Resource Allocation under Sports Management Systems: Game-theoretical Methods

Yu-Hsien Liao, Pang-Tung Liu, Ruey-Rong Huang, Kun-Chang Lee, and En-Cheng Chi

Abstract—**In general, participators and its operational strategies (decisions) might be essential factors simultaneously under sports management systems. Thus, a new principle is introduced by both considering the participators and its operational strategies (decisions) in this paper. Based on a specific reduction and excess function, we further adopt axiomatic and dynamic results to present the rationality for this principle. Finally, the game-theoretical results are also applied to analyze resource allocation rules under sports management systems.**

Index Terms—**Operational strategy, principle, resource allocation, sports management system.**

I. INTRODUCTION

Recently, the game-theoretical characterizations could be adopted to analyze various interaction relationship and related models among participators and coalitions by applying mathematical results. In addition to theoretical analysis, game-theoretical results also could be applied to offer optimal results or equilibrium conditions for some real-word models, including sports management. On the other hand, management techniques related to combinations of different theoretical methods have become the main notion in the field of sports management. The use of several game-theoretical results could promote the management completeness no matter for training of sports skills or management of sports industry.

Under the researches of coalitional cooperative games, a characteristic map might be defined under whole the subcollections of the collection of participators. This means that the choices available for every participator are either to join completely in a condition or not to join at all. In real conditions of *sports management systems*, however, every participator takes different operational levels (or strategies) to distribute related resource. A *multi-choice condition* can be deemed as a reasonable extension of a coalitional condition in which every participator takes different operational levels (or strategies). Several principles have been introduced in the context of multi-choice conditions. By determining overall values for a particular participator on multi-choice conditions and fuzzy conditions, Cheng et al. [1], Hwang and Liao [4], Liao [6], [7], Nouweland et al. [10] and Wei et al. [12] submitted some extended principles by respectively applying the notions of the core, the EANSC and the Shapley value. By focusing on both the participators and its operational levels (or strategies), Hwang and Liao [5] defined an extended Shapley value [11] on fuzzy conditions.

In this research, we focus on the principle of the *pseudo equal allocation of non-separable costs* (PEANSC) due to Hsieh and Liao [3]. In the context of coalitional conditions, Hsieh and Liao [3] defined a notion of reduced condition and related property of conformance to show that the PEANSC is a stabilizing principle that matches the properties of completeness, equal treatment property, covariance and conformance. These mentioned above yield an inspiration in the context of multi-choice conditions:

- whether the pre-existing researches of the PEANSC could be improved under multi-choice conditions and sports management.

This research is aimed at solving the inspiration. The main results of this research are introduced as follows.

1) Inspired by the results of Hwang and Liao [5], a multi-choice generalization of the PEANSC, the *multi-choice pseudo equal allocation of non-separable costs* (MPEANSC), is defined by focusing on the participators and the actions simultaneously in Section 2.
2) In Section 3, we define alternative extended properties of Hsieh and Liao [3] to axiomatize the MPEANSC on multi-choice conditions.
3) In Section 4, the notion of excess is applied to arise a dynamic result for the MPEANSC on multi-choice conditions.
4) In Section 5, the condition-theoretical results are also applied to analyze resource allocation rules under sports management systems. Related connections and comparisons are also mentioned in Section 6.

II. THE MULTI-CHOICE PSEUDO EQUAL ALLOCATION OF NON-SEPARABLE COSTS

Let UL be the universe of participators. For $m \in UL$ and $d_m \in \mathbb{N}$, $D_m = \{0, 1, \cdots, d_m\}$ could be treated as the activity level (strategy, decision) collection of participator m and $D_m^+ = d_m \setminus \{0\}$, where 0 represents no participation. Let $D^L = \prod_{m \in L} D_m$ be the product collection of the activity level (strategy, decision) collections of all participators of L. for every $T \subseteq L$, we define $\theta^T \in D^L$ is the vector with $\theta_m^T = 1$ if $m \in T$, and $\theta_m^T = 0$ if $m \in L \setminus T$. Denote 0_L the zero vector in $\mathbb{R}^L$. For $z \in \mathbb{N}$, let 0_z be the zero vector in $\mathbb{R}^z$ and $\mathbb{L}_z = \{1, \cdots, z\}$.

