The Optimal Estimation of the Turán-type Inequality for the Gamma Function and Its Numerical Method

Zhongfeng Sun and Huizeng Qin

***Abstract*—In this paper, we demonstrate that**

$$\Delta_n(x) = \Gamma^{(n-1)}(x)\,\Gamma^{(n+1)}(x) - \left[\Gamma^{(n)}(x)\right]^2$$

are strictly convex on $(0,+\infty)$ for all $n = 1,3,\ldots$, and there exists a unique global minimum x_n of $\Delta_n(x)$ for each odd positive integer n. On the other hand, we develop the algorithm for calculating the global minimum or the zero point of $\Delta_n(x)$ based on the Newton's method and the recurrence relation of $\Gamma^{(n)}(x)$ associated with the Digamma function, the numerical results show that it's more effective than other two algorithms. Furthermore, we find that $\Delta_n(x_n)$ is strictly increasing and $\Delta_n(x_n) \gg \alpha = 0.6359\ldots$ for all odd integers $n \geq 7$ through the numerical experiments.

***Index Terms*—Turán-type Inequality, Gamma Function, Digamma Function, Newton's Method.**

I. Introduction

THE Turán inequality for the Legendre polynomials $P_n(x)$ was introduced by Turán[1] as follows,

$$P_n^2(x) \geq P_{n-1}(x)P_{n+1}(x),\ -1 \leq x \leq 1, \tag{1}$$

where n is the positive integer. In the last ten years, the issue of the Turán-type inequality for the most important special functions has attracted the attention of scholars.

For convenience, we introduce the following notation

$$f^{(n)}(x) := \frac{d^n}{dx^n} f(x), \quad n = 0,1,2,3,\ldots,$$

where $f^{(0)}(x)$ represents $f(x)$. The gamma function $\Gamma(x)$ and its derivatives are defined by the following integral formula

$$\Gamma^{(n)}(x) = \int_0^\infty t^{x-1}e^{-t}\ln^n t\,dt, \tag{2}$$

where $x > 0$ and $n = 0,1,2,3,\ldots$.

Alzer and Felder[2] proved the Turán-type inequality for the gamma function, and the main result is given in the following Lemma.

Lemma 1.1 For $n = 1,3,5,\ldots$ and real numbers $x > 0$, one has

$$\Delta_n(x) = \Gamma^{(n-1)}(x)\,\Gamma^{(n+1)}(x) - \left[\Gamma^{(n)}(x)\right]^2 \geq \alpha, \tag{3}$$

with the best possible constant

$$\alpha = \min_{\frac{3}{2}\leq x\leq 2} \Gamma^2(x)\,\psi'(x) = 0.6359\cdots, \tag{4}$$

where $\psi(x)$ is the Digamma (psi) function defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{5}$$

Merovci[3] developed new Turán-type inequalities for the (q,k)-polygamma functions expressed as follows,

$$\left[\psi_{q,k}^{(m)}(x)\right]^{\frac{1}{a}}\left[\psi_{q,k}^{(n)}(y)\right]^{\frac{1}{b}} \geq \psi_{q,k}^{(\frac{m}{a}+\frac{n}{b})}\left(\frac{x}{a}+\frac{y}{b}\right), \tag{6}$$

where $\psi_{q,k}(x)$ is the (q,k)-analogue of the psi function. Subsequently, Nantomah[4] extended (6) to the general case with the aid of the Hölder's inequality and the generalized Minkowski's inequality. Based on proving monotonicity for special ratio of sections for series of Mittag-Leffler functions, Sitnik and Mehrez[5] proved that the following Turán-type inequalities hold true,

$$\mathbb{E}_{\alpha,\beta}(x)\mathbb{E}_{\alpha,\beta+2}(x) \geq \left[\mathbb{E}_{\alpha,\beta+1}(x)\right]^2, \tag{7}$$

where $\mathbb{E}_{\alpha,\beta}(x) = \Gamma(\beta)E_{\alpha,\beta}(x)$ and $E_{\alpha,\beta}(x)$ is the Mittag-Leffler function defined by

