The Optimal Estimation of the Turán-type Inequality for the Gamma Function and Its Numerical Method

Zhongfeng Sun and Huizeng Qin

Abstract—In this paper, we demonstrate that
\[
\Delta_n(x) = \Gamma^{(n-1)}(x) \Gamma^{(n+1)}(x) - \left[ \Gamma^{(n)}(x) \right]^2
\]
are strictly convex on \((0, +\infty)\) for all \(n = 1, 3, \ldots\), and there exists a unique global minimum \(x_n\) of \(\Delta_n(x)\) for each odd positive integer \(n\). On the other hand, we develop the algorithm for calculating the global minimum or the zero point of \(\Delta_n(x)\) based on the Newton’s method and the recurrence relation of \(\Gamma^{(n)}(x)\) associated with the Digamma function, the numerical results show that it’s more effective than other two algorithms. Furthermore, we find that \(\Delta_n(x_n)\) is strictly increasing and \(\Delta_n(x_n) \geq \alpha = 0.6359 \ldots\) for all odd integers \(n \geq 7\) through the numerical experiments.

Index Terms—Turán-type Inequality, Gamma Function, Digamma Function, Newton’s Method.

I. INTRODUCTION

The Turán inequality for the Legendre polynomials \(P_n(x)\) was introduced by Turán[1] as follows,
\[
P_n^2(x) \geq P_{n-1}(x)P_{n+1}(x), \quad -1 \leq x \leq 1,
\]
where \(n\) is the positive integer. In the last ten years, the issue of the Turán-type inequality for the most important special functions has attracted the attention of scholars.

For convenience, we introduce the following notation
\[
f^{(n)}(x) := \frac{d^n}{dx^n} f(x), \quad n = 0, 1, 2, 3, \ldots,
\]
where \(f^{(0)}(x)\) represents \(f(x)\). The gamma function \(\Gamma(x)\) and its derivatives are defined by the following integral formula
\[
\Gamma^{(n)}(x) = \int_0^\infty t^n e^{-t} t^{n-1} \ln^n t dt,
\]
where \(x > 0\) and \(n = 0, 1, 2, 3, \ldots\).

Alzer and Felder[2] proved the Turán-type inequality for the gamma function, and the main result is given in the following Lemma.

Lemma 1.1 For \(n = 1, 3, 5, \ldots\) and real numbers \(x > 0\), one has
\[
\Delta_n(x) = \Gamma^{(n-1)}(x) \Gamma^{(n+1)}(x) - \left[ \Gamma^{(n)}(x) \right]^2 \geq \alpha,
\]
with the best possible constant
\[
\alpha = \min_{\frac{3}{2} \leq x \leq 2} \Gamma^2(x) \psi'(x) = 0.6359 \ldots
\]
where \(\psi(x)\) is the Digamma (psi) function defined by
\[
\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.
\]

Merovc[i3] developed new Turán-type inequalities for the \((q, k)\)-polygamma functions expressed as follows,
\[
\psi_{q,k}^{(m)}(x) \psi_{q,k}^{(n)}(y) \geq \psi_{q,k}^{(m+q)} \left( \frac{x}{a} + \frac{y}{b} \right),
\]
where \(\psi_{q,k}(x)\) is the \((q, k)\)-analogue of the psi function. Subsequently, Nantomah[4] extended (6) to the general case with the aid of the Hölder’s inequality and the generalized Minkowski’s inequality. Based on proving monotonicity for special ratio of sections for series of Mittag-Leffler functions, Sitnik and Mehrez[5] proved that the following Turán-type inequalities hold true,
\[
E_{\alpha,\beta}(x)E_{\alpha,\beta+2}(x) \geq [E_{\alpha,\beta+1}(x)]^2,
\]
where \(E_{\alpha,\beta}(x) = \Gamma(\beta)E_{\alpha,\beta}(x)\) and \(E_{\alpha,\beta}(x)\) is the Mittag-Leffler function defined by
\[
E_{\alpha,\beta}(x) = \sum_{n=0}^\infty \frac{x^n}{\Gamma(\beta + na)}.
\]