A **multi-choice system** is a triple (L, d, h), where $L \neq \emptyset$ is a finite collection of participators, $d = (d_m)_{m \in L}$ is the vector that presents the highest operational levels for every participator, and $h : D^L \to \mathbb{R}$ is a characteristic map with $h(0_L) = 0$ which assigns to every $\lambda = (\lambda_m)_{m \in L} \in D^L$ the amount that the participators can get when every participator

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m takes level λ_m. As d is fixed throughout this research, we take (L,h) rather than (L,d,h).

Let (L,h) be a multi-choice system and $\lambda \in D^L$. We let $J(\lambda)=\{m\in L|\ \lambda_m\neq 0\}$ and λ_H to be the restriction of λ at H for every $H\subseteq L$. We also define that $h_*(H)=\max_{\lambda\in D^L}\{h(\lambda)|J(\lambda)=H\}$ is the **maximal-utility**[1] among every level vector λ with $J(\lambda)=H$.

Define the family of total multi-choice systems to be Ω. Given $(L,h)\in\Omega$, let $A^L=\{(m,k_m)\mid m\in L, k_m\in D_m^+\}$. A **principle** on Ω is a function χ assigning to every $(L,h)\in\Omega$ a constituent

$$\chi(L,h)=\Big(\chi_{m,k_m}(L,h)\Big)_{(m,k_m)\in A^L}\in\mathbb{R}^{A^L}.$$

Here $\chi_{m,k_m}(L,h)$ is the amount of the participator m when it participates operational level k_m to join system h.

In the following we define a generalized form of the pseudo equal allocation of non-separable costs on multi-choice systems.

Definition 1: The **multi-choice pseudo equal allocation of non-separable costs (MPEANSC)** of multi-choice systems, $\overline{\tau}$, is the map on Ω which assigns to every $(L,h)\in\Omega$, every participator $m\in L$ and every $k_m\in D_m$ the amount

$$\overline{\tau}_{m,k_m}(L,h)=\tau_{m,k_m}(L,h)+\frac{1}{|L|}\cdot\Big[h(d)-\sum_{n\in L}\tau_{n,d_n}(L,h)\Big],$$

where $\tau_{m,k_m}(L,h)=h\big(k_m,0_{L\setminus\{m\}}\big)$ means the **individual distinction** of the participator m and its level k_m.

III. Axiomatic outcomes

Here we present that a reduced system could be provided to axiomatize the MPEANSC.

Let χ be a principle on Ω.

- χ matches **completeness (COM)** if for every $(L,h)\in\Omega$, $\sum_{m\in L}\chi_{m,d_m}(L,h)=h(d)$.
- χ matches **rule for two-person systems (RTPS)** if for every $(L,h)\in\Omega$ with $|L|\leq 2$, $\chi(L,h)=\overline{\tau}(L,h)$.
- χ matches **coincident treatment effect (CTE)** if for every $(L,h)\in\Omega$ with $h(\lambda,k_m,0)=h(\lambda,0,k_n)$ for some $(m,k_m),(n,k_n)\in A^L$ and for every $\lambda\in D^{N\setminus\{m,n\}}$, $\chi_{m,k_m}(L,h)=\chi_{n,k_n}(L,h)$.
- χ matches **synchronization (SYNC)** if for every $(L,h),(L,u)\in\Omega$ with $h(\lambda)=u(\lambda)+\sum_{i\in J(\lambda)}b_{m,\lambda_m}$ for some $b\in\mathbb{R}^{A^L}$ and for every $\lambda\in D^L$, $\chi(L,h)=\chi(L,u)+b$.

Situations COM and SYNC are important and widely receivable. Definitely, many researches focus on only principles that match COM. COM means that all participators distribute whole utility entirely. RTPS is a generalized axiom of the two-person axiom due to Hart and Mas-Colell [2]. RTPS means that the value received by every participator should be applied the principle $\overline{\tau}$ in one-person or two-person systems. Property CTE means that the values among two participators should be equal if the individual distinctions among these two participators are coincident. In the following, we would like to show that the MPEANSC matches COM, RTPS, CTE and SYNC.

