\sigma)},$$

then the equilibrium (a^*, h^*) is stable for (3) if condition $(H4)$ or $(H5)$ is satisfied and is unstable for (3) if condition $(H6)$ is satisfied.

$$\begin{aligned} &(H4)\ d > \frac{1-\sigma}{1+\sigma}; \\ &(H5)\ m^2 < \frac{1-\sigma}{d(1+\sigma)} \leq (m+1)^2, d < D_m; \\ &(H6)\ m^2 < \frac{1-\sigma}{d(1+\sigma)} \leq (m+1)^2, d > D_m. \end{aligned}$$

Remark 1. It is easy to observe that $\min\limits_{1 \leq k \leq m} \frac{k^2(1-\sigma)-\mu(1+\sigma)}{k^2(k^2+\mu)(1+\sigma)} < 0$ for $k = 1$ and $\mu > \mu_0$. If $D_m > 0$, condition $(H5)$ may be satisfied; If $D_m < 0$, condition $(H5)$ can't hold due to $d > 0$. So, condition $(H5)$ can't be satisfied due to $d > 0$. In this paper, we deem that $D_m \geq 0$.

B. Turing Instability of the Limit Cycle

In this section, we apply the normal form and center manifold theory to the diffusion system (3). The stability of the limit cycle under inhomogeneous disturbances in space in theorem 3 is analyzed. Here suppose condition $(H3)$ holds, then the supercritical Hopf bifurcation occurs at $\mu = \mu_0$. And, the limit cycle is stable under spatially homogeneous perturbation.

According to the reference [29], let $u_1 = a - a^*, u_2 = h - h^*, \mu = \mu_0$, and at $U = (u_1, u_2)^T$ system (3) becomes

$$\begin{cases} U_t = \left[J(\mu_0) + D \begin{pmatrix} \partial_{xx} & 0 \\ 0 & \partial_{xx} \end{pmatrix}\right] U + F(U, \mu_0), \\ U_x(0,t) = U_x(\pi, t) = (0,0)^T, \end{cases} \tag{16}$$

where

$$J(\mu_0) = \begin{pmatrix} \dfrac{1-\sigma}{1+\sigma} & -\dfrac{1}{(1+\sigma)^2} \\ 2(1-\sigma) & \dfrac{-1+\sigma}{1+\sigma} \end{pmatrix},$$

$$D = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}.$$

According to the reference [30], $F(U, \mu_0)$ has the following form

$$F(U, \mu_0) = \frac{1}{2}Q(U,U) + \frac{1}{6}C(U,U,U) + O\left(|U|^4\right),$$

and

$$Q(U,V) = \begin{pmatrix} Q_1(U,V) \\ Q_2(U,V) \end{pmatrix},$$

$$C(U,V,W) = \begin{pmatrix} C_1(U,V,W) \\ C_2(U,V,W) \end{pmatrix},$$

with

$$
\begin{aligned}
Q_1(U,V) &= \frac{2u_1v_1}{(1+\sigma)^2} - \frac{2(u_2v_1+u_1v_2)}{(1+\sigma)^3} + \frac{2u_2v_2}{(1+\sigma)^4},\\
Q_2(U,V) &= \frac{2(1-\sigma)u_1v_1}{1+\sigma},\\
C_1(U,V,W) &= -\frac{2}{(1+\sigma)^6}\left((1+2\sigma)u_2v_1w_1\right.\\
&\quad + (1+2\sigma)u_1v_2w_1 + (1+2\sigma)u_1v_1w_2\\
&\quad - 2(1+\sigma)u_2v_1w_2 - 2(1+\sigma)u_1v_2w_2\\
&\quad - 2(1+\sigma)u_2v_2w_1 + \sigma^2u_1v_2w_1\\
&\quad \left.+ \sigma^2u_2v_1w_1 + \sigma^2u_1v_1w_2 + 3u_2v_2w_2\right),\\
C_2(U,V,W) &= 0.
\end{aligned}
$$

For any $U=(u_1,u_2)^T, V=(v_1,v_2)^T, W=(w_1\ ,w_2)^T$, and $U,V,W \in \mathrm{H}^2((0,\pi)) \times \mathrm{H}^2((0,\pi))$, the linear operator L defined in (9) for $\mu=\mu_0$ is

$$
LU = \left[J(\mu_0) + D\begin{pmatrix} \partial_{xx} & 0 \\ 0 & \partial_{xx}\end{pmatrix}\right]U,
$$

for $U \in \mathrm{H}^2((0,\pi)) \times \mathrm{H}^2((0,\pi))$.