Consequently, Dou and Yin[6] extended (7) to the general case with four parameters represented as,
\[
\frac{(\beta + k)(\beta + qk)}{\beta + (q + 1)k} E_{\alpha,\beta,q,k}(x)E_{\alpha,\beta+2k,q,k}(x) > [E_{\alpha,\beta+k,q,k}(x)]^2.
\]

By means of the new Cauchy-Bunyakovsky-Schwarz inequality, Bhandari and Bisui[7] derived some Turán-type inequalities for the \(n\)-th derivative of gamma function, the polygamma function, the exponential integral function and the Abramowitz’s function. Baricz, Bhandari et al.[8], [9], [10], [11] established some Turán-type inequalities for the Bessel, the modified Bessel, the Krätzel functions and the Lommel functions, respectively. Baricz[12] constructed the Turán-type and the reverse Turán-type inequalities of the generalized complete elliptic integrals, which leads to a sharp lower bound for the generalized complete elliptic integral of the first kind.

Motivated by [2], we aim to construct the optimal estimations \(\Delta_n(x) \geq \alpha_n \geq \alpha, x \in (0, +\infty)\) for each odd positive integer \(n\) and develop the algorithm to calculate \(\alpha_n\).

The paper is organized as follows. In Section II, we demonstrate that \(\Delta_n(x)\) is strictly convex on \((0, +\infty)\) and
This, combining with (12), implies that there exists a unique global minimum $x_n$ for each odd positive integer $n$. In Section III, we develop three algorithms based on the Newton’s method for calculating the global minimum of $\Delta_n(x)$ for odd positive integer $n$ and the zero point of $\Delta_n(x)$ for even positive integer $n$. A final conclusion is given in Section IV.

II. THE GLOBAL MINIMUM OF $\Delta_n(x)$ FOR ODD POSITIVE INTEGER $n$

**Theorem 2.1** For each odd positive integer $n$ and $x > 0$, we have $\Delta_n'(x) > 0$; that is, $\Delta_n'(x)$ is strictly increasing on $(0, +\infty)$.

**Proof.** Differentiating $\Delta_n(x)$ in (3) with respect to $x$, one has

$$\Delta_n'(x) = \Gamma^{(n-1)}(x) \Gamma^{(n+2)}(x) - \Gamma^{(n)}(x) \Gamma^{(n+1)}(x),$$

(10)

and

$$\Delta_n''(x) = \Gamma^{(n-1)}(x) \Gamma^{(n+3)}(x) - [\Gamma^{(n)}(x)]^2,$$

(11)

for $n = 1, 2, \ldots$. With the aid of (2), $\Delta_n''(x)$ can be expressed by double integral as follows,

$$\Delta_n''(x) = \int_0^\infty t x^{-1} e^{-t} \ln t \ dt dl$$

$$\times \int_0^\infty s x^{-1} e^{-s} \ln t^2 \ ds ds$$

$$- \int_0^\infty t x^{-1} e^{-t} \ln t \ dt dl$$

$$\times \int_0^\infty s x^{-1} e^{-s} \ln t^2 \ ds ds$$

(12)

$$= \int_0^\infty \int_0^\infty (st)^{x-1} e^{-t - s} (\ln t) dtds,$$

Exchanging variables $t$ and $s$, we obtain

$$\Delta_n''(x) = \int_0^\infty \int_0^\infty (st)^{x-1} e^{-t - s} (\ln t) dtds,$$

(13)

This, combining with (12), implies that

$$\Delta_n''(x) = \frac{1}{2} \int_0^\infty \int_0^\infty (st)^{x-1} e^{-t - s} (\ln t) dtds,$$

(14)

which leads to $\Delta_n''(x) > 0$ for $n = 1, 3, \ldots$ and $x > 0$. Thus, $\Delta_n''(x)$ is strictly increasing on $(0, +\infty)$ for each odd positive integer $n$. ■

**Theorem 2.2** For every odd integer $n \geq 1$, there exists a unique global minimum $x_n \in (0, +\infty)$ making $\alpha_n := \Delta_n(x_n) = \min_{x \in (0, +\infty)} \Delta_n(x) \geq \alpha$ and $\Delta_n'(x_n) = 0$.