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\beta+n\alpha)}. \tag{8}$$

Consequently, Dou and Yin[6] extended (7) to the general case with four parameters represented as,

$$\begin{aligned} &\frac{(\beta+k)(\beta+qk)}{\beta(\beta+(q+1)k)}E_{\alpha,\beta,q,k}(x)E_{\alpha,\beta+2k,q,k}(x) \\ > \ &\left[E_{\alpha,\beta+k,q,k}(x)\right]^2. \end{aligned} \tag{9}$$

By means of the new Cauchy-Bunyakovsky-Schwarz inequality, Bhandari and Bissu[7] derived some Turán-type inequalities for the n-th derivative of gamma function, the polygamma function, the exponential integral function and the Abramowitz's function. Baricz, Bhandari et al.[8], [9], [10], [11] established some Turán-type inequalities for the Bessel, the modified Bessel, the Krätzel functions and the Lommel functions, respectively. Baricz[12] constructed the Turán-type and the reverse Turán-type inequalities of the generalized complete elliptic integrals, which leads to a sharp lower bound for the generalized complete elliptic integral of the first kind.

Motivated by [2], we aim to construct the optimal estimation $\Delta_n(x) \geq \alpha_n \geq \alpha, x \in (0,+\infty)$ for each odd positive integer n and develop the algorithm to calculate α_n.

The paper is organized as follows. In Section II, we demonstrate that $\Delta_n(x)$ is strictly convex on $(0,+\infty)$ and

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Zhongfeng Sun is with School of Mathematics and Statistics, Shandong University of Technology, Zibo, Shandong, 255000, P. R. China.

Huizeng Qin is with School of Mathematics and Statistics, Shandong University of Technology, Zibo, Shandong, 255000, P. R. China. Huizeng Qin is the corresponding author. (e-mail: qin_hz@163.com)

has a unique global minimum x_n for each odd positive integer n. In Section III, we develop three algorithms based on the Newton's method for calculating the global minimum of $\Delta_n(x)$ for odd positive integer n and the zero point of $\Delta_n(x)$ for even positive integer n. A final conclusion is given in Section IV.

II. The global minimum of $\Delta_n(x)$ for odd positive integer n

Theorem 2.1 For each odd positive integer n and $x > 0$, we have $\Delta_n''(x) > 0$; that is, $\Delta_n'(x)$ is strictly increasing on $(0, +\infty)$.

Proof. Differentiating $\Delta_n(x)$ in (3) with respect to x, one has

$$\begin{aligned} &\Delta_n'(x) \\ =\ & \Gamma^{(n-1)}(x)\,\Gamma^{(n+2)}(x) - \Gamma^{(n)}(x)\,\Gamma^{(n+1)}(x), \end{aligned} \tag{10}$$

and

$$\Delta_n''(x) = \Gamma^{(n-1)}(x)\,\Gamma^{(n+3)}(x) - \left[\Gamma^{(n+1)}(x)\right]^2, \tag{11}$$

for $n = 1, 2, \ldots$. With the aid of (2), $\Delta_n''(x)$ can be expressed by double integral as follows,

$$\begin{aligned} \Delta_n''(x) = & \int_0^\infty t^{x-1}e^{-t}\ln^{n-1} t dt \\ & \times \int_0^\infty s^{x-1}e^{-s}\ln^{n+3} s ds \\ & - \int_0^\infty t^{x-1}e^{-t}\ln^{n+1} t dt \\ & \times \int_0^\infty s^{x-1}e^{-s}\ln^{n+1} s ds \\ = & \int_0^\infty \int_0^\infty (st)^{x-1}e^{-t-s}(\ln s \ln t)^{n-1} \\ & \times \left(\ln^4 s - \ln^2 t \ln^2 s\right) dtds. \end{aligned} \tag{12}$$

Exchanging variables t and s, we obtain

$$\begin{aligned} \Delta_n''(x) = & \int_0^\infty \int_0^\infty (st)^{x-1}e^{-t-s}(\ln s \ln t)^{n-1} \\ & \times \left(\ln^4 t - \ln^2 s \ln^2 t\right) dsdt. \end{aligned} \tag{13}$$