Let L^* be the adjoint operator of L, then

$$
L^*U = \left[J^*(\mu_0) + D\begin{pmatrix} \partial_{xx} & 0 \\ 0 & \partial_{xx}\end{pmatrix}\right]U,
$$

where

$$
J^*(\mu_0) = \begin{pmatrix} \dfrac{1-\sigma}{1+\sigma} & 2(1-\sigma) \\ -\dfrac{1}{(1+\sigma)^2} & \dfrac{-1+\sigma}{1+\sigma} \end{pmatrix}.
$$

Obviously, $\langle L^*U,V\rangle = \langle U,LV\rangle$ and the inner product in $\mathrm{H}^2((0,\pi)) \times \mathrm{H}^2((0,\pi))$ is defined as $\langle U,V\rangle = \frac{1}{\pi}\int_0^\pi \bar{U}^T V dx$. The linearized system of (16) at the equilibrium $(0,0)$ is

$$
\begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} = L\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{17}
$$

with Neumann boundary conditions

$$
U_x(0,t) = U_x(\pi,t) = (0,0)^T. \tag{18}
$$

The solution (u_1,u_2) of (17)-(18) in $\mathrm{H}^2((0,\pi)) \times \mathrm{H}^2((0,\pi))$ has the following form

$$
\begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} = \sum_{k=0}^{\infty}\begin{pmatrix} a_k \\ h_k \end{pmatrix} e^{\lambda t}cos(kx), \tag{19}
$$

where $\lambda \in \mathrm{C}$ is the temporal spectrum, k is the wave number and a_k, h_k are complex numbers for $k=0,1,2,\cdots,n,\cdots$. Substituting (19) into (17), we have

$$
\sum_{k=0}^{\infty}\lambda\begin{pmatrix} a_k \\ h_k \end{pmatrix} e^{\lambda t}cos(kx) = \sum_{k=0}^{\infty}L_k\begin{pmatrix} a_k \\ h_k \end{pmatrix} e^{\lambda t}cos(kx).
$$

We collect items equal to the power of k and in that way we have

$$
(\lambda I - L_k)\begin{pmatrix} a_k \\ h_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ k=0,1,2,\cdots,n,\cdots \tag{20}
$$

where

$$
L_k = \begin{pmatrix} \dfrac{1-\sigma}{1+\sigma} - dk^2 & -\dfrac{1}{(1+\sigma)^2} \\ 2(1-\sigma) & \dfrac{-1+\sigma}{1+\sigma} - k^2 \end{pmatrix}.
$$

Equation (23) has a nonzero solution if and only if

$$
det(\lambda I - L_k) = 0.
$$

Nextly, the equation can be rewritten as

$$
\lambda^2 - T_k\lambda + D_k = 0,\ k=0,1,2,\cdots,n,\cdots
$$

where

$$
\begin{aligned}
T_k &= -(1+d)k^2,\\
D_k &= \left(k^2 - \frac{(1-\sigma)}{d(1+\sigma)}\right)dk^2 + \frac{dk^2(1-\sigma)}{1+\sigma} + \omega_0^2.
\end{aligned}
$$

Here, $T_0=0$, $T_k<0$ for all $k=1,2,\cdots,n,\cdots$ and $D_0=\omega_0^2>0$. Then, it follows that for $k=0$, the eigenvalues of L are with zero real parts. We need to proceed the center manifold reduction. Firstly, if $d>\frac{1-\sigma}{1+\sigma}$ then $D_k > \frac{dk^2(1-\sigma)}{1+\sigma} + \omega_0^2 > 0$ for all $k=0,1,2,\cdots,n,\cdots$. The stable limit cycle is still stable for system (3). Then, if $m^2 < \frac{1-\sigma}{d(1+\sigma)} \le (m+1)^2$ and $d<\overline{D}$, where $\overline{D} = \min\limits_{1\le k\le m}\frac{(k^2-1)(1-\sigma)}{k^2(1-\sigma+k^2(1+\sigma))}$. The stable limit cycle is still stable for system (3). Furthermore, if $m^2 < \frac{1-\sigma}{d(1+\sigma)} \le (m+1)^2$ and where $\overline{D} = \min\limits_{1\le k\le m}\frac{(k^2-1)(1-\sigma)}{k^2(1-\sigma+k^2(1+\sigma))}$, then there exists at least one negative in $D_1, D_2,\cdots,D_m$.