**Proof.** From (2), we have

$$\Gamma^{(n)}(x) = \int_1^\infty t x e^{-t} \ln t \ dt dl$$

$$+ \int_1^\infty t x e^{-t} \ln t \ dt dl$$

(15)

$$= \int_1^\infty t x e^{-t} \ln t \ dt dl + I_n,$$

where

$$I_n = \int_0^1 t x e^{-t} \ln t \ dt dl, \quad x > 0, \quad n = 0, 1, 2, \ldots.$$  

(16)

It follows that

$$e^{-1}J_n \leq I_n \leq J_n, \quad n = 0, 2, 4, \ldots$$

(17)

$$J_n \leq I_n \leq e^{-1}J_n, \quad n = 1, 3, 5, \ldots$$

where

$$J_n = \int_0^1 t x e^{-t} \ln t \ dt dl, \quad x > 0, \quad n = 0, 1, 2, \ldots.$$  

(18)

Using the transformation $y = \ln t$ and integration by part, (18) becomes

$$J_n = \int_0^\infty e^{xy} y^n dy$$

$$= \frac{1}{x} \int_0^\infty y^n e^{xy} dy$$

$$= \frac{n}{x} \int_0^\infty e^{xy} y^{n-1} dy$$

$$= \frac{n}{x} J_n$$

$$= \cdots$$

$$= (1)^n \frac{n!}{x^n} J_0$$

$$= (1)^n \frac{n!}{x^n} x^n = x.$$  

(19)

Merging (3) with (15), we find that

$$\Delta_{2k-1}(x) = \left(\int_1^\infty t x e^{-t} \ln t\ln(2k-2)\ dt dl + I_{2k-2}\right)$$

$$\times \left(\int_1^\infty t x e^{-t} \ln t\ln(2k)\ dt dl + I_{2k}\right)$$

$$- \left(\int_1^\infty t x e^{-t} \ln t\ln(2k-1)\ dt dl + I_{2k-1}\right)^2$$

(20)

$$\geq \left(\int_1^\infty t x e^{-t} \ln t\ln(2k-2)\ dt dl \right)$$

$$\times \left(\int_1^\infty t x e^{-t} \ln t\ln(2k)\ dt dl \right)$$

$$- \left(\int_1^\infty t x e^{-t} \ln t\ln(2k-1)\ dt dl \right) - I_{2k-1}^2$$

$$= \hat{\Delta}_{2k-1}(x) - I_{2k-1}^2, \quad k = 1, 2, \ldots,$$

where

$$\hat{\Delta}_{2k-1}(x) = \left(\int_1^\infty t x e^{-t} \ln t\ln(2k-2)\ dt dl \right)$$

$$\times \left(\int_1^\infty t x e^{-t} \ln t\ln(2k)\ dt dl \right)$$

$$- \left(\int_1^\infty t x e^{-t} \ln t\ln(2k-1)\ dt dl \right)^2 - I_{2k-1}^2$$

(21)

Volume 28, Issue 4: December 2020
Combining (17), (19) with (20), we yield
\[ \Delta_{2k-1}(x) \geq \hat{\Delta}_{2k-1}(x) - J_{2k-1}^2 = \hat{\Delta}_{2k-1}(x) - \left( \frac{(2k-1)!}{x^{2k}} \right)^2. \] (22)

It is noted that \( \hat{\Delta}_{2k-1}(x) (k = 1, 2, \ldots) \) can be expressed by double integral as follows,
\[
\hat{\Delta}_{2k-1}(x) = \int_1^\infty \int_1^\infty \Gamma\left( \frac{1}{2} - k, \ln t + \ln s \right) e^{-t-s} \ln^2 t \, dt \, ds
\]
\[ \times \int_1^\infty \int_1^\infty t^{x-1} e^{-t} \ln^2 t \, dt \, ds
\]
\[ \times \int_1^\infty \int_1^\infty s^{x-1} e^{-s} \ln^2 s \, ds \, ds.
\] (23)