Lemma 1: The MPEANSC matches COM.

Proof: Let $(L,h)\in\Omega$.

$$\begin{aligned}&\sum_{m\in L}\overline{\tau}_{m,d_m}(L,h)\\
=\ &\sum_{m\in L}\tau_{m,d_m}(L,h)+\sum_{m\in L}\tfrac{1}{|L|}\cdot\Big[h(d)-\sum_{n\in L}\tau_{n,d_n}(L,h)\Big]\\
=\ &\sum_{m\in L}\tau_{m,d_m}(L,h)+\tfrac{|L|}{|L|}\cdot\Big[h(d)-\sum_{n\in L}\tau_{n,d_n}(L,h)\Big]\\
=\ &\sum_{m\in L}\tau_{m,d_m}(L,h)+h(d)-\sum_{n\in L}\tau_{n,d_n}(L,h)\\
=\ &h(d).\end{aligned}$$

So, the principle $\overline{\tau}$ matches COM. ■

Lemma 2: The MPEANSC matches RTPS.

Proof: By the definitions of the MPEANSC and RTPS, the proof could be completed. ■

Lemma 3: The MPEANSC matches CTE.

Proof: Let $(L,h)\in\Omega$. Assume that $h(\lambda,k_m,0)=h(\lambda,0,k_n)$ for some $(m,k_m),(n,k_n)\in A^L$ and for every $\lambda\in D^{L\setminus\{m,n\}}$. By taking $\lambda=0_{L\setminus\{m,n\}}$,

$$h(k_m,0_{L\setminus\{m\}})=h(\lambda,k_m,0)=h(\lambda,0,k_n)=h(k_n,0_{L\setminus\{n\}}),$$

i,e,. $\tau_{m,k_m}(L,h)=h(k_m,0_{L\setminus\{m\}})=h(k_n,0_{L\setminus\{n\}})=\tau_{n,k_n}(L,h)$. So,

$$\begin{aligned}&\overline{\tau}_{m,k_m}(L,h)\\
=\ &\tau_{m,k_m}(L,h)+\tfrac{1}{|L|}\cdot\Big[h(d)-\sum_{t\in L}\tau_{t,d_t}(L,h)\Big]\\
=\ &\tau_{n,k_n}(L,h)+\tfrac{1}{|L|}\cdot\Big[h(d)-\sum_{t\in L}\tau_{t,d_t}(L,h)\Big]\\
=\ &\overline{\tau}_{n,k_n}(L,h).\end{aligned}$$

So, the principle $\overline{\tau}$ matches CTE. ■

Lemma 4: The MPEANSC matches SYNC.

Proof: Let $(L,h),(L,u)\in\Omega$ with $h(\lambda)=u(\lambda)+\sum_{t\in J(\lambda)}b_{t,\lambda_t}$ for some $b\in\mathbb{R}^{A^L}$ and for every $\lambda\in D^L$. for every $(m,k_m)\in A^L$,

$$\begin{aligned}\tau_{m,k_m}(L,h)&=h(k_m,0_{L\setminus\{m\}})\\
&=u(k_m,0_{L\setminus\{m\}})+b_{m,k_m}\\
&=\tau_{m,k_m}(L,u)+b_{m,k_m}.\end{aligned}$$

So,

$$\begin{aligned}&\overline{\tau}_{m,k_m}(L,h)\\
=\ &\tau_{m,k_m}(L,h)+\tfrac{1}{|L|}\cdot\Big[h(d)-\sum_{n\in L}\tau_{n,d_n}(L,h)\Big]\\
=\ &\tau_{m,k_m}(L,u)+b_{m,k_m}+\tfrac{1}{|L|}\cdot\Big[u(d)+\sum_{t\in L}b_{t,d_t}\\
&\qquad-\sum_{n\in L}\tau_{n,d_n}(L,u)-\sum_{n\in L}b_{n,d_n}\Big]\\
=\ &\tau_{m,k_m}(L,u)+b_{m,k_m}+\tfrac{1}{|L|}\cdot\Big[u(d)-\sum_{n\in L}\tau_{n,d_n}(L,u)\Big]\\
=\ &\overline{\tau}_{m,k_m}(L,u)+b_{m,k_m}.\end{aligned}$$

