An $M^X/G/1$ G-queue with Single Vacation, Setup Times and Working Breakdown

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Abstract— **An $M^X/G/1$ G-queue with single vacation, setup times and working breakdown is analyzed in this paper. In a normal busy period, the positive customer being served is taken away by the negative customer who arrives. And the system malfunctioned by reason of the arrival of negative customers. During the repair time, the server is maintained without stopping the service. If the system is not idle at the end of the repair or the vacation, the system starts a new busy period. Otherwise, the system turns off. And the system starts the setup when the positive customers arrive. By applying the matrix-analytic theory, the probability generating functions(PGF) for queue length are obtained. The key performance indicators about the system are also expatiated. In addition, the sensitivity analysis and cost analysis of the queueing model are demonstrated by numerical examples.**

Index Terms—**working breakdown, setup times, G-queue, embedded Markov chain, supplementary variable method.**

I. Introduction

THE queueing system with negative customers was proposed by Erol et al. [1] in 1991, which is called G-queues. Usually, the negative customer removes an ordinary customer (called positive customer) and causes the equipment break down. Negative customers have been explained as synchronous signals virus on external system in the communication. For instance, the intrusive computer viruses can leads the file system astray and causes damage to the operation of the computer. Based on the notion of G-queues, Harrison et al. [2] built the reliability model of the M/M/1 queue and obtained PGF for queue length. In recent years, Xu et al. [3] discussed an M/M/1 G-queues with working vacation. Combined G-queue with retrial model, Yang et al. [4] established an $M^X/G/1$ unreliable model and analyzed it by the state transfer analyses. Refer to Zhang and Liu [5], Vijayashree and Anjuka [6] and Rajadurai et al. [7] for the complete understanding of the queueing systems with negative customers.

The unreliable model has used in many applications such as telecommunication market, computer industry and manufacturing system. The queues with server breakdown are characterized by the server stopping the service under the repair period. In 2012, Kalidass and Kasturi [8] first introduced working breakdown strategy, where the failed system provides service at relatively low rate. Kim and Lee [9] applied working breakdown in a M/M/1 system and investigated the probabilistic property about this system. Rajadurai [10] investigated an retrial queue with working breakdowns and working vacations. The sensitivity analysis was carried out on main parameters of the model. Li and Zhang [11] presented an retrial G-queue with working breakdowns, and they derived the steady-state solutions based on the matrix calculation. Gao et al. [12] introduced a new sort of discrete-time queue with working breakdowns, where the substitute server is available to work when the primary server goes wrong. Considered the concept of reliability cost, Yang and Chen [13] constructed a cost model of an working breakdown queue. They also gave the measurement interpretation of the system performance. The queueing model with working breakdown has been widely studied and the relevant conclusions of which can be referred to Jiang and Liu [14], Liu and Song [15], Li and Zhang [16], and many others.

The queueing system with setup time is applied widely in production area, such as preventive maintenance of the production line and the check-in counters close before the customer arrives and so on. Considered the widespread use of queueing models in many practical life, Choudhury [17] worked out the $M^X/X/1$ queues with setup times, where the server turns off if the system becomes empty. In addition, the server starts the set-up before it works during each busy period. Zhou et al. [18] considered the single working vacation G-queue with setup times and derived the PGF of queue length.

As far as the author knows, there are many researches on various queueing models, but none on the queueing model of $M^X/G/1$ with working breakdown and setup times. In the production system, the standby machine with a slower service rate will replace the machine when the machine suddenly fails. And the machines are maintained preventively to reduce the cost before starting new production. While the server is temporarily unavailable for service, the machine can take a vacation to reduce resource consumption. So motivated by above situations, we analyze an $M^X/G/1$ G-queue with single vacation, setup times and working breakdown. The introduced queueing model has certain application value for production system with the machine replacement strategy.

This paper has six parts. The remaining five sections are expatiated as following. With the requisite assumptions, Section II gives a detailed explanation of this model. In Section III, the stability condition is obtained by establishing and analyzing transition probability matrix. The stationary distribution of the system is worked out in SectionIV. The performance analysis is also presented. In Section V, the sensitivity and cost optimization of the model are studied numerically. In the end, Section VI draws conclusions.

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II. Description of the queueing model

The details of an $M^X/G/1$ G-queue with single vacation, setup times and working breakdown are described here.

1) The customer input process. There are positive customers and negative customers in this system. The arrive process of positive customers in batches is a compound Possion process. The rate of this process is λ_1 and the batch size X is a random variable. The distribution of X is $P(Y=m)=p_m$, m=1,2,... and the PGF of Y is $p(z)=\sum\limits_{k=1}^{\infty}p_k z^k$. The primary and secondly moments of Y are expressed as $E[Y]=p'(1)$ and $E[Y(Y-1)]=p''(1)$. The arrival process of negative customers is a Possion process with parameter λ_2.

2) The single vacation process. In a normal busy period, if the system becomes idle after a service completion and no negative customers arrive during the service period, the server takes a single vacation. The distribution function of the vacation time U is $U(x)=1-exp\{-\int_0^x u(t)dt\}$. In addition, the server turns off if the system becomes idle at a vacation completion instant. Otherwise, the server boots into it's normal working period and works at a normal service rate.

3) The setup process. The service of the first positive customer must take a setup time from the turned-off server, when the setup time is over, the system will start to work. The setup time H has a distribution function $H(x)=1-exp\{-\int_0^x h(t)dt\}$.

4) Service process during the normal busy period. The distribution function of the normal service time S_1 is $S_1(x)=1-exp\{-\int_0^x \mu_1(t)dt\}$, Laplace-Stieltjes transform (LST) is $\widetilde{S_1}(s)$ and nth moments α_n, $n\geq 1$. Obviously, $\alpha_1=E(S_1)\triangleq 1/\mu_1$.