We choose $q = (\frac{1}{2(1+\sigma)} + \frac{iw_0}{2(1-\sigma)}, 1), q* = (\frac{(1-\sigma)}{\omega_0}i, \frac{1}{2} - \frac{(1-\sigma)i}{2\omega_0(1+\sigma)})$, so that $Lq = i\omega_0 q, L^*q^* = -i\omega_0 q^*, \langle q^*,q\rangle = 1$ and $\langle q^*,\bar{q}\rangle = 0$.

According to the reference [30], we write $U = zq + \bar{z}\bar{q} + w, z = \langle q^*,U\rangle, w=(w_1,w_2)^T$ and

$$
\begin{cases} u_1 = \dfrac{1}{2(1+\sigma)}(z-\bar{z}) + \dfrac{iw_0}{2(1-\sigma)}(z-\bar{z}) + w_1,\\ u_2 = z+\bar{z}+w_2. \end{cases}
$$

The system (16) in (z,w) coordinates can be expressed as

$$
\begin{cases} \dot{z} = i\omega_0 z + \left\langle q^*, \tilde{f}\right\rangle,\\ \dot{w} = Lw + H(z,\bar{z},w), \end{cases} \tag{21}
$$

where

$$
\begin{aligned}
\tilde{f} &= F(zq+\bar{z}\bar{q}+w,\mu_0),\\
H(z,\bar{z},w) &= \widetilde{f} - \left\langle q^*,\widetilde{f}\right\rangle q - \left\langle \bar{q}^*,\widetilde{f}\right\rangle \bar{q}.
\end{aligned}
$$

By calculation we have

$$\begin{aligned}
\tilde{f} &= \frac{1}{2}Q(q,q)z^2 + Q(q,\bar{q})z\bar{z} + \frac{1}{2}Q(\bar{q},\bar{q})\bar{z}^2 \\
&\quad + O\left(|z|^3, |z|\cdot|w|, |w|^2\right), \\
\left\langle q^*, \tilde{f}\right\rangle &= \frac{1}{2}\left\langle q^*, Q(q,q)\right\rangle z^2 + \left\langle q^*, Q(q,\bar{q})\right\rangle z\bar{z} \\
&\quad + \frac{1}{2}\left\langle q^*, Q(\bar{q},\bar{q})\right\rangle \bar{z}^2 + O\left(|z|^3, |z|\cdot|w|, |w|^2\right), \\
\left\langle \bar{q}^*, \tilde{f}\right\rangle &= \frac{1}{2}\left\langle \bar{q}^*, Q(q,q)\right\rangle z^2 + \left\langle \bar{q}^*, Q(q,\bar{q})\right\rangle z\bar{z} \\
&\quad + \frac{1}{2}\left\langle \bar{q}^*, Q(\bar{q},\bar{q})\right\rangle \bar{z}^2 + O\left(|z|^3, |z|\cdot|w|, |w|^2\right), \\
H(z,\bar{z},w) &= \frac{1}{2}z^2H_{20} + z\bar{z}H_{11} + \frac{1}{2}\bar{z}^2H_{02} \\
&\quad + O\left(|z|^3, |z|\cdot|w|, |w|^2\right),
\end{aligned}$$

where

$$\begin{cases}
H_{20} = Q(q,q) - \left\langle q^*, Q(q,q)\right\rangle q - \left\langle \bar{q}^*, Q(q,q)\right\rangle \bar{q}, \\
H_{11} = Q(q,\bar{q}) - \left\langle q^*, Q(q,\bar{q})\right\rangle q - \left\langle \bar{q}^*, Q(q,\bar{q})\right\rangle \bar{q}, \\
H_{02} = Q(\bar{q},\bar{q}) - \left\langle q^*, Q(\bar{q},\bar{q})\right) q - \left\langle \bar{q}^*, Q(\bar{q},\bar{q})\right\rangle \bar{q}.
\end{cases}$$

Furthermore,

$$H_{20}{=}H_{11} = H_{02} = (0,0)^T,$$

which implies $H(z,\bar{z},w) = O\left(|z|^3, |z|\cdot|w|, |w|^2\right)$.

The system (21) has a central manifold, which can be written as

$$w = \frac{1}{2}z^2w_{20} + z\bar{z}w_{11} + \frac{1}{2}\bar{z}^2w_{02} + O\left(|z|^3\right).$$

Together with $Lw + H(z,\bar{z},w) = \dot{w} = \frac{\partial w}{\partial z}\dot{z} + \frac{\partial w}{\partial \bar{z}}\dot{\bar{z}}$, we can get

$$\begin{aligned}
w_{20} &= \left[2i\omega_0 - L\right]^{-1}H_{20} = \left[2i\omega_0 - J\left(\mu_0\right)\right]^{-1}H_{20} \\
&= (0,0)^T, \\
w_{11} &= -L^{-1}H_{11} = -J^{-1}\left(\mu_0\right)H_{11} \\
&= (0,0)^T, \\
w_{02} &= \left[-2i\omega_0 - L\right]^{-1}H_{02} = \left[-2i\omega_0 - J\left(\mu_0\right)\right]^{-1}H_{02} \\
&= (0,0)^T,
\end{aligned}$$

then, $w = O\left(|z|^3\right)$.