For \( x \geq 1 \), the equality above leads to
\[ \hat{\Delta}_{2k-1}(x) \geq \frac{1}{2} \int_1^\infty \int_1^\infty \Gamma\left( \frac{1}{2} - k, \ln t + \ln s \right) e^{-t-s} \ln^2 t \, dt \, ds
\]
\[ \times \left( \ln \ln t \right)^{2k-2} \ln^2 t \, dt \, ds
\]
\[ \geq 4^{x-1} \frac{1}{2} \int_1^\infty \int_1^\infty e^{-t-s} \ln^2 t \, dt \, ds \]
\[ \times \left( \ln \ln t \right)^{2k-2} \ln^2 t \, dt \, ds,
\] (24)
which means that
\[ \lim_{x \to +\infty} \hat{\Delta}_{2k-1}(x) = +\infty (k = 1, 2, \ldots). \]
This together with the inequality (22) implies
\[ \lim_{x \to +\infty} \Delta_{2k-1}(x) = +\infty, \quad k = 1, 2, \ldots. \] (25)

Moreover, the authors in [2] proved that
\[ \lim_{x \to +\infty} \Delta_n(x) = +\infty, \quad n = 1, 2, \ldots, \] (26)
which is (3.5) in [2].

Combining (3), (25), (26) with Theorem 2.1, we can conclude that there exists a unique global minimum \( x_{2k-1} \in (0, +\infty) \) such that
\[ \Delta_{2k-1}(x_{2k-1}) = \min_{x \in (0, +\infty)} \Delta_{2k-1}(x) \geq \alpha \]
and \( \Delta'_{2k-1}(x_{2k-1}) = 0 \) for each positive integer \( k \). \[ \blacksquare \]

### III. Algorithms for Calculating the Global Minimum or the Zero Point of \( \Delta_n(x) \)

In this section, we apply three different forms of \( \Gamma^{(n)}(x) \)\(^{[13]} \), \( \Gamma^{(n)}(x) \)\(^{[14]} \), \( \Gamma^{(n)}(x) \)\(^{[15]} \) to represent \( \Delta_n(x) \), \( \Delta'_n(x) \) and \( \Delta''_n(x) \). Based on those results, we will establish three algorithms for calculating the global minimum \( x_n \) of \( \Delta_n(x) \) by using the Newton’s method.

**Algorithm I.** The following equality was given in [13]
\[ \frac{d}{dx} e^{f(x)} = f(x) e^{f(x)}, \] (27)
where
\[ Y_r(x_1, x_2, \ldots, x_r) \] is the (exponential) complete Bell polynomial.

We set \( f(x) = \ln \Gamma(x) \) with \( x > 0 \), then \( f^{(k)}(x) = \psi^{(k-1)}(x) (k = 1, 2, \ldots) \). This together with (27) leads to
\[ \Gamma^{(n)}(x) = \Gamma(x) \tilde{Y}_n(x), \] (28)
for \( n = 1, 2, \ldots \) and \( \tilde{Y}_0(x) = 1 \). Inserting (28) into (3), (10) and (11), we obtain
\[ \Delta_n(x) = \Gamma^2(x) \left( \tilde{Y}_{n-1}(x) \tilde{Y}_{n+1}(x) - \tilde{Y}_n^2(x) \right), \] (30)
\[ \Delta'_n(x) = \Gamma^2(x) \left( \tilde{Y}_{n-1}(x) \tilde{Y}_{n+2}(x) - \tilde{Y}_n(x) \tilde{Y}_{n+1}(x) \right), \] (31)
and
\[ \Delta''_n(x) = \Gamma^2(x) \left( \tilde{Y}_{n-2}(x) \tilde{Y}_{n+1}(x) - \tilde{Y}_n(x) \tilde{Y}_{n+1}(x) \right), \] (32)
respectively. Now we establish the Newton’s method for calculating the zero point \( x_n \) of (31) as follows,
\[ x_n^{(m+1)} = \frac{\tilde{Y}_{n-1}(x) \tilde{Y}_{n+2}(x) - \tilde{Y}_n(x) \tilde{Y}_{n+1}(x)}{\Gamma^{(n)}(x)}, \] (33)
for \( m = 0, 1, 2, \ldots \).