5) The breakdown rule and repair process. When the server is in a normal working state, the positive customer who served is taken away and the server is down if a negative customer arrives. The negative customer will disappear automatically if it arrives during other period. A repair procedure starts immediately when the server fails. If the system has no customer after the repair is completed, the server takes vacation. Otherwise, it starts a new normal busy period. The repair time R obeys the exponential distribution with parameter r.

6) The working breakdown process. In a working breakdown period, the service rate of the server decreased from μ_1 to μ_2 and the lower service time S_2 has a distribution function $S_2(x)=1-exp\{-\int_0^x \mu_2(t)dt\}$, LST $\widetilde{S_2}(s)$ and nth moments η_n, $n\geq 1$. And $\eta_1=E(S_2)\triangleq 1/\mu_2$.

We assume that the variables are referred in this paper are independent. Other assumptions are $S_1(x)$, $S_2(x)$, $V(x)$, $H(x)$ are continuous at $x=0$ and take the value of 0 at $x=0$, and all of these functions have a value of 1 at $x=\infty$. For any of the distribution function $G(x)$ in this paper, we define that the LST of $G(x)$ is $\tilde{G}(s)=\int_0^\infty e^{-sx}dG(x)$. We also denote $\bar{G}(x)=1-G(x)$ and $\bar{G}^*(s)=\int_0^\infty e^{-sx}\bar{G}(x)dx$, so, we have $\bar{G}^*(s)=\frac{1-\tilde{G}(s)}{s}$.

The number of positive customers in the queueing system and state of the server at time t are presented by $N(t)$ and $J(t)$, respectively. At time t, define

$$J(t)=\begin{cases}0, & the\ system\ is\ in\ a\ vacation\ state,\\ 1, & the\ system\ is\ in\ a\ setup\ or\ turn\ off\ state,\\ 2, & the\ system\ is\ in\ a\ regular\ work\ state,\\ 3, & the\ server\ is\ in\ a\ working\ breakdown\ state.\end{cases}$$

At time $t\geq 0$, the random variable $\varphi(t)$ is defined as :

$$\varphi(t)=\begin{cases}the\ elapsed\ vacation\ time, & J(t)=0,\\ the\ elapsed\ setup\ time, & J(t)=1,\\ the\ elapsed\ normal\ service\ time, & J(t)=2,\\ the\ elapsed\ lower\ service\ time, & J(t)=3.\end{cases}$$

So $X(t)=\{J(t),N(t),\varphi(t),t\geq 0\}$ is a Markov process, and the state space is $\{(1,0)\}\cup\{(3,0)\}\cup\{(0,0,x),x\geq 0\}\cup\{(j,k,x),j=0,1,2,3,k\geq 1,x\geq 0\}$.

Make $\{t_n,n=1,2,3\cdots\}$ for a time series at which a vacation or a service or setup is complete or a breakdown occurs. Define $Y_n=\{J(t_n^+),N(t_n^+)\}$. Clearly, the serial of stochastic variables $\{Y_n;n\geq 1\}$ is an embedded MC and the state space is $\{(0,0)\}\cup\{(1,0)\}\cup\{(2,n),n\geq 1\}\cup\{(3,n),n\geq 0\}$.

III. Stable condition and stationary distribution

We first define the following probability for the transition matrix of $\{Y_n;n\geq 1\}$.

a) Define

$$b_k=\sum_{i=0}^{k}p_k^{(i)}\int_0^\infty\frac{(\lambda_1 x)^i}{i!}e^{-\lambda_1 x}e^{\lambda_2 x}dS_1(x),\quad k\geq 0,$$

$p_k^{(i)}$ is the probability of k positive customers arrive in i batches and is the i-fold convolution of p_k. We assume $p_0^{(0)}=1$. So, $\{b_k;k\geq 0\}$ is the probability of k positive customers arrive without negative customers arrive during S_1. As know from the calculation,

$$B(z)\triangleq\sum_{k=0}^{\infty}b_k z^k=\widetilde{S}_1[\lambda_1[1-p(z)]+\lambda_2],$$

$$B(1)=\widetilde{S}_1(\lambda_2),\quad B'(1)=\lambda_1 p'(1)\int_0^\infty xe^{-\lambda_2 x}dS_1(x),$$

$$B''(1)=\lambda_1 p''(1)\int_0^\infty xe^{-\lambda_2 x}dS_1(x)+[\lambda_1 p'(1)]^2\int_0^\infty x^2e^{-\lambda_2 x}dS_1(x).$$

b) Define

$$d_k=\sum_{i=0}^{k}p_k^{(i)}\int_0^\infty\frac{(\lambda_1 x)^i}{i!}e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2 x}[1-S_1(x)]dx,$$

where $k\geq 0$. Thus, $\{d_k;k\geq 0\}$ is the probability of negative customers arrival results in incomplete service and k positive customers arrive in the mean time. As know from the calculation,

$$D(z) \triangleq \sum_{k=0}^{\infty} d_k z^k = \frac{\lambda_2}{\lambda_1[1-p(z)]+\lambda_2}[1-B(z)],$$

$$D(1) = 1 - B(1), \quad D'(1) = \frac{\lambda_1 p'(1)}{\lambda_2} D(1) - B'(1),$$

$$D''(1) = \frac{2[\lambda_1 p'(1)]^2 + \lambda_1\lambda_2 p''(1)}{(\lambda_2)^2}[1-B(1)] - \frac{2\lambda_1 p'(1)}{\lambda_2} B'(1) - B''(1).$$

c) Define

$$c_k = \sum_{i=0}^{k} p_k^{(i)} \int_0^\infty \frac{(\lambda_1 x)^i}{i!} e^{-\lambda_1 x} e^{-rx} dS_2(x), \quad k \ge 0.$$