This, combining with (12), implies that

$$\begin{aligned} \Delta_n''(x) = & \frac{1}{2}\int_0^\infty \int_0^\infty (st)^{x-1}e^{-t-s}(\ln s \ln t)^{n-1} \\ & \times \left[\ln^2 s - \ln^2 t\right]^2 dtds, \end{aligned} \tag{14}$$

which leads to $\Delta_n''(x) > 0$ for $n = 1, 3, \ldots$ and $x > 0$. Thus, $\Delta_n'(x)$ is strictly increasing on $(0, +\infty)$ for each odd positive integer n. ■

Theorem 2.2 For every odd integer $n \geq 1$, there exists a unique global minimum $x_n \in (0, +\infty)$ makes $\alpha_n := \Delta_n(x_n) = \min\limits_{x\in(0,+\infty)} \Delta_n(x) \geq \alpha$ and $\Delta_n'(x_n) = 0$.

Proof. From (2), we have

$$\begin{aligned} \Gamma^{(n)}(x) = & \int_1^\infty t^{x-1}e^{-t}\ln^n t dt \\ & + \int_0^1 t^{x-1}e^{-t}\ln^n t dt \\ = & \int_1^\infty t^{x-1}e^{-t}\ln^n t dt + I_n, \end{aligned} \tag{15}$$

where

$$I_n = \int_0^1 t^{x-1}e^{-t}\ln^n t dt, \quad x > 0, \quad n = 0, 1, 2, \ldots. \tag{16}$$

It follows that

$$\begin{aligned} & e^{-1}J_n \leqslant I_n \leqslant J_n, \qquad n = 0, 2, 4, \ldots, \\ & J_n \leqslant I_n \leqslant e^{-1}J_n, \qquad n = 1, 3, 5, \ldots, \end{aligned} \tag{17}$$

where

$$J_n = \int_0^1 t^{x-1}\ln^n t dt, \quad x > 0, \quad n = 0, 1, 2, \ldots. \tag{18}$$

Using the transformation $y = \ln t$ and integration by part, (18) becomes

$$\begin{aligned} J_n = & \int_{-\infty}^0 e^{xy}y^n dy \\ = & \frac{1}{x}\int_{-\infty}^0 y^n de^{xy} \\ = & -\frac{n}{x}\int_{-\infty}^0 e^{xy}y^{n-1}dy \\ = & -\frac{n}{x}J_{n-1} \\ = & \cdots \\ = & (-1)^n\frac{n!}{x^n}J_0 \\ = & (-1)^n\frac{n!}{x^{n+1}}. \end{aligned} \tag{19}$$

Merging (3) with (15), we find that

$$\begin{aligned} & \Delta_{2k-1}(x) \\ = & \left(\int_1^\infty t^{x-1}e^{-t}\ln^{2k-2} t dt + I_{2k-2}\right) \\ & \times \left(\int_1^\infty t^{x-1}e^{-t}\ln^{2k} t dt + I_{2k}\right) \\ & - \left(\int_1^\infty t^{x-1}e^{-t}\ln^{2k-1} t dt + I_{2k-1}\right)^2 \\ \geqslant & \int_1^\infty t^{x-1}e^{-t}\ln^{2k-2} t dt \\ & \times \int_1^\infty t^{x-1}e^{-t}\ln^{2k} t dt \\ & - \left(\int_1^\infty t^{x-1}e^{-t}\ln^{2k-1} t dt\right)^2 - I_{2k-1}^2 \\ = & \widehat{\Delta}_{2k-1}(x) - I_{2k-1}^2, \quad k = 1, 2, \ldots, \end{aligned} \tag{20}$$

where

$$\begin{aligned} & \widehat{\Delta}_{2k-1}(x) \\ = & \int_1^\infty t^{x-1}e^{-t}\ln^{2k-2} t dt \cdot \int_1^\infty t^{x-1}e^{-t}\ln^{2k} t dt \\ & - \left(\int_1^\infty t^{x-1}e^{-t}\ln^{2k-1} t dt\right)^2. \end{aligned} \tag{21}$$

Combining (17), (19) with (20), we yield

$$\begin{aligned}\Delta_{2k-1}(x) \geq & \ \widehat{\Delta}_{2k-1}(x) - J_{2k-1}^2 \\ = & \ \widehat{\Delta}_{2k-1}(x) - \left(\frac{(2k-1)!}{x^{2k}}\right)^2. \end{aligned} \tag{22}$$