Therefore, the diffusion system restricted to the center manifold is

$$\dot{z} = i\omega_0 z + \left\langle q^*, \tilde{f}\right\rangle = i\omega_0 z + \sum_{2\le i+j\le 3}\frac{g_{ij}}{i!j!}z^i\bar{z}^j + O\left(\left|z^4\right|\right) \tag{22}$$

where

$$\begin{aligned}
g_{20} &= \left\langle q^*, Q(q,q)\right\rangle, \\
g_{11} &= \left\langle q^*, Q(q,\bar{q})\right\rangle, \\
g_{02} &= \left\langle q^*, Q(\bar{q},\bar{q})\right\rangle, \\
g_{21} &= 2\left\langle q^*, Q\left(w_{11},q\right)\right\rangle + \left\langle q^*, Q\left(w_{20},\bar{q}\right)\right\rangle \\
&\quad + \left\langle q^*, C(q,q,\bar{q})\right\rangle \\
&= \left\langle q^*, C(q,q,\bar{q})\right\rangle.
\end{aligned}$$

For the system (21), we know that it is determined by (22). The $Poincar\acute{e}$ normal form of (16) can be written as follows

$$\dot{z} = (\alpha(\mu) + i\omega(\mu))z + z\sum_{j=1}^{M}c_j(\mu)(z\bar{z})^j \tag{23}$$

where z is a complex variable, $M \ge 1$, and $c_j(\mu)$ is a complex-valued coefficient.

Nextly, we derive the key expression of Turing stability on the central manifold. According to the reference [30], we have

$$\begin{aligned}
c_1(\mu_0) &= \frac{g_{20}g_{11}[3\alpha(\mu_0) + i\omega(\mu_0)]}{2\left[\alpha^2(\mu_0) + \omega^2(\mu_0)\right]} + \frac{\left|g_{11}\right|^2}{\alpha(\mu_0) + i\omega(\mu_0)} \\
&\quad + \frac{\left|g_{02}\right|^2}{2[\alpha(\mu_0) + 3i\omega(\mu_0)]} + \frac{g_{21}}{2}.
\end{aligned}$$

Since $\alpha\left(\mu_0\right) = 0$ and $\omega\left(\mu_0\right) = \omega_0 > 0$, it is easy to obtain that

$$\operatorname{Re}\left(c_1\left(\mu_0\right)\right) = \operatorname{Re}\left[\frac{g_{20}g_{11}}{2\omega_0}i + \frac{g_{21}}{2}\right].$$

Due to

$$\begin{aligned}
g_{20} &= \left\langle q^*, Q(q,q)\right\rangle = \frac{i - 3\omega_0}{2(1+\sigma)^3\omega_0}, \\
g_{11} &= \left\langle q^*, Q(q,\bar{q})\right\rangle = \frac{-i + \omega_0}{2(1+\sigma)^3\omega_0}, \\
g_{21} &= \left\langle q^*, C(q,q,\bar{q})\right\rangle = \frac{(2-\sigma)i + \omega_0(1+\sigma)}{(1+\sigma)^6\omega_0},
\end{aligned}$$

then, $\operatorname{Re}\left[c_1\left(\mu_0\right)\right] = \frac{-\sigma}{2(1-\sigma)(1+\sigma)^5} < 0$, which implies that the spatially homogeneous periodic solution is stable.

According to the above analysis, we obtain Theorem 4.

Theorem 4. *Assume condition $(H3)$ is satisfied, let*

$$\overline{D} = \min_{1\le k\le m}\frac{(k^2-1)(1-\sigma)}{k^2(1-\sigma+k^2(1+\sigma))},$$

the spatially homogeneous periodic solution for system (3) is stable if condition $(H7)$ or $(H8)$ holds, the spatially homogeneous periodic solution for system (3) is unstable if condition $(H9)$ holds.

$$\begin{aligned}
&(H7)\ \ d > \frac{1-\sigma}{1+\sigma}; \\
&(H8)\ \ m^2 < \frac{1-\sigma}{d(1+\sigma)} \le (m+1)^2, d < \overline{D}; \\
&(H9)\ \ m^2 < \frac{1-\sigma}{d(1+\sigma)} \le (m+1)^2, d > \overline{D}.
\end{aligned}$$

Remark 2. It is easy to observe that $\min\limits_{1\le k\le m}\frac{(k^2-1)(1-\sigma)}{k^2(1-\sigma+k^2(1+\sigma))} = 0$ for $k = 1$. So, condition $(H8)$ can't be satisfied due to $d > 0$.