Hence, $\{c_k; k \ge 0\}$ is the probability of $R \ge S_2$ and k positive customers arrive during S_2. We have

$$C(z) \triangleq \sum_{k=0}^{\infty} c_k z^k = \widetilde{S}_2[\lambda_1[1-p(z)]+r],$$

$$C(1) = \widetilde{S}_2(r), \quad C'(1) = \lambda_1 p'(1) \int_0^\infty x e^{-rx} dS_2(x),$$

$$C''(1) = [\lambda_1 p'(1)]^2 \int_0^\infty x^2 e^{-rx} dS_2(x) + \lambda_1 p''(1) \int_0^\infty x e^{-rx} dS_2(x).$$

d) Define

$$q_k = \sum_{i=0}^{k} p_k^{(i)} \int_0^\infty \frac{(\lambda_1 x)^i}{i!} e^{-\lambda_1 x} r e^{-rx} [1 - S_2(x)] dx,$$

where $k \ge 0$. Thus, $\{q_k; k \ge 0\}$ is the probability of $R \le S_2$ and k positive customers arrive during R. We get

$$Q(z) \triangleq \sum_{k=0}^{\infty} q_k z^k = \frac{r}{\lambda_1[1-p(z)]+r}[1-C(z)],$$

$$Q(1) = 1 - C(1), \quad Q'(1) = \frac{\lambda_1 p'(1)}{r} Q(1) - C'(1),$$

$$Q''(1) = \frac{\lambda_1 r p''(1) + 2[\lambda_1 p'(1)]^2}{r^2} Q(1) - \frac{2\lambda_1 p'(1)}{r} C'(1) - C''(1).$$

e) Define

$$u_k = \sum_{i=0}^{k} p_k^{(i)} \int_0^\infty \frac{(\lambda_1 x)^i}{i!} e^{-\lambda_1 x} dU(x), \quad k \ge 0.$$

Then, $\{u_k; k \ge 0\}$ is the probability that there are k positive customers arriving in U. It is easy to know

$$U(z) \triangleq \sum_{k=0}^{\infty} u_k z^k = \widetilde{U}[\lambda_1[1-p(z)]],$$

$$U(1) = 1, \quad U'(1) = \lambda_1 p'(1) \int_0^\infty x dU(x),$$

$$U''(1) = [\lambda_1 p'(1)]^2 \int_0^\infty x^2 dU(x) + \lambda_1 p''(1) \int_0^\infty x dU(x).$$

f) Define

$$h_k = \sum_{i=0}^{k} p_k^{(i)} \int_0^\infty \frac{(\lambda_1 x)^i}{i!} e^{-\lambda_1 x} dH(x), \quad k \ge 0.$$

Then, $\{h_k; k \ge 0\}$ is the probability of k positive customers arrive in H. It is know from the calculation,

$$H(z) \triangleq \sum_{k=0}^{\infty} h_k z^k = \widetilde{H}[\lambda_1[1-p(z)]],$$

$$H(1) = 1, \quad H'(1) = \lambda_1 p'(1) \int_0^\infty x dH(x),$$

$$H''(1) = [\lambda_1 p'(1)]^2 \int_0^\infty x^2 dH(x) + \lambda_1 p''(1) \int_0^\infty x dH(x).$$

g) Define

$$l_k = \sum_{i=0}^{k} q_i b_{k-i}, \quad k \ge 0.$$

Hence, $\{l_k; k \ge 0\}$ is the probability that $R \le S_2$ and in the meantime there are k positive customers during R plus S_1. It is known from the analysis,

$$L(z) \triangleq \sum_{k=0}^{\infty} l_k z^k = Q(z) B(z), L(1) = Q(1) B(1),$$

$$L'(1) = Q'(1) B(1) + Q(1) A'(1),$$

$$L''(1) = Q''(1) B(1) + 2Q'(1) B'(1) + Q(1) B''(1).$$

h) Define

$$m_k = \sum_{i=0}^{k} q_i d_{k-i}, \quad k \ge 0.$$

Then $\{m_k; k \ge 0\}$ is the probability of $R \le S_2$ and the negative customer arrival result in incomplete the new started service, and there are k positive customers arrive in this period. It is easy to know that,

$$M(z) \triangleq \sum_{k=0}^{\infty} m_k z^k = Q(z) D(z), M(1) = Q(1) D(1),$$

$$M'(1) = Q'(1) D(1) + Q(1) D'(1),$$

$$M''(1) = Q''(1) D(1) + 2Q'(1) D'(1) + Q(1) D''(1).$$

According to the lexicographical sequence, we represent the transition probability matrix of MC in the following block-Jacobi matrix.

$$P = \begin{pmatrix} W_0 & W_1 & W_2 & W_3 & \cdots \\ A_{00} & A_1 & A_2 & A_3 & \cdots \\ & A_0 & A_1 & A_2 & \cdots \\ & & \ddots & \ddots & \vdots \end{pmatrix},$$

where

$$W_0 = \begin{pmatrix} 0 & u_0 & 0 \\ 0 & 0 & 0 \\ \frac{\lambda_1 p_1}{\lambda_1 + r} l_0 & \frac{r}{\lambda_1 + r} u_0 & \frac{\lambda_1 p_1}{\lambda_1 + r}(c_0 + m_0) \end{pmatrix},$$

$$A_{00} = \begin{pmatrix} b_0 & 0 & d_0 \\ l_0 & 0 & c_0 + m_0 \end{pmatrix},$$

$$A_k = \begin{pmatrix} b_k & d_k \\ l_k & c_k + m_k \end{pmatrix}, \quad k \geq 0,$$

$$W_k = \begin{pmatrix} u_k & 0 \\ \sum_{i=1}^{k} p_i h_{k-i} & 0 \\ \omega_{k1} & \omega_{k2} \end{pmatrix}, \quad k \geq 1,$$

and

$$\omega_{k1} = \frac{r}{\lambda_1 + r} u_k + \sum_{i=1}^{k+1} \frac{\lambda_1 p_i}{\lambda_1 + r} l_{k-i+1},$$

$$\omega_{k2} = \sum_{i=1}^{k+1} \frac{\lambda_1 p_i}{\lambda_1 + r} (c_{k-i+1} + m_{k-i+1}).$$