It is noted that $\widehat{\Delta}_{2k-1}(x)(k=1,2,\ldots)$ can be expressed by double integral as follows,

$$\begin{aligned}\widehat{\Delta}_{2k-1}(x) = & \int_1^\infty t^{x-1}e^{-t}\ln^{2k-2}t dt \\ & \times \int_1^\infty s^{x-1}e^{-s}\ln^{2k}s ds \\ & - \int_1^\infty t^{x-1}e^{-t}\ln^{2k-1}t dt \\ & \times \int_1^\infty s^{x-1}e^{-s}\ln^{2k-1}s ds \\ = & \int_1^\infty \int_1^\infty (st)^{x-1}(\ln s \ln t)^{2k-2} \\ & \times e^{-t-s}\left(\ln^2 s - \ln t \ln s\right) dtds \\ = & \frac{1}{2}\int_1^\infty \int_1^\infty (st)^{x-1}e^{-t-s} \\ & \times (\ln s \ln t)^{2k-2}\ln^2\frac{t}{s}dtds. \end{aligned} \tag{23}$$

For $x \geqslant 1$, the equality above leads to

$$\begin{aligned}\widehat{\Delta}_{2k-1}(x) \geqslant & \ \frac{1}{2}\int_2^\infty \int_2^\infty (st)^{x-1}e^{-t-s} \\ & \times (\ln s \ln t)^{2k-2}\ln^2\frac{t}{s}dtds \\ \geqslant & \ 4^{x-1}\cdot\frac{1}{2}\int_2^\infty \int_2^\infty e^{-t-s} \\ & \times (\ln s \ln t)^{2k-2}\ln^2\frac{t}{s}dtds, \end{aligned} \tag{24}$$

which means that $\lim\limits_{x\to+\infty}\widehat{\Delta}_{2k-1}(x) = +\infty (k=1,2,\ldots)$. This together with the inequality (22) implies

$$\lim_{x\to+\infty}\Delta_{2k-1}(x) = +\infty, \quad k=1,2,\ldots. \tag{25}$$

Moreover, the authors in [2] proved that

$$\lim_{x\to 0+0}\Delta_n(x) = +\infty, \quad n=1,2,\ldots, \tag{26}$$

which is (3.5) in [2].

Combining (3), (25), (26) with Theorem 2.1, we can conclude that there exists a unique global minimum $x_{2k-1} \in (0,+\infty)$ such that $\Delta_{2k-1}(x_{2k-1}) = \min\limits_{x\in(0,+\infty)}\Delta_{2k-1}(x) \geq \alpha$ and $\Delta'_{2k-1}(x_{2k-1}) = 0$ for each positive integer k. ■

III. Algorithms for calculating the global minimum or the zero point of $\Delta_n(x)$

A. *The global minimum of $\Delta_n(x)$ for odd positive integer n*

In this section, we apply three different forms of $\Gamma^{(n)}(x)$[13], [14], [15] to represent $\Delta_n(x)$, $\Delta'_n(x)$ and $\Delta''_n(x)$. Based on those results, we will establish three algorithms for calculating the global minimum x_n of $\Delta_n(x)$ by using the Newton's method.

Algorithm I. The following equality was given in [13]

$$\begin{aligned}& \frac{d^r}{dx^r}e^{f(x)} \\ = & \ e^{f(x)}\ Y_r\left(f^{(1)}(x), f^{(2)}(x), \ldots, f^{(r)}(x)\right), \end{aligned} \tag{27}$$

where $Y_r(x_1, x_2, \ldots, x_r)\,(r=1,2,\ldots)$ is the (exponential) complete Bell polynomial.