So, the principle $\overline{\tau}$ matches SYNC. ■

A natural extension of the reduction due to Hsieh and Liao [3] on multi-choice systems is as follows. Let $(L,h)\in\Omega$, $S\subseteq L$ and χ be a principle. The **reduced system** (S,h_S^{χ})

[1] In this article we focus on bounded multi-choice systems, defined as those systems (L,d,h) such that, there exists $R_h\in\mathbb{R}$ such that $h(\lambda)\leq R_h$ for every $\lambda\in D^L$. We apply it to assure that $h_*(H)$ could be well-defined.

related to S **and** χ is defined as for every $\lambda \in D^S$,

$$h_S^{\chi}(\lambda) = \begin{cases} 0 & \lambda = 0_S, \\ h(\lambda_m, 0_{L\setminus\{m\}}) & S \geq |2| \text{ and } J(\lambda) = \{m\} \text{ for some } m, \\ h(\lambda, d_{L\setminus S}) - \sum_{m \in L\setminus S} \chi_{m,d_m}(L,h) & \text{otherwise.} \end{cases}$$

The *bilateral conformance* axiom can be stated as follows. Let χ be a principle on Ω. For any category of two participators under a system, one takes a "reduced system" between them by focusing on the utilities remaining after the rest of the participators are allocated the amounts based on χ. Thus, χ is *bilateral consistent* if, when it is taken to arbitrary reduced system, it arises the same amounts as in the original system always. Formally, a principle χ matches **bilateral conformance (BCFE)** if for every $(L,h) \in \Omega$ with $|L| \geq 3$, for every $S \subseteq L$ with $|S| = 2$ and for every $(m, k_m) \in A^S$, $\chi_{m,k_m}(L,h) = \chi_{m,k_m}(S, h_S^{\chi})$.

Lemma 5: The MPEANSC $\overline{\tau}$ matches BCFE.

Proof: Let $(L,h) \in \Omega$ with $|L| \geq 3$ and $S \subseteq L$ with $|S| = 2$. Assume that $S = \{m,n\}$. By the definition of $\overline{\tau}$, for every $(p, k_p) \in A^S$,

$$\begin{aligned} &\overline{\tau}_{p,k_p}(S, h_S^{\overline{\tau}}) \\ = \;& \tau_{p,k_p}(S, h_S^{\overline{\tau}}) + \tfrac{1}{|S|} \cdot \Big[h_S^{\overline{\tau}}(d_S) - \sum_{t\in S} \tau_{t,d_t}(S, h_S^{\overline{\tau}})\Big]. \end{aligned} \tag{1}$$

By definitions of τ and $h_S^{\overline{\tau}}$, for every $k_m \in D_m$,

$$\begin{aligned} \tau_{m,k_m}(S, h_S^{\overline{\tau}}) &= h_S^{\overline{\tau}}(k_m, 0) \\ &= h\big(k_m, 0_{L\setminus\{m\}}\big) \\ &= \tau_{m,k_m}(L,h). \end{aligned} \tag{2}$$

By definitions of $h_S^{\overline{\tau}}$, $\overline{\tau}$ and equations (1), (2),

$$\begin{aligned} &\overline{\tau}_{m,k_m}(S, h_S^{\overline{\tau}}) \\ = \;& \tau_{m,k_m}(L,h) + \tfrac{1}{|S|} \cdot \Big[h_S^{\overline{\tau}}(d_S) - \sum_{t\in S} \tau_{t,d_t}(L,h)\Big] \\ = \;& \tau_{m,k_m}(L,h) + \tfrac{1}{|S|} \cdot \Big[h(d) - \sum_{t\in L\setminus S} \overline{\tau}_{t,d_t}(L,h) \\ & \qquad - \sum_{t\in S} \tau_{t,d_t}(L,h)\Big] \\ = \;& \tau_{m,k_m}(L,h) + \tfrac{1}{|S|} \cdot \Big[\sum_{t\in S} \overline{\tau}_{t,d_t}(L,h) - \sum_{t\in S} \tau_{t,d_t}(L,h)\Big] \\ & \Big(\text{by COM of } \overline{\tau}\Big) \\ = \;& \tau_{m,k_m}(L,h) + \tfrac{1}{|S|} \cdot \Big[\tfrac{|S|}{|L|} \cdot \big[h(d) - \sum_{t\in L} \tau_{t,d_t}(L,h)\big]\Big] \\ = \;& \tau_{m,k_m}(L,h) + \tfrac{1}{|L|} \cdot \Big[h(d) - \sum_{t\in L} \tau_{t,d_t}(L,h)\Big] \\ = \;& \overline{\tau}_{m,k_m}(L,h). \end{aligned}$$