IV. Numerical Simulation for Pattern formation

In this section, the numerical simulations are employed to verify the theoretical analysis in section II and section III.

First of all, the Hopf bifurcation curve in parameter space (σ, μ_0) and the supercritical Hopf bifurcation diagram in three-dimensional space (μ, a, h) of system (5) are shown in Figure 1 (a) and Figure 1 (b), respectively. The supercritical Hopf bifurcation diagram illustrates the stable equilibrium (black curve), unstable equilibrium (red curve), stable limit cycles (blue surface) and specific limit cycles (rose red curve) for $\mu = 0.6$.

Nextly, we choose $\sigma = 0.2$, then $\mu_0 = 0.6667$ and $(a^*, h^*) = (1.20, 1.44)$. In order to illustrate the conclusion of Theorem 1, we let $\mu = 0.68$ $(\mu > \mu_0)$ and $\mu = 0.64$ $(\mu < \mu_0)$ make it satisfy the conditions of $(H2)$ and $(H3)$ in Theorem 1, respectively. The corresponding dynamics of the equilibrium of system (5) are shown in Figure 2. Figure 2 (a) illustrates that equilibrium (a^*, h^*) is asymptotically stable when $\mu = 0.68$. Figure 2 (b) illustrates that the equilibrium (a^*, h^*) is unstable and the phase orbit converges to the stable limit cycle when $\mu = 0.64$.

Furthermore, we adopt numerical simulations to explain the influences of diffusions on the stability of equilibrium and the spatial homogeneous periodic solution bifurcating from Hopf bifurcation. Let $\sigma = 0.2$, $\mu = 0.68$, then $\frac{1-\sigma}{1+\sigma} = 0.6667$. Particularly, let $d = 0.68$ make the conditions $(H2)$ and $(H4)$ of Theorem 3 hold. The equilibrium of system (3) is still stable in the above situations. The patterns are illustrated in Figure 3 (a) and Figure 3 (b). Let $\sigma = 0.2, \mu = 0.68, d = 0.05$, then $\frac{1-\sigma}{d(1+\sigma)} = 13.333$, $D_m = 0.0610652$, the conditions $(H2)$ and $(H5)$ of Theorem 3 hold. The equilibrium of system (3) is still stable illustrated in Figure 4 (a) and Figure 4 (b). Let $\sigma = 0.2, \mu = 0.68, d = 0.04$, then $\frac{1-\sigma}{d(1+\sigma)} = 16.66667$, $D_m = 0.03742006$, the conditions $(H2)$ and $(H6)$ of Theorem 3 hold. The stable equilibrium of system (5) becomes unstable for system (3) in the above situations, which means that Turing instability occurs. Please see Figure 5 (a) and Figure 5 (b). It is easy to find the stripe patterns.

In the following, we illustrate the Turing instability in homogeneous limit cycles. Let $\sigma = 0.2$ and $\mu = 0.64$, then $\frac{1-\sigma}{1+\sigma} = 0.6667$. Particularly, let $d = 0.68$ make conditions $(H3)$ and $(H7)$ of Theorem 4 hold. The stable limit cycle of system (3) is still stable in the above situations. Please see Figure 6 (a) and Figure 6 (b). Let $\sigma = 0.2$, $\mu = 0.64$ and $d = 0.1$, then $\overline{D} = 0$, conditions $(H3)$ and $(H9)$ of Theorem 4 hold. The stable limit cycle of system (5) becomes unstable for system (3) due to the effects of diffusions. Please see Figure 7 (a) and Figure 7 (b). The Turing patterns of spot stripe or spot are clearly observed by enlarging Figure 7 (b) into Figure 8 (a) and Figure 8 (b) respectively.