We set $f(x) = \ln\Gamma(x)$ with $x > 0$, then $f^{(k)}(x) = \psi^{(k-1)}(x)\,(k=1,2,\ldots)$. This together with (27) leads to

$$\Gamma^{(n)}(x) = \Gamma(x)\widetilde{Y}_n(x), \tag{28}$$

where

$$\widetilde{Y}_n(x) = Y_n\left(\psi(x), \psi^{(1)}(x), \ldots, \psi^{(n-1)}(x)\right), \tag{29}$$

for $n=1,2,\ldots$ and $\widetilde{Y}_0(x) = 1$. Inserting (28) into (3), (10) and (11), we obtain

$$\Delta_n(x) = \Gamma^2(x)\left(\widetilde{Y}_{n-1}(x)\widetilde{Y}_{n+1}(x) - \widetilde{Y}_n^2(x)\right), \tag{30}$$

$$\begin{aligned}\Delta'_n(x) = & \ \Gamma^2(x)\left(\widetilde{Y}_{n-1}(x)\widetilde{Y}_{n+2}(x)\right. \\ & \left. - \widetilde{Y}_n(x)\widetilde{Y}_{n+1}(x)\right), \end{aligned} \tag{31}$$

and

$$\begin{aligned}& \Delta''_n(x) \\ = & \ \Gamma^2(x)\left(\widetilde{Y}_{n-1}(x)\widetilde{Y}_{n+3}(x) - \widetilde{Y}_{n+1}^2(x)\right), \end{aligned} \tag{32}$$

respectively. Now we establish the Newton's method for calculating the zero point x_n of (31) as follows,

$$\begin{aligned}& x_n^{(m+1)} \\ = & \ x_n^{(m)} - \frac{\widetilde{Y}_{n-1}\cdot\widetilde{Y}_{n+2} - \widetilde{Y}_n\cdot\widetilde{Y}_{n+1}}{\widetilde{Y}_{n-1}\cdot\widetilde{Y}_{n+3} - \widetilde{Y}_{n+1}^2}\left(x_n^{(m)}\right), \end{aligned} \tag{33}$$

for $m=0,1,2,\ldots$.

Algorithm II. In [14], we constructed the recursive relations of the derivatives of the Gamma function as follows,

$$\Gamma^{(n)}(\alpha) = \Gamma(\alpha)\widehat{\Gamma}_n(\alpha), \quad n=0,1,2,\ldots, \tag{34}$$

where

$$\begin{aligned}\widehat{\Gamma}_0(x) = & \ 1, \widehat{\Gamma}_1(x) = \psi(x), \\ \widehat{\Gamma}_n(x) = & \ \psi(x)\widehat{\Gamma}_{n-1}(x) + (n-1)!\sum_{k=0}^{n-2}\frac{(-1)^{n-k}}{k!} \\ & \times\zeta(n-k,x)\widehat{\Gamma}_k(x), \quad n \geq 2, \end{aligned} \tag{35}$$

and $\zeta(n-k,x)$ is the Hurwitz zeta function.

Combining (3), (10), (11) with (34), we obtain

$$\Delta_n(x) = \Gamma^2(x)\left(\widehat{\Gamma}_{n-1}(x)\widehat{\Gamma}_{n+1}(x) - \widehat{\Gamma}_n^2(x)\right), \quad (36)$$

$$\begin{aligned}\Delta_n'(x) = \quad & \Gamma^2(\alpha)\left(\widehat{\Gamma}_{n-1}(x)\widehat{\Gamma}_{n+2}(x)\right.\\ & \left.-\widehat{\Gamma}_n(x)\widehat{\Gamma}_{n+1}(x)\right),\end{aligned} \quad (37)$$

and

$$\Delta_n''(x) = \Gamma^2(\alpha)\left(\widehat{\Gamma}_{n-1}(x)\widehat{\Gamma}_{n+3}(x) - \widehat{\Gamma}_{n+1}^2(x)\right). \quad (38)$$

Consequently, the Newton's method for calculating the zero point x_n of (37) is expressed by

$$x_n^{(m+1)} = x_n^{(m)} - \frac{\widehat{\Gamma}_{n-1}\cdot\widehat{\Gamma}_{n+2} - \widehat{\Gamma}_n\cdot\widehat{\Gamma}_{n+1}}{\widehat{\Gamma}_{n-1}\cdot\widehat{\Gamma}_{n+3} - \widehat{\Gamma}_{n+1}^2}\left(x_n^{(m)}\right), \quad (39)$$

for $m = 0, 1, 2, \ldots$.