Let $e_0 = (1,1)^T$, $e = (1,1,1)^T$, it is easy to prove that

$$A_{00} e + \sum_{k=1}^{\infty} A_k e_0 = e_0,$$

$$W_0 e + \sum_{k=1}^{\infty} W_k e_0 = e,$$

$$\sum_{k=0}^{\infty} A_k e_0 = e_0.$$

Theorem 1. *The sufficient and necessary condition of the embedded MC $\{Y_n; n \geq 1\}$ is ergodic is*

$$\left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{r}\right) \cdot \frac{[1 - \tilde{S}_1(\lambda_1)][1 - \tilde{S}_r(r)] p'(1)}{1 - \tilde{S}_r(\lambda_2) \tilde{S}_2(r)} < 1.$$

Proof: It can easily check that MC $\{Y_n; n \geq 1\}$ is an unreduced and non-periodic Markov chain. Therefore we just have to prove the sufficient and necessary condition that this MC is an positive recurrent chain is that

$$\left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{r}\right) \cdot \frac{p'(1)[1 - \tilde{S}_1(\lambda_2)][1 - \tilde{S}_2(r)]}{1 - \tilde{S}_1(\lambda_2) \tilde{S}_2(r)} < 1.$$

Denote

$$A = \sum_{k=0}^{\infty} A_k = \begin{pmatrix} B(1) & D(1) \\ L(1) & C(1) + M(1) \end{pmatrix},$$

then A is a degenerative stochastic matrix.
Let $\tau = (\tau_1, \tau_2)$ is the invariant probability vector of A, we can get

$$\tau_1 = \frac{L(1)}{L(1) + D(1)}, \quad \tau_2 = \frac{D(1)}{L(1) + D(1)}.$$

The vector η is denoted by $\eta = \sum_{k=0}^{\infty} k A_k e_0$. Thus η is explicitly given by

$$\eta = \begin{pmatrix} \frac{\lambda_1 p'(1)}{\lambda_2} \left[1 - \tilde{S}_1(\lambda_2)\right] \\ \left[\frac{\lambda_1 p'(1)}{r} + \frac{\lambda_1 p'(1)}{\lambda_2} \left[1 - \tilde{S}_1(\lambda_2)\right]\right] \cdot \left[1 - \tilde{S}_2(r)\right] \end{pmatrix}.$$

It is obvious from Neuts [19] that the sufficient and necessary condition that $\{Y_n; n \geq 1\}$ is positive recurrent is

$$\tau\eta < 1 \Leftrightarrow \left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{r}\right) \frac{p'(1)[\tilde{S}_1(\lambda_2) - 1][\tilde{S}_2(r) - 1]}{1 - \tilde{S}_1(\lambda_2) \tilde{S}_2(r)} < 1.$$

And with the Burke's theorem [20], the sufficient and necessary condition for Markov process $X(t)$ have the steady state probabilities is $\left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{r}\right) \cdot \frac{[\tilde{S}_1(\lambda_2) - 1][\tilde{S}_2(r) - 1] p'(1)}{1 - \tilde{S}_1(\lambda_2) \tilde{S}_2(r)} < 1$ holds. ■

The limiting probabilities and limiting probability densities are defined as:

$$\begin{aligned} &P_{1,0} = \lim_{t \to \infty} P(J(t) = 1, N(t) = 0), \\ &P_{3,0} = \lim_{t \to \infty} P(J(t) = 3, N(t) = 0), \\ &P_{0,k}(x)\,dx = \lim_{t \to \infty} P(J(t) = 0, N(t) = k, \\ &x \leq \varphi(t) < x + dx), \quad k \geq 0, \\ &P_{j,k}(x)\,dx = \lim_{t \to \infty} P(J(t) = j, N(t) = k, \\ &x \leq \varphi(t) < x + dx), \quad k \geq 1, j = 1,2,3. \end{aligned}$$

IV. Steady state analysis

First we give the following equilibrium equations of the queueing system.

$$\lambda_1 P_{1,0} = r P_{3,0} + \int_0^{\infty} P_{0,0}(x) u(x) dx, \tag{1}$$

$$(\lambda_1 + r) P_{3,0} = \lambda_2 \int_0^{\infty} P_{2,1}(x) dx + \int_0^{\infty} P_{3,1}(x) \mu_2(x) dx, \tag{2}$$

$$\frac{dP_{0,k}(x)}{dx} = -[\lambda_1 + u(x)] P_{0,k}(x) + (1 - \delta_{k,0}) \sum_{i=1}^{k} \lambda_1 p_i P_{0,k-i}(x), \quad k \geq 0, \tag{3}$$

$$\frac{dP_{1,k}(x)}{dx} = -[\lambda_1 + h(x)] P_{1,k}(x) + (1 - \delta_{k,1}) \sum_{i=1}^{k-1} \lambda_1 p_i P_{1,k-i}(x), \quad k \geq 1, \tag{4}$$

$$\frac{dP_{2,k}(x)}{dx} = -[\lambda_1 + \lambda_2 + \mu_1(x)] P_{2,k}(x) + (1 - \delta_{k,1}) \sum_{i=1}^{k-1} \lambda_1 p_i P_{2,k-i}(x), \quad k \geq 1, \tag{5}$$

$$\frac{dP_{3,k}(x)}{dx} = -[\lambda_1 + r + \mu_2(x)] P_{3,k}(x) + (1 - \delta_{k,1}) \sum_{i=1}^{k-1} \lambda_1 p_i P_{3,k-i}(x), \quad k \geq 1, \tag{6}$$

where $\delta_{k,0}$ and $\delta_{k,1}$ are the kronecker delta.
Then we give the boundary conditions.