Similarly, $\overline{\tau}_{n,k_n}(S, h_S^{\overline{\tau}}) = \overline{\tau}_{n,k_n}(L,h)$ for every $k_n \in D_n$. So, the MPEANSC matches BCFE. ∎

Lemma 6: If a principle χ matches RTPS and BCFE, then it matches COM also.

Proof: Let χ be a principle on Ω matching RTPS and BCFE, and $(L,h) \in \Omega$. It is done for $|L| \leq 2$ by RTPS. Suppose that $|L| \geq 3$. Let $n \in L$, consider $\big(\{n\}, h_{\{n\}}^{\chi}\big)$. Based on definition of $h_{\{n\}}^{\chi}$,

$$h_{\{n\}}^{\chi}(d_n) = h(d) - \sum_{m\in L\setminus\{n\}} \chi_{m,d_m}(L,h).$$

Since χ matches BCFE, $\chi_{n,k_n}(L,h) = \chi_{n,k_n}\big(\{n\}, h_{\{n\}}^{\chi}\big)$ for every $k_n \in D_n$. Especially, $\chi_{n,d_n}(L,h) = \chi_{n,d_n}\big(\{n\}, h_{\{n\}}^{\chi}\big)$. Further, by RTPS of χ, $\chi_{n,d_n}(L,h) = h_{\{n\}}^{\chi}(d_n)$. Hence, $\sum_{m\in L} \chi_{m,d_m}(L,h) = h(d)$, i.e., χ matches COM. ∎

Subsequently, we axiomatize the MPEANSC by applying axioms of RTPS and bilateral conformance.

Theorem 1: A principle χ on Ω matches RTPS and BCFE if and only if $\chi = \overline{\tau}$.

Proof: By Lemma 2, $\overline{\tau}$ matches RTPS. By Lemma 5, $\overline{\tau}$ matches BCFE.

To provide uniqueness, suppose that χ matches RTPS and BCFE on Ω. By Lemma 6, χ matches COM. Let $(L,h) \in \Omega$. If $|L| \leq 2$, then by RTPS of χ, $\chi(L,h) = \overline{\tau}(L,h)$. The situation $|L| > 2$: Let $m \in L$ and $S = \{m,n\}$ for some $n \in L \setminus \{m\}$, then for every $k_m \in D_m,\ k_n \in D_n$,

$$\begin{aligned} & \chi_{m,k_m}(L,h) - \chi_{n,k_n}(L,h) \\ = \;& \chi_{m,k_m}(S, h_S^{\chi}) - \chi_{n,k_n}(S, h_S^{\chi}) \\ & \Big(\text{BCFE of } \chi\Big) \\ = \;& \overline{\tau}_{m,k_m}(S, h_S^{\chi}) - \overline{\tau}_{n,k_n}(S, h_S^{\chi}) \\ & \Big(\text{RTPS of } \chi\Big) \\ = \;& \tau_{m,k_m}(S, h_S^{\chi}) - \tau_{n,k_n}(S, h_S^{\chi}) \\ & \Big(\text{Definition 1}\Big) \\ = \;& \Big[h_S^{\chi}(k_m, 0) - h_S^{\chi}(0, k_n)\Big] \\ & \Big(\text{Definition 1}\Big) \\ = \;& \Big[h(k_m, 0_{L\setminus\{m\}}) - h(k_n, 0_{L\setminus\{n\}})\Big] \\ & \Big(\text{Definition of } h_S^{\chi}\Big) \end{aligned} \tag{3}$$