Algorithm III. In [15], we have shown that the following recurrence relation of $\Gamma^{(n)}(x)$ is satisfied

$$\Gamma^{(n)}(x) = \Gamma(x)\, H_n(x), \quad (40)$$

where

$$H_0(x) = 1, \quad H_n(x) = \sum_{l=0}^{n-1} H_{n,l}(x), \quad (41)$$

and

$$\begin{aligned} H_{n,0}(x) &= \psi^{(n-1)}(x), \\ H_{n,l}(x) &= \sum_{i=1}^{n-1} C_{n-1}^{i}\psi^{(n-1-i)}(x)\, H_{i,l-1}(x), \end{aligned} \quad (42)$$

for $n = 2, 3, \ldots$ and $l = 1, 2, \ldots, n-1$.

Merging (3), (10), (11) with (40), one has

$$\Delta_n(x) = \Gamma^2(x)\left(H_{n-1}H_{n+1} - H_n^2\right)(x), \quad (43)$$

and

$$\Delta_n'(x) = \Gamma^2(x)\left(H_{n-1}H_{n+2} - H_nH_{n+1}\right)(x), \quad (44)$$

and

$$\Delta_n''(x) = \Gamma^2(x)\left(H_{n-1}H_{n+3} - H_{n+1}^2\right)(x). \quad (45)$$

Hence, the Newton's method for calculating the zero point x_n of (44) can be expressed by

$$x_n^{(m+1)} = x_n^{(m)} - \frac{H_{n-1}H_{n+2} - H_nH_{n+1}}{H_{n-1}H_{n+3} - H_{n+1}^2}\left(x_n^{(m)}\right), \quad (46)$$

for $m = 0, 1, 2, \ldots$.

The comparison between algorithms I, II and III by using Mathematica are listed in Table I (*Computer Systems: Intel(R) Core(TM) i5-3470 CPU@3.20GHz 3.20GHz RAM 3.47GB*). Suppose that ε is the absolute error limit and T represents the running time(unit: second), we take $x_n^{(m+1)}$ as an approximation of x_n when $|x_n^{(m+1)} - x_n^{(m)}| < \varepsilon$.

Due to the strict convexity of $\Delta_n(x)$ in Theorem 2.1, the global minimum x_n is equal to the stationary point of $\Delta_n(x)$ for each odd integer n.

The numerical experiments indicate that we can gain almost the same approximations of x_n, $\alpha_n = \min\limits_{x\in(0,+\infty)} \Delta_n(x)$ by using three different algorithms (33), (39) and (46), so only the approximations of x_n and α_n calculated by (46) are displayed in Table I.

TABLE I
THE CALCULATION OF THE GLOBAL MINIMUM x_n AND α_n FOR ODD POSITIVE INTEGER n ($\varepsilon = 10^{-32}$)

n	x_n	α_n	$T_{(33)}$	$T_{(39)}$	$T_{(46)}$
1	$1.87453732473659026420497567626 73\cdots$	$0.63599431991168388868806147590357\cdots$	0.0312	0.0156	0.0
3	$2.32968195685189471877525272595 15\cdots$	$1.08857631976209154416351077095 60\cdots$	0.0468	0.0	0.0
5	$2.67241861600620417190103097926 67\cdots$	$7.75328973941717319413381666171 88\cdots$	0.1093	0.0156	0.0156
7	$2.95315602418978950955953218368 21\cdots$	$112.44753903772367060128386530429\cdots$	0.1250	0.0312	0.0156
9	$3.20288639022393408441562636740 98\cdots$	$2616.04810832096008469909696097 31\cdots$	0.1718	0.0312	0.0
11	$3.43515926060634764410678073364 93\cdots$	$8.62347888464810000014216420\cdots * 10^4$	0.1718	0.0781	0.0156
13	$3.65621484763498727227555268141 30\cdots$	$3.73325280213784916281872 83\cdots * 10^6$	0.2187	0.0625	0.0312
15	$3.86926807291655991543786744956 49\cdots$	$2.01736199837302386060885 08\cdots * 10^8$	0.2500	0.0937	0.0312