$$P_{0,k}(0) = P_{0,0}(0) = \int_0^{\infty} P_{2,1}(x) \mu_1(x) dx, \quad k \geq 0, \tag{7}$$

$$P_{1,k}(0) = \lambda_1 p_k P_{1,0}, \quad k \geq 1, \tag{8}$$

$$P_{2,k}(0) = \int_0^{\infty} P_{2,k+1}(x) \mu_1(x) dx + r \int_0^{\infty} P_{3,k}(x) dx + \int_0^{\infty} P_{0,k}(x) u(x) dx + \int_0^{\infty} P_{1,k}(x) h(x) dx, \quad k \geq 1, \tag{9}$$

$$P_{3,k}(0) = \lambda_2 \int_0^{\infty} P_{2,k+1}(x) dx + \int_0^{\infty} P_{3,k+1}(x) \mu_2(x) dx + \lambda_1 p_k P_{3,0}, \quad k \geq 1, \tag{10}$$

Finally we get the normalization condition.

$$P_{1,0}+P_{3,0}+\sum_{k=0}^{\infty}\int_{0}^{\infty}P_{0,k}\left(x\right)dx+\sum_{j=1}^{3}\sum_{k=1}^{\infty}P_{j,k}\left(x\right)dx$$
$$=1. \tag{11}$$

According to the generating functions

$$P_j\left(x,z\right)=\sum_{k=a}^{\infty}P_{j,k}\left(x\right)z^k, j=0,a=0; j=1,2,3,a=1.$$

Based on (3)-(6), we can deduce that

$$P_0\left(x,z\right)=e^{-[\lambda_1[1-p(z)]]x}\cdot\left[1-U(x)\right]P_0\left(0,z\right), \tag{12}$$
$$P_1\left(x,z\right)=e^{-[\lambda_1[1-p(z)]]x}\cdot\left[1-H(x)\right]P_1\left(0,z\right), \tag{13}$$
$$P_2\left(x,z\right)=e^{-[\lambda_1[1-p(z)]+\lambda_2]x}\cdot\left[1-S_1(x)\right]P_2\left(0,z\right), \tag{14}$$
$$P_3\left(x,z\right)=e^{-[\lambda_1[1-p(z)]+r]x}\cdot\left[1-S_2(x)\right]P_3\left(0,z\right). \tag{15}$$

By (1), (2), (7)-(10), after some computations, we can have

$$\lambda_1 P_{1,0}=rP_{3,0}+U(0)P_{0,0}(0), \tag{16}$$
$$(\lambda_1+r)P_{3,0}=D(0)P_{2,1}(0)+C(0)P_{3,1}(0), \tag{17}$$
$$P_0(0,z)=B(0)P_{2,1}(0)=P_{0,0}(0), \tag{18}$$
$$P_1(0,z)=\lambda_1 P_{1,0}p(z), \tag{19}$$
$$P_2(0,z)=\frac{P_{1,0}f_1(z)+P_{3,0}f_2(z)}{g(z)}, \tag{20}$$
$$P_3(0,z)=P_{1,0}f_3(z)+P_{3,0}f_4(z), \tag{21}$$

where

$$\begin{aligned}
g\left(z\right)&\triangleq\left[B(z)-z\right]\cdot\left[C(z)-z\right]-zQ(z)D(z),\\
f_1(z)&\triangleq\frac{\lambda_1\left[z-C(z)\right]}{U(0)}\cdot zU(0)\left[p(z)H(z)-1\right]\\
&\quad+\frac{\lambda_1\left[C(z)-z\right]}{U(0)}\cdot\left[1-zU(z)\right],\\
f_2(z)&\triangleq\frac{r}{U(0)}\cdot\left[C(z)-z\right]\left[zU(z)-zU(0)-1\right]\\
&\quad+zQ(z)\left[\lambda_1[zp(z)-1]-r\right],\\
f_3(z)&\triangleq\frac{D(z)f_1(z)}{[z-C(z)]g(z)},\\
f_4(z)&\triangleq\frac{D(z)f_2(z)+g(z)\left[\lambda_1[zp(z)-1]-r\right]}{[z-C(z)]g(z)}.
\end{aligned}$$

Before we calculate $P_j(0,z), j=0,1,2,3$, we give the following lemma to analyze the roots of $g(z)=0$ when $z\in(0,1)$.

Lemma 1. *In the interval* $(0,1)$, *the equation* $g(z)=0$ *has a root* $z=\delta$ *if* $\left(\frac{\lambda_1}{\lambda_2}+\frac{\lambda_1}{r}\right)\cdot\frac{[1-\tilde{S}_1(\lambda_2)][1-\tilde{S}_2(r)]p'(1)}{1-\tilde{S}_1(\lambda_2)\tilde{S}_2(r)}<1$.

Proof: Denote $f_0(z)=z-C(z)$ and we have

$$f_0(0)<0, \quad f_0(1)>0.$$

Moreover, for any $0<z<1$, we know analytically that

$$f_0'(z)=1-C'(z), \quad f_0''(z)=-C''(z),$$

which shows in the interval $(0,1)$, $f_0(z)$ is a convex function. Then $f_0(0)<0$ and $f_0(1)>0$ mean $f_0(z)$ has the unique root $z=\gamma$ when $z\in(0,1)$.

Clearly, $g(0)=D(0)C(0)>0$, and

$$g(\gamma)=[\gamma-B(\gamma)][\gamma-C(\gamma)]-\gamma Q(\gamma)D(\gamma)<0,$$

which mean that $g(z)=0$ has the root $z=\delta$ when $z\in(0,1)$. ■

Then, plug $z=\delta$ in (20) and we get $f_1(\delta)P_{1,0}+f_2(\delta)P_{3,0}=0$.