V. Conclusion

Turing patterns dynamics of Gierer-Meinhardt of the activator-inhibitor model has been theoretically and numerically studied in this paper. The center manifold theory was first denoted to analyzing the existence of Hopf bifurcation. It was found that the system (5) undergoes a supercritical Hopf bifurcation at the equilibrium (a^*, h^*). Afterward, based on the normal form and the center manifold reduction, some conditions of the Turing instability are obtained under the diffusive effects. Analysis results show that the system (3) will undergo Turing instability in equilibrium and periodic solutions, and some spot or stripe patterns will be possibly formed. In addition to theoretical analysis, the numerical simulation methods were also employed to verify the conclusions of the theoretical analysis. The results indicate that complex dynamics do happen in Gierer-Meinhardt model. In particular, it was found that under certain conditions, diffusion could cause instability of the equilibrium and the limit cycle, which are otherwise stable without diffusion. To sum up, the results are conducive to understand the formation of biological patterns, and the method provides us with an understanding of the dynamical complexity of space and time in the activator-inhibitor model. More interesting and complex behavior about such a model will further be explored in the future.

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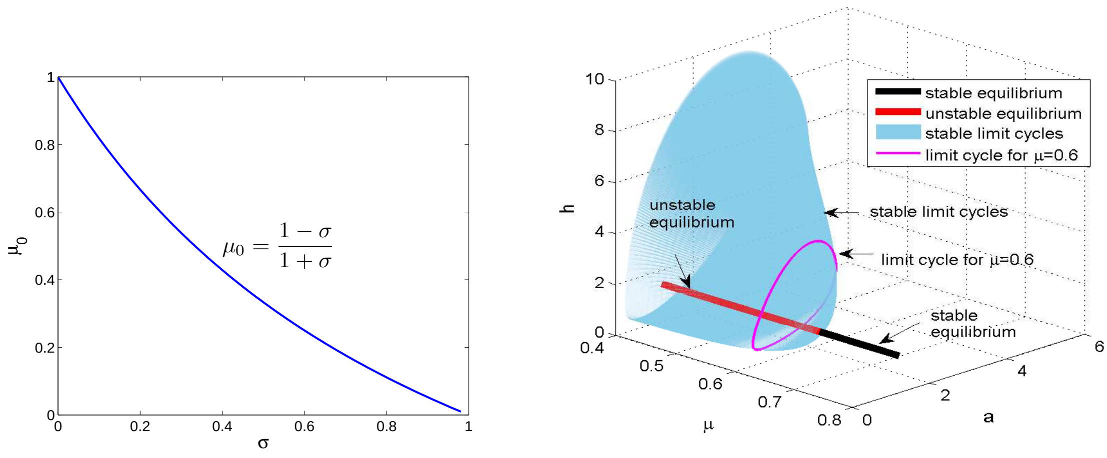
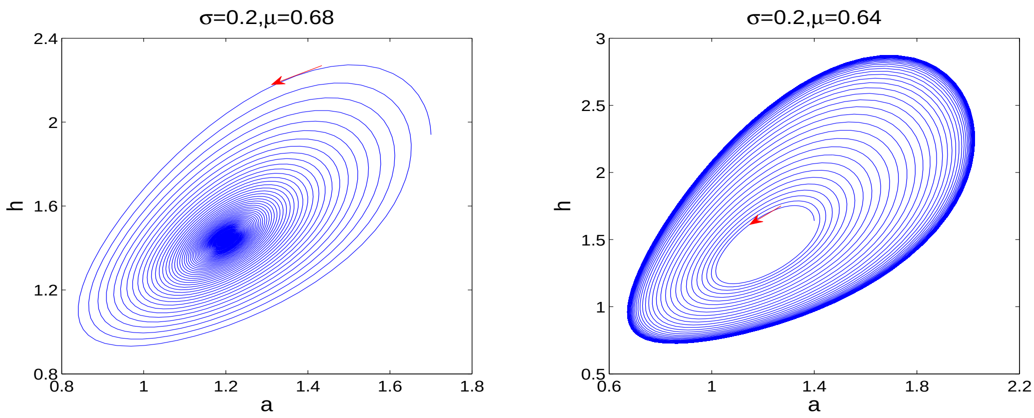


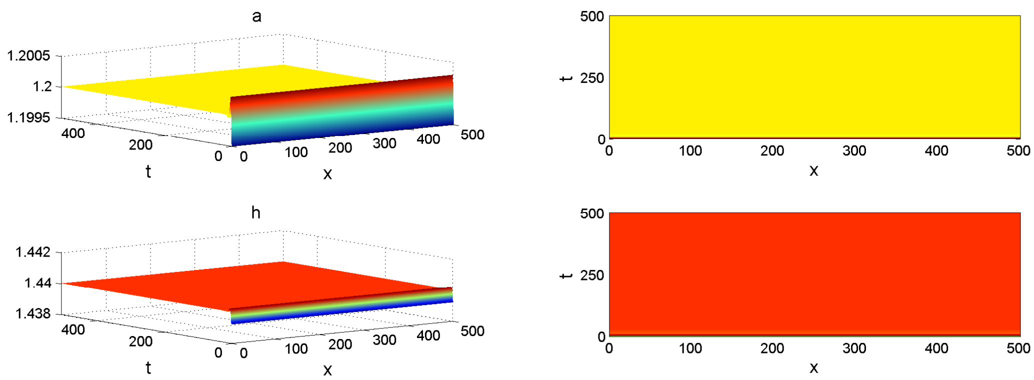
(a) (b)