TABLE II
THE STATIONARY POINT x_n OF $\Delta_n(x)$ AND $\Delta_n(x_n)$ FOR EVEN POSITIVE INTEGER n ($\varepsilon = 10^{-32}$)

n	x_n	$\Delta_n(x_n)$	$T_{(33)}$	$T_{(39)}$	$T_{(46)}$
2	$1.13678842444949996673851058441 46\cdots$	$-0.91970598993403994545274100023 3\cdots$	0.0468	0.0156	0.0156
4	$1.42194476841481277372345476964 08\cdots$	$-6.29618748252811107448898043929 8\cdots$	0.1093	0.0156	0.0156
6	$1.68792118380277765725403987398 97\cdots$	$-97.56531284006192228367361232344\cdots$	0.1406	0.0312	0.0156
8	$1.93975225054912028930220128655 05\cdots$	$-2493.08132669118492333293893708 9\cdots$	0.1406	0.0468	0.0156
10	$2.18053387146818489015596217285 40\cdots$	$-9.07018197378046424483575 16\cdots * 10^4$	0.1718	0.0625	0.0156
12	$2.41235223809375832021384119041 48\cdots$	$-4.32195061170568975674244 6\cdots * 10^6$	0.2500	0.0625	0.0156
14	$2.63669904050165746993840659062 07\cdots$	$-2.55730144329763301287944 4\cdots * 10^8$	0.2968	0.0937	0.0156
16	$2.85468692013307910675685464936 35\cdots$	$-1.81123969390992764827483\cdots * 10^{10}$	0.3437	0.0937	0.0312

Table I show that the algorithm III is much faster than algorithms I and II, which means that the recurrence relation of $\Gamma^{(n)}(x)$ (40)$\sim$(42) developed in [15] is robust from the view point of the numerical simulation.

B. The numerical analysis of $\Delta_n(x)$ for even positive integer n

Lemma 1.1 guarantee $\alpha_n \geq \alpha > 0$ for odd positive integer n, however, the numerical results in Table II indicate that $\Delta_n(x) < 0$ at the stationary point x_n for even positive integer n. Unfortunately, for even positive integer n, the existence of the global minimum of $\Delta_n(x)$ on $(0, +\infty)$ can't be proved at present. Thus, properties of $\Delta_n(x)$ for even positive integer n can only be analyzed by numerical experiments.

Based on (30), (34), (43) and the Newton's method, we establish three algorithms for calculating the zero point of $\Delta_n(x)$ as follows,

$$y_n^{(m+1)} = y_n^{(m)} - \frac{\widetilde{Y}_{n-1}\widetilde{Y}_{n+1} - \widetilde{Y}_n^2}{\widetilde{Y}_{n-1}\widetilde{Y}_{n+2} - \widetilde{Y}_n\widetilde{Y}_{n+1}}\left(y_n^{(m)}\right), \quad (47)$$

$$y_n^{(m+1)} = y_n^{(m)} - \frac{\widehat{\Gamma}_{n-1}\widehat{\Gamma}_{n+1} - \widehat{\Gamma}_n^2}{\widehat{\Gamma}_{n-1}\widehat{\Gamma}_{n+2} - \widehat{\Gamma}_n\widehat{\Gamma}_{n+1}}\left(y_n^{(m)}\right), \quad (48)$$

and

$$y_n^{(m+1)} = y_n^{(m)} - \frac{H_{n-1}H_{n+1} - H_n^2}{H_{n-1}H_{n+2} - H_nH_{n+1}}\left(y_n^{(m)}\right), \quad (49)$$

for $m = 0, 1, 2, \ldots$.

We take $y_n^{(m+1)}$ as an approximation of y_n when $|y_n^{(m+1)} - y_n^{(m)}| < \varepsilon$, the numerical results are displayed in Tables III and IV.