Thus, we have

$$P_0(0,z)=\frac{\lambda_1 f_2(\delta)+rf_1(\delta)}{U(0)f_2(\delta)}P_{1,0}, \tag{22}$$
$$P_1(0,z)=\lambda_1 p(z)P_{1,0}, \tag{23}$$
$$P_2(0,z)=\frac{f_2(\delta)f_1(z)-f_1(\delta)f_2(z)}{f_2(\delta)g(z)}P_{1,0}, \tag{24}$$
$$P_3(0,z)=\frac{f_2(\delta)f_3(z)-f_1(\delta)f_4(z)}{f_2(\delta)}P_{1,0}. \tag{25}$$

Next we need to introduce the following lemma before find $P_{1,0}$, and we omit the proof of this.

Lemma 2.

$$\begin{aligned}
&g(1)=0, \quad f_1(1)=0, \quad f_2(1)=0,\\
&g'(1)=1-B(1)C(1)-\left[\frac{\lambda_1 p'(1)}{\lambda_2}+\frac{\lambda_1 p'(1)}{r}\right]D(1)Q(1),\\
&g''(1)=2[1-B'(1)][1-C'(1)]-2Q'(1)D'(1)\\
&\quad-Q(1)\left[B''(1)+2D'(1)+D''(1)\right]\\
&\quad-D(1)\left[C''(1)+2Q'(1)+Q''(1)\right],\\
&f_1'(1)=\frac{\lambda_1 Q(1)}{U(0)}\left[1+U'(1)+U(0)[p'(1)+H'(1)]\right],\\
&f_1''(1)=\frac{\lambda_1 Q(1)}{U(0)}[2U'(1)+U''(1)]+2\lambda_1 U(0)p'(1)\\
&\quad+\lambda_1 U(0)\left[2H'(1)+p''(1)+H''(1)+2p'(1)H'(1)\right]\\
&\quad+\frac{2\lambda_1[1-C'(1)]}{U(0)}\left[1+U'(1)+U(0)[p'(1)+H'(1)]\right],\\
&f_2'(1)=r[1-C'(1)-Q'(1)]+\lambda_1 Q(1)\left[p'(1)+1\right]\\
&\quad-\frac{rQ(1)[1+U'(1)]}{U(0)},\\
&f_2''(1)=\lambda_1[Q(1)+Q'(1)][p'(1)+1]+p''(1)+2p'(1)\\
&\quad+\frac{2r}{U(0)}[C'(1)-1][1-U(0)+U'(1)]\\
&\quad-r[2Q'(1)+Q''(1)+C''(1)],\\
&f_3(1)=\frac{D(1)f_1'(1)}{Q(1)g'(1)}, \quad f_4(1)=\frac{D(1)f_2'(1)-rg'(1)}{Q(1)g'(1)},\\
&f_3'(1)=g'(1)Q(1)[2D'(1)f_1'(1)+D(1)f_1''(1)]\\
&\quad-D(1)f_1'(1)\left[2g'(1)[1-C'(1)]+Q(1)g''(1)\right],\\
&f_4'(1)=\frac{D(1)f_2''(1)-rg''(1)+2g'(1)[\lambda_1+\lambda_1 p'(1)]}{2Q(1)g'(1)}\\
&\quad-\frac{[D(1)H_2'(1)-rg'(1)][2g'(1)[1-C'(1)]]}{(g'(1)Q(1))^2}\\
&\quad-\frac{[D(1)H_2'(1)-rg'(1)]Q(1)g''(1)}{(Q(1)g'(1))^2}.
\end{aligned}$$

By definition of the marginal generating functions $\Phi_j(z)=\int_0^\infty P_j(x,z)dx$, $j=0,1,2,3$. Plug(22)-(25) into (12)-(15), respectively. Then the following theorem is obtained by calculation.

Theorem 2.

$$\Phi_0(z) = \int_0^\infty P_0(x,z)dx = P_{1,0}\cdot \frac{\lambda_1 f_2(\delta) + r f_1(\delta)}{U(0) f_2(\delta)} V^* \left[{}_1[1-p(z)]\right],$$

$$\Phi_1(z) = \int_0^\infty P_1(x,z)dx = \lambda_1 p(z) H^* \left[\lambda_1[1-p(z)]\right] P_{1,0},$$

$$\Phi_2(z) = \int_0^\infty P_2(x,z)dx = P_{1,0}\cdot \frac{f_1(z)f_2(\delta) - f_2(z)f_1(\delta)}{g(z)f_2(\delta)} S_1^* \left[\lambda_1[1-p(z)] + \lambda_2\right],$$

$$\Phi_3(z) = \int_0^\infty P_3(x,z)dx = P_{1,0}\cdot \frac{f_3(z)f_2(\delta) - f_4(z)f_1(\delta)}{f_2(\delta)} S_2^* \left[\lambda_2[1-p(z)] + r\right].$$

Based on the normalization condition

$$P_{1,0} + P_{3,0} + \sum_{j=0}^{3} \Phi_j(1) = 1.$$

$P_{1,0}$ *is calculated by*

$$P_{1,0} = \left[\Delta_0 + \Delta_1 + \Delta_2 + \Delta_3 + \frac{f_2(\delta) - f_1(\delta)}{f_2(\delta)}\right]^{-1},$$

where

$$\Delta_0 = \frac{U'(1)}{\lambda_1 p'(1)} \frac{\lambda_1 f_2(\delta) + r f_1(\delta)}{U(0) f_2(\delta)},$$
$$\Delta_1 = \frac{H'(1)p(1)}{p'(1)},$$
$$\Delta_2 = \frac{D(1)[f_1'(1)f_2(\delta) - f_2'(1)f_1(\delta)]}{\lambda_2 f_2(\delta) g'(1)},$$
$$\Delta_3 = \frac{Q(1)[f_3(1)f_2(\delta) - f_1(\delta)f_4(1)]}{r f_2(\delta)}.$$