**Algorithm II.** In [14], we constructed the recursive relations of the derivatives of the Gamma function as follows,
\[ \Gamma^{(n)}(x) = \Gamma^{(n)}(x) \tilde{\Gamma}_n(x), \quad n = 0, 1, 2, \ldots., \] (34)
where
\[ \tilde{\Gamma}_0(x) = 1, \tilde{\Gamma}_1(x) = \psi(x), \]
\[ \tilde{\Gamma}_n(x) = \psi(x) \tilde{\Gamma}_{n-1}(x) + (n-1)! \sum_{k=0}^{n-2} (-1)^{n-k} \frac{(n-k)}{k!}, \] (35)
\[ \times \zeta(n-k, x) \tilde{\Gamma}_k(x), \quad n \geq 2, \]
and \( \zeta(n-k,x) \) is the Hurwitz zeta function. Combining (3), (10), (11) with (34), we obtain
\[
\Delta_n(x) = \Gamma^2(x) \left( \hat{\Gamma}_{n-1}(x) \hat{\Gamma}_{n+1}(x) - \hat{\Gamma}_n^2(x) \right),
\]
(36)
\[
\Delta'_n(x) = \Gamma^2(\alpha) \left( \hat{\Gamma}_{n-1}(x) \hat{\Gamma}_{n+2}(x) - \hat{\Gamma}_n \hat{\Gamma}_{n+1}(x) \right),
\]
(37)
and
\[
\Delta''_n(x) = \Gamma^2(\alpha) \left( \hat{\Gamma}_{n-1}(x) \hat{\Gamma}_{n+3}(x) - \hat{\Gamma}_{n+2}^2(x) \right).
\]
(38)
Consequently, the Newton’s method for calculating the zero point \( x_n \) of (37) is expressed by
\[
x_n^{(m+1)} = x_n^{(m)} - \frac{\Gamma_{n-1} \cdot \hat{\Gamma}_{n+1} - \Gamma_n \cdot \hat{\Gamma}_{n+1}}{\Gamma_{n-1} \cdot \hat{\Gamma}_{n+1} - \Gamma_n \cdot \hat{\Gamma}_{n+1}} (x_n^{(m)}),
\]
(39)
for \( m = 0, 1, 2, \ldots \).

Algorithm III. In [15], we have shown that the following recurrence relation of \( \Gamma^{(n)}(x) \) is satisfied
\[
\Gamma^{(n)}(x) = \Gamma(x) H_n(x),
\]
(40)
where
\[
H_0(x) = 1, \quad H_n(x) = \sum_{i=0}^{n-1} H_{n,i}(x),
\]
(41)
and
\[
H_{n,0}(x) = \psi^{(n-1)}(x),
\]
\[
H_{n,i}(x) = \sum_{i=1}^{C_{n-1}^{(n-1-1)}} \psi^{(n-1-1)}(x) H_{i-1,i}(x),
\]
(42)
for \( n = 2, 3, \ldots \) and \( l = 1, 2, \ldots, n - 1 \).

Merging (3), (10), (11) with (40), one has
\[
\Delta_n(x) = \Gamma^2(x) \left( H_{n-1} H_{n+1} - H_n^2(x) \right),
\]
(43)
and
\[
\Delta'_n(x) = \Gamma^2(x) \left( H_{n-1} H_{n+2} - H_n H_{n+1}(x) \right),
\]
(44)
and
\[
\Delta''_n(x) = \Gamma^2(x) \left( H_{n-1} H_{n+3} - H_{n+2}^2(x) \right).
\]
(45)
Hence, the Newton’s method for calculating the zero point \( x_n \) of (44) can be expressed by
\[
x_n^{(m+1)} = x_n^{(m)} - \frac{H_{n-1} H_{n+2} - H_n H_{n+1}(x_n^{(m)})}{H_{n-1} H_{n+3} - H_{n+2}^2(x_n^{(m)})},
\]
(46)
for \( m = 0, 1, 2, \ldots \).