Fig. 1: (a) The supercritical Hopf bifurcation diagram of system (5). (b) The supercritical Hopf bifurcation diagram in three-dimensional space (μ, a, h) of (5).



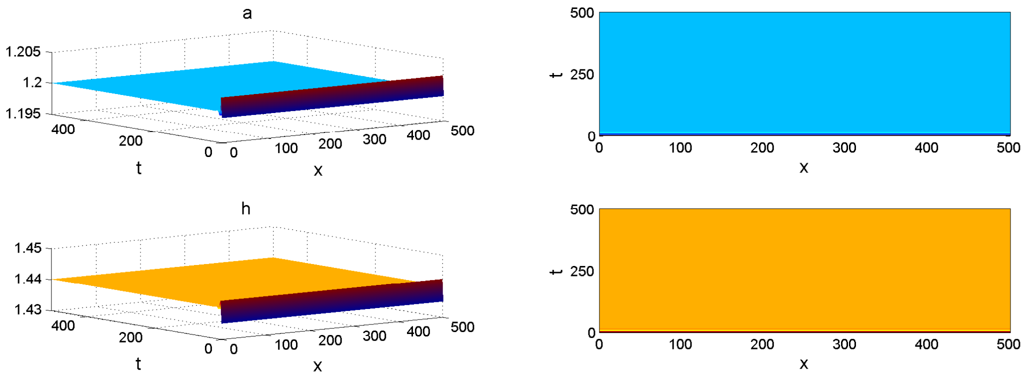
(a) (b)

Fig. 2: (a) The phase orbit starting from $(a_0, h_0) = (1.70, 1.94)$ converges to equilibrium $(a^*, h^*) = (1.20, 1.44)$. (b) The equilibrium is unstable while phase orbit starting from $(a_0, h_0) = (1.40, 1.64)$ converges to stable limit cycle.



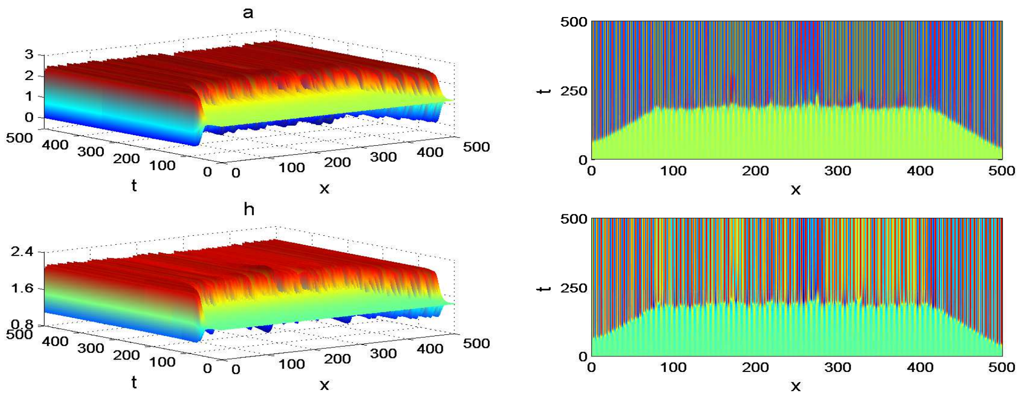
(a) (b)

Fig. 3: (a) The stable equilibrium $(a^*, h^*) = (1.20, 1.44)$ of system (3) is still stable under conditions $(H2)$ and $(H4)$. (b) The projection of (a) in (x, t) coordinates.