TABLE III
THE ZERO POINT a_n OF $\Delta_n(x)(n = 2, 4, \ldots)$ $(\varepsilon = 10^{-32})$

n	a_n	$T_{(47)}$	$T_{(48)}$	$T_{(49)}$
2	$0.8803166472752543788796343 67\cdots$	0.0312	0.0468	0.0156
4	$1.2097706008768241374722332 10\cdots$	0.0781	0.0781	0.0156
6	$1.5006025724591567979598542 78\cdots$	0.0781	0.1250	0.0312
8	$1.7687570889458984734837764 01\cdots$	0.1093	0.1875	0.0625
10	$2.0212748565513694037597725 20\cdots$	0.1406	0.2500	0.0625
12	$2.2620456993602055235488544 81\cdots$	0.1875	0.3593	0.0937
14	$2.4935122782545795051940568 36\cdots$	0.2500	0.4843	0.1250
16	$2.7173400655338742348895708 81\cdots$	0.2656	0.6093	0.1718

TABLE IV
THE ZERO POINT b_n OF $\Delta_n(x)(n = 2, 4, \ldots)$ $(\varepsilon = 10^{-32})$

n	b_n	$T_{(47)}$	$T_{(48)}$	$T_{(49)}$
2	$2.8886772243420627038495053 04\cdots$	0.0312	0.0468	0.0156
4	$2.9953190714312407560871278 27\cdots$	0.0468	0.0625	0.0156
6	$3.1558517843202871756780504 06\cdots$	0.0781	0.1250	0.0312
8	$3.3360428644069602108686374 72\cdots$	0.1093	0.1718	0.0468
10	$3.5241956911856454988739758 70\cdots$	0.1250	0.2343	0.0625
12	$3.7153176627566270625092907 50\cdots$	0.1562	0.2968	0.0937
14	$3.9070190113046251729882859 02\cdots$	0.1875	0.3906	0.1093
16	$4.0980838480267825135953248 68\cdots$	0.2656	0.5625	0.1562

Numerical results show that $\Delta_n(x)$ have two zero points a_n and b_n for $n = 2, 4, 6, \ldots$. Tables II, III and IV indicate that the algorithm III is much faster than algorithms I and II for dealing with both the stationary and zero point of $\Delta_n(x)(x > 0,\ n = 2, 4, \ldots)$.

Remark 3.1 Theoretical analysis and numerical experiments show that $\Delta_n(x)(x > 0,\ n = 2, 4, \ldots)$ have the following properties:
(1) $\lim\limits_{x\to 0+0} \Delta_n(x) = +\infty$;
(2) $\Delta_n(x)$ have at least two zero points a_n and b_n;
(3) there exists $x_n \in (a_n, b_n)$, such that $\Delta_n'(x_n) = 0$ and $\Delta_n(x_n) < 0$.

Conjecture According to the numerical results above-mentioned, we guess that $\Delta_n(x)(x > 0,\ n = 2, 4, \ldots)$ may be satisfied the following properties:

(1) $\Delta_n(x) < 0$ for $x \in (a_n, b_n)$ and $\Delta_n(x) > 0$ for $x \in (0, a_n) \cup (b_n, +\infty)$;

(2) The inequality $\Delta_n(x) \geq \Delta_n(x_n)$ holds true on $(0, +\infty)$, where $x_n \in (a_n, b_n)$ is the stationary point of $\Delta_n(x)$; that is, x_n is the global minimum of $\Delta_n(x)$ on $(0, +\infty)$;

(3) $\lim\limits_{x\to+\infty} \Delta_n(x) = +\infty$.

IV. CONCLUSION

In [2], the authors proved that $\Delta_n(x) \geq \alpha = 0.6359\ldots$ for all odd positive integers n and $x > 0$. In this paper, we show that $\Delta_n(x)$ is strictly convex on $(0, +\infty)$ and has a unique global minimum x_n for each odd positive integers n; that is, $\Delta_n(x) \geq \alpha_n = \Delta_n(x_n), x \in (0, +\infty)$. Moreover, we find that α_n is strictly increasing and $\alpha_n \gg \alpha$ as $n \geq 7$ through the numerical experiments, which means that α_n is a sharp lower bound of $\Delta_n(x)$. Therefore, we think that the estimate of the Turán-type inequality for the n-th derivative of the Gamma function in this paper is better than the result in [2].

In Section III, we develop three algorithms for calculating $\Delta_n(x)$ based on the Newton's method and the recurrence relation of $\Gamma^{(n)}(x)$. Numerical results in Tables I $\sim$ IV indicate that the algorithm III is much faster than algorithms I and II, which implies that the recurrence relation of $\Gamma^{(n)}(x)$ associated with the Digamma function[15] is more effectively.

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