The comparison between algorithms I, II and III by using Mathematica are listed in Table I (Computer Systems: Intel(R) Core(TM) i5-3470 CPU@3.20GHz 3.20GHz RAM 3.47GB). Suppose that \( \varepsilon \) is the absolute error limit and \( T \) represents the running time(unit: second), we take \( x_n \) as an approximation of \( x_n \) when \( |x_n^{(m+1)} - x_n^{(m)}| < \varepsilon \).

Due to the strict convexity of \( \Delta_n(x) \) in Theorem 2.1, the global minimum \( x_n \) is equal to the stationary point of \( \Delta_n(x) \) for each odd integer \( n \).

The numerical experiments indicate that we can gain almost the same approximations of \( x_n \), \( \alpha_n = \min \Delta_n(x) \) by using different algorithms (33), (39) and (46), so only the approximations of \( x_n \) and \( \alpha_n \) calculated by (46) are displayed in Table I.

<table>
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<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( \alpha_n )</th>
<th>( T_{(33)} )</th>
<th>( T_{(39)} )</th>
<th>( T_{(46)} )</th>
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<td>1</td>
<td>1.8745372449082242014975672673 \ldots</td>
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<td>0.0312</td>
<td>0.0156</td>
<td>0.0</td>
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<td>7.35289374141719143815866617188 \ldots</td>
<td>0.1093</td>
<td>0.0156</td>
<td>0.0156</td>
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<tr>
<td>7</td>
<td>9.3551550214897905595521836821 \ldots</td>
<td>112.447593077237600128386530492 \ldots</td>
<td>0.1250</td>
<td>0.0312</td>
<td>0.0156</td>
</tr>
<tr>
<td>9</td>
<td>3.20828639022343841415623674098 \ldots</td>
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<td>0.1718</td>
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<tr>
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<tr>
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<td>0.0312</td>
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Table II

<table>
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<th>( x_n )</th>
<th>( \Delta_n(x_n) )</th>
<th>( T_{(33)} )</th>
<th>( T_{(39)} )</th>
<th>( T_{(46)} )</th>
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</thead>
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<td>2</td>
<td>-0.91705989934039945452741000233 \ldots</td>
<td>0.0468</td>
<td>0.0156</td>
<td>0.0156</td>
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<tr>
<td>4</td>
<td>-6.2961874825811107488890439298 \ldots</td>
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<td>0.0156</td>
<td>0.0156</td>
</tr>
<tr>
<td>6</td>
<td>-9.756531284061922836736123244 \ldots</td>
<td>0.1406</td>
<td>0.0312</td>
<td>0.0156</td>
</tr>
<tr>
<td>8</td>
<td>-2493.01363266918392332939837089 \ldots</td>
<td>0.1406</td>
<td>0.0468</td>
<td>0.0156</td>
</tr>
<tr>
<td>10</td>
<td>-2.9010159815423021378044244357516 \ldots \times 10^4</td>
<td>0.1718</td>
<td>0.0625</td>
<td>0.0156</td>
</tr>
<tr>
<td>12</td>
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</tr>
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<tr>
<td>16</td>
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<td>0.3437</td>
<td>0.0937</td>
<td>0.0312</td>
</tr>
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</table>
Table I show that the algorithm III is much faster than algorithms I and II, which means that the recurrence relation of \( \Gamma^{(n)}(x) \) (40)~(42) developed in [15] is robust from the view point of the numerical simulation.

B. The numerical analysis of \( \Delta_n(x) \) for even positive integer \( n \)

Lemma 1.1 guarantee \( \alpha_n \geq \alpha > 0 \) for odd positive integer \( n \), however, the numerical results in Table II indicate that \( \Delta_n(x) < 0 \) at the stationary point \( x_n \) for even positive integer \( n \). Unfortunately, for even positive integer \( n \), the existence of the global minimum of \( \Delta_n(x) \) on \((0, +\infty)\) can’t be proved at present. Thus, properties of \( \Delta_n(x) \) for even positive integer \( n \) can only be analyzed by numerical experiments.