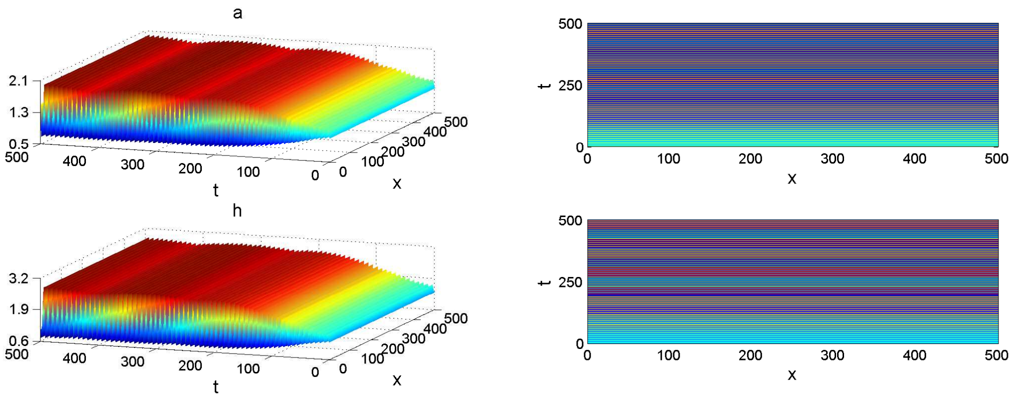
(a) (b)

Fig. 4: (a) The stable equilibrium $(a^*, h^*) = (1.20, 1.44)$ of system (3) is still stable under conditions $(H2)$ and $(H5)$. (b) The projection of (a) in (x, t) coordinates.



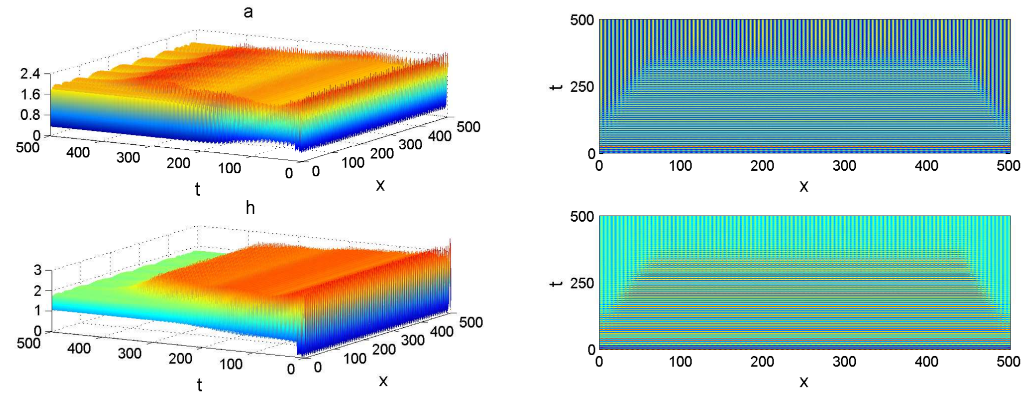
(a) (b)

Fig. 5: (a) The stable equilibrium $(a^*, h^*) = (1.20, 1.44)$ becomes unstable for system (3) under conditions $(H2)$ and $(H6)$. (b) The projection of (a) in (x, t) coordinates.



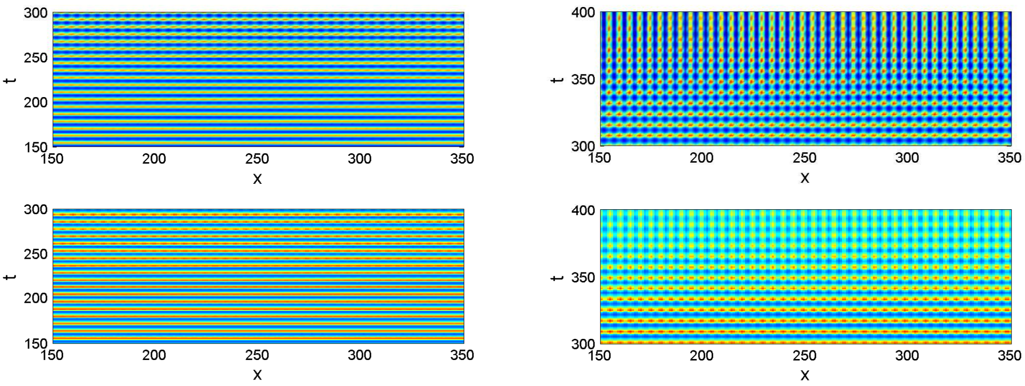
(a) (b)

Fig. 6: (a) The stable limit cycle of system (5) is still stable for system (3) under conditions $(H3)$ and $(H7)$. (b) The projection of (a) in (x, t) coordinates.



(a) (b)

Fig. 7: (a) The stable limit cycle becomes unstable for system (3) under conditions $(H3)$ and $(H9)$. (b) The projection of (a) in (x, t) coordinates.



(a) (b)

Fig. 8: The local enlargement of Fig. 7(b).

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