Apparently, PGF for queue length is

$$\Phi(z) = P_{1,0} + P_{3,0} + \sum_{j=0}^{3} \Phi_j(z).$$

The probability of the server in the vacation state is

$$P_u = \Phi_0(1) = \Delta_0 P_{1,0}.$$

The probability of the server in the normal busy state is

$$P_1 = \Phi_2(1) = \Delta_2 P_{1,0}.$$

The probability of the server is in a turn off or setup period is

$$P_h = P_{1,0} + \Phi_1(1) = (1 + \Delta_1) P_{1,0}.$$

The probability of the server in a working breakdown state is

$$P_2 = P_{3,0} + \Phi_3(1) = \left[\Delta_3 - \frac{f_1(\delta)}{f_2(\delta)}\right] P_{1,0}.$$

$E[L_j]$ is defined as the average number of positive customers when the system is in state j, $j = 0, 1, 2, 3$. Based on Theorem 2, it can be calculated easily that

$$E[L_0] = \lim_{z\to 1} \Phi_0'(z) = \varepsilon_0 P_{1,0},$$
$$E[L_1] = \lim_{z\to 1} \Phi_1'(z) = \varepsilon_1 P_{1,0},$$
$$E[L_2] = \lim_{z\to 1} \Phi_2'(z) = \varepsilon_2 P_{1,0},$$
$$E[L_3] = \lim_{z\to 1} \Phi_3'(z) = \varepsilon_3 P_{1,0}.$$

where

$$\varepsilon_0 = \frac{[p'(1)U''(1) - p''(1)U'(1)][\lambda_1 f_2(\delta) + r f_1(\delta)]}{2\lambda_1 U(0) f_2(\delta)[p'(1)]^2},$$
$$\varepsilon_1 = \frac{H'(1)p'(1) + p'(1) + H'(1)}{2p'(1)} + \frac{p''(1)[1 - 2H'(1)]}{2[p'(1)]^2},$$
$$\varepsilon_2 = \frac{D(1)[f_1'(1)f_2(\delta) - f_2'(1)f_1(\delta)]}{2\lambda_2 f_1(\delta)[g'(1)]^2} \cdot [g'(1)f_2(\delta)g''(1)] - \frac{D(1)[f_2(\delta)f_1'(1) - f_2'(1)f_1(\delta)]}{2\lambda_2 f_1(\delta)[g'(1)]^2} \cdot [2g''(1) + 2g'(1)] + \frac{D'(1)f_2(\delta)[f_2(\delta)f_1'(1) - f_2'(1)f_1(\delta)]}{\lambda_2 f_1(\delta)} + \frac{D'(1)g'(1)[f_2(\delta)f_1'(1) - f_2'(1)f_1(\delta)]}{2\lambda_2 f_1(\delta)[g'(1)]^2},$$
$$\varepsilon_3 = \frac{1}{r f_2(\delta)}\{Q'(1)[f_2(\delta)f_3(1) - f_1(\delta)f_4(1)]\} + \frac{1}{r f_2(\delta)}\{Q(1)[f_2(\delta)f_3'(1) - f_1(\delta)f_4'(1)]\}.$$

Hence, the mean system length ($E[L]$) can be get by means of

$$E[L] = \lim_{z\to 1} \Phi'(z) = E[L_0] + \sum_{j=0}^{3} E[L_j].$$

Define $E[W]$ as the expected sojourn time of a positive customer. And it is easy to obtain that $E[W] = \frac{E[L]}{\lambda_1}$ by the Little's formula.

V. Numerical results

We discuss the effect of operating parameters on $E[L]$ by analyzing numerical examples. It is assumed that $S_1(x)$, $S_2(x)$, $U(x)$, $H(x)$ and $R(x)$ obey exponential distribution with parameter μ_1, μ_2, θ, h and r, respectively. Moreover, we assume that the distribution function of the arrival batch size Y is $P(Y = k) = p(1-p)^{k-1}$. Clearly, $p'(1) = 1/p$ and $p''(1) = 2(1-p)/p^2$.

Under the stable condition $\lambda_1 r + \lambda_1\lambda_2 < p(\lambda_1 + \mu_1)(r + \mu_2) - \mu_1\mu_2$, the values of some parameters in the model are chosen as $\lambda_1 = 1.4$, $\lambda_2 = 1$, $\theta = 2$, $p = 0.8$, $r = 0.5$, $\mu_1 = 6$, $\mu_2 = 1$, $h = 2$, unless they are selected as independent variables in numerical analysis.

A. Sensitivity Analysis

Fig.1 indicates that $E[L]$ will become bigger with the increase of negative customer input rate λ_2 when the value of r is constant. The reason for this trend is that the system enters the repair mode when the server fails due to the arrival of negative customer. In addition, the rate of service during a working breakdown period is lower than the normal service rate. Fig.1 also shows the influence of r on $E[L]$. It reflects that the increase of r reduces the value of $E[L]$, this is because that the expected repair time is $1/r$.