Based on (30), (34), (43) and the Newton’s method, we establish three algorithms for calculating the zero point of \( \Delta_n(x) \) as follows,

\[
y_n^{(m+1)} = y_n^{(m)} - \frac{\tilde{Y}_n - \tilde{Y}_n+1 - \tilde{Y}_2}{\tilde{Y}_n - \tilde{Y}_n+1 - \tilde{Y}_2} \left( y_n^{(m)} \right),
\]

and

\[
y_n^{(m+1)} = y_n^{(m)} - \frac{\tilde{H}_n - \tilde{H}_n+1 - \tilde{H}_2}{\tilde{H}_n - \tilde{H}_n+1 - \tilde{H}_2} \left( y_n^{(m)} \right),
\]

for \( m = 0, 1, 2, \ldots \).

We take \( y_n^{(m+1)} \) as an approximation of \( y_n \) when \( |y_n^{(m+1)} - y_n^{(m)}| < \epsilon \), the numerical results are displayed in Tables III and IV.

![Table III](image)

Numerical results show that \( \Delta_n(x) \) have two zero points \( a_n \) and \( b_n \) for \( n = 2, 4, 6, \ldots \) Tables II, III and IV indicate that the algorithm III is much faster than algorithms I and II for dealing with both the stationary and zero point of \( \Delta_n(x) \) \((x > 0, n = 2, 4, \ldots)\).

Remark 3.1 Theoretical analysis and numerical experiments show that \( \Delta_n(x) \) have the following properties:

1. \( \lim_{x \to 0^+} \Delta_n(x) = +\infty \);
2. \( \Delta_n(x) \) have at least two zero points \( a_n \) and \( b_n \);
3. There exists \( x_n \in (a_n, b_n) \), such that \( \Delta_n'(x_n) = 0 \) and \( \Delta_n(x_n) < 0 \).

Conjecture According to the numerical results above-mentioned, we guess that \( \Delta_n(x) \) \((x > 0, n = 2, 4, \ldots)\) may be satisfied the following properties:

1. \( \Delta_n(x) < 0 \) for \( x \in (a_n, b_n) \) and \( \Delta_n(x) > 0 \) for \( x \in (0, a_n) \cup (b_n, +\infty) \);
2. The inequality \( \Delta_n(x) \geq \Delta_n(x_n) \) holds true on \((0, +\infty)\), where \( (x_n \in (a_n, b_n) \) is the stationary point of \( \Delta_n(x) \); that is, \( x_n \) is the global minimum of \( \Delta_n(x) \) on \((0, +\infty)\);
3. \( \lim_{x \to +\infty} \Delta_n(x) = +\infty \).

IV. Conclusion

In [2], the authors proved that \( \Delta_n(x) \geq \alpha = 0.6359 \ldots \) for all odd positive integers \( n \) and \( x > 0 \). In this paper, we show that \( \Delta_n(x) \) is strictly convex on \((0, +\infty)\) and has a unique global minimum \( x_n \) for each odd positive integers \( n \); that is, \( \Delta_n(x) \geq \alpha_n = \Delta_n(x_n), x \in (0, +\infty) \). Moreover, we find that \( \alpha_n \) is strictly increasing and \( \alpha_n \geq \alpha \) as \( n \geq 7 \) through the numerical experiments, which means that \( \alpha_n \) is a sharp lower bound of \( \Delta_n(x) \). Therefore, we think that the estimate of the Turán-type inequality for the \( n \)-th derivative of the Gamma function in this paper is better than the result in [2].

In Section III, we develop three algorithms for calculating \( \Delta_n(x) \) based on the Newton’s method and the recurrence relation of \( \Gamma^{(n)}(x) \). Numerical results in Tables I ~ IV indicate that the algorithm III is much faster than algorithms I and II, which implies that the recurrence relation of \( \Gamma^{(n)}(x) \) associated with the Digamma function [15] is more effectively.

![Table IV](image)

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