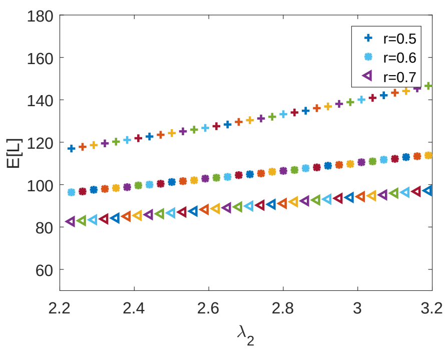

Fig. 1: The effect of λ_2 on $E[L]$.(different r)

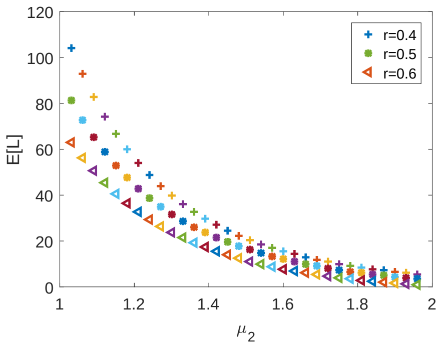

Fig. 2: The effect of μ_2 on $E[L]$.(different r)

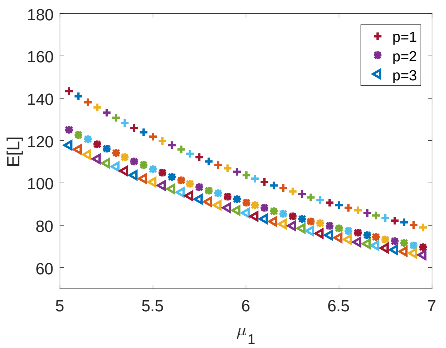

Fig. 3: The effect of μ_1 on $E[L]$.(different p)

Fig.2 presents $E[L]$ versus μ_2 for different r, it illustrates that the bigger the value of μ_2 is, the smaller the mean system length $E[L]$ is. It will be sent to repair immediately if the server fails. And the expected repair time is $1/r$. Therefore, from Fig.2, the increase of r causes the value of $E[L]$ to decrease. It also reflects that the influence degree of r on $E(L)$ gets smaller as μ_2 gets larger.

Fig.3 reveals the trend of $E[L]$ with respect to μ_1, it is fairly easy to see that $E[L]$ decreases with increasing values of μ_1. Furthermore, the smaller the batch size Y is, the shorter the average queue length is, that is, $E[L]$ decreases as p increases.

B. Cost Analysis

Cost minimization has a very important theory and practical value in actual production. Based on the model studied, we seek the the optimal service rate μ_2 that minimizes the expected cost per unit of time. Before establishing the cost function, consider the following definitions to represent cost per unit of time in different situations.

C_L: cost for each positive customer present in the system;
C_{μ_1}: cost for service during a normal service period;
C_{μ_2}: cost for service during a working breakdown period;
C_θ: cost during a vacation period;
C_r: cost during a repair period;
C_h: cost during a setup or turn off period.

According to the above cost definition and performance indicators of the system, the expected operating cost function per unit time can be established as

$$\min_{\mu_2} : f(\mu_2) = C_L E[L] + C_{\mu_1}\mu_1 + C_{\mu_2}\mu_2 + C_\theta\theta + C_r r + C_h h.$$

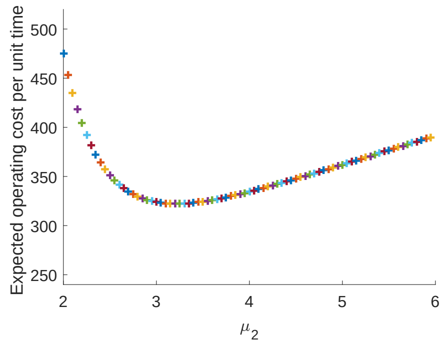

Fig. 4: The effect of μ_2 on the expected cost per unit time.

TABLE I: The Result of the Parabolic Method.

iterations	0	1	2	3
x_1	3.000000	3.200000	3.200000	3.200000
x_2	3.200000	3.214022	3.207614	3.207576
x_3	3.300000	3.300000	3.214022	3.207614
$f(x_1)$	323.870228	322.284346	322.284346	322.284346
$f(x_2)$	322.284346	322.283847	322.282538	322.282537
$f(x_3)$	322.534546	322.534546	322.283847	322.282538
x^*	3.214022	3.207614	3.207576	3.207558
$f(x^*)$	322.283847	322.282538	322.282537	322.282537
tolerance	0.014022	0.006407	0.000037	0.000008

Because of the complexity and highly non-linearity of the above cost function, the parabolic method is used to solve the optimization problem. Based on the polynomial approximation theory, the quadratic function has the unique optimum at 3-point pattern $\{x_1, x_2, x_3\}$ and it occurs at

$$x^* = \frac{1}{2}\frac{(x_2^2 - x_3^2)f(x_1) + (x_3^2 - x_1^2)f(x_2) + (x_1^2 - x_2^2)f(x_3)}{(x_2 - x_3)f(x_1) + (x_3 - x_1)f(x_2) + (x_1 - x_2)f(x_3)}.$$

Then we assume $C_L = 40$, $C_{\mu_1} = 35$, $C_{\mu_2} = 20$, $C_\theta = 25$, $C_r = 25$,$C_h = 20$, and search for the optimal value μ_r^* according to the specific steps of the parabolic method [21].

As can be seen from Fig.4 that the value of the cost function decreases at first and then increases. So there is an optimal value of μ_2 that minimizes the cost function. The process and results of the parabolic method are shown in Table.1. After three iterations, the results of optimal service rate and the minimum expected cost per unit of time are $\mu_2^* = 3.207558$ and $f(\mu_2^*) = 322.282537$ respectively when the error is controlled within $\varepsilon = 10^{-5}$.

VI. Conclusion

This paper discusses an $M^X/G/1$ G-queue with single vacation, setup times and working breakdown. Considering the practical applications, we present a detailed description of this model. The steady-state conditions of the system are derived. Queues generating function is derived by solving the state transfer equation. In addition, we discuss the various representative indicators about the model. The effects of different parameters on the queue length are explained by means of numerical examples and optimization drawings. The cost optimization problem of the model is also solved. Based on this model, we can further study that both working breakdown and working vacation are introduced into the $M^X/G/1$ G-queue.

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