$\overline{\tau}$ instead of χ in equation (3) similarly, we derive that

$$\overline{\tau}_{m,k_m}(L,h) - \overline{\tau}_{n,k_n}(L,h) = \Big[h(k_m, 0_{L\setminus\{m\}}) - h(k_n, 0_{L\setminus\{n\}})\Big]. \tag{4}$$

Based on (3) and (4),

$$\chi_{m,k_m}(L,h) - \chi_{n,k_n}(L,h) = \overline{\tau}_{m,k_m}(L,h) - \overline{\tau}_{n,k_n}(L,h).$$

Thus, $\chi_{m,k_m}(L,h) - \overline{\tau}_{m,k_m}(L,h) = c$ for every (m, k_m). It remains to show that $c = 0$. By COM of χ and $\overline{\tau}$,

$$0 = \sum_{m\in L} \Big[\chi_{m,d_m}(L,h) - \overline{\tau}_{m,d_m}(L,h)\Big] = |L| \cdot c.$$

Therefore, $c = 0$. ∎

Next, we axiomatize the MPEANSC by taking axioms of completeness, coincident treatment effect, synchronization and bilateral conformance.

Lemma 7: If a principle χ on Ω matches COM, CTE and SYNC, then χ matches RTPS.

Proof: Assume that a principle χ matches COM, CTE and SYNC. Let $(L,h) \in \Omega$ with $L = \{m,n\}$ for some $m \neq n$. We define (L,u) to be that for every $\lambda \in D^L$,

$$u(\lambda) = h(\lambda) - \sum_{m\in J(\lambda)} h(\lambda_m, 0_{L\setminus\{m\}}).$$

By definition of u, for every $k_m \in D_m$,

$$\begin{aligned} u(k_m, 0) &= h(k_m, 0) - h(k_m, 0) \\ &= 0. \end{aligned}$$

Similarly, $u(k_n, 0) = 0$ for every $k_n \in D_n$. Since $u(k_m, 0) = 0 = u(k_n, 0)$, by CTE of χ, $\chi_{m,k_m}(L, u) = \chi_{n,k_n}(L, u)$. By COM of χ,

$$u(d) = \chi_{m,d_m}(L, u) + \chi_{n,d_n}(L, u) = 2 \cdot \chi_{m,d_m}(L, u). \quad (5)$$

By equation (5) and definition of u,

$$\chi_{m,d_m}(L, u) = \frac{u(d)}{2} = \frac{1}{2} \cdot \big[h(d) - \tau_{m,d_m}(L, h) - \tau_{n,d_n}(L, h)\big].$$

By SYNC of χ,

$$\begin{aligned} & \chi_{m,k_m}(L, h) \\ = \; & \chi_{m,k_m}(L, u) + \tau_{m,k_m}(L, h) \\ = \; & \tfrac{1}{2} \cdot \big[h(d) - \tau_{m,d_m}(L, h) - \tau_{n,d_n}(L, h)\big] + \tau_{m,k_m}(L, h) \\ = \; & \overline{\tau}_{m,k_m}(L, h). \end{aligned}$$

Similarly, $\chi_{n,k_n}(L, h) = \overline{\tau}_{n,k_n}(L, h)$ for every $k_n \in D_n$. Hence, χ matches RTPS. ■

Theorem 2: A principle χ on Ω matches COM, CTE, SYNC and BCFE if and only if $\chi = \overline{\tau}$.

Proof: Based on Lemmas 1, 3, 4, $\overline{\tau}$ matches COM, CTE and SYNC. The remaining proof could be resulted by Lemmas 1, 7 and Theorem 1. ■

In the following we show that each of the properties appeared in Theorems 1, 2 is independent of the remaining properties.

Example 1: Define a principle χ on Ω by for every $(L, h) \in \Omega$ and for every $(m, k_m) \in A^L$,

$$\chi_{m,k_m}(L, h) = \begin{cases} \overline{\tau}_{m,k_m}(L, h) & \text{, if } |L| \leq 2, \\ 0 & \text{, otherwise.} \end{cases}$$

χ matches RTPS, but it violates BCFE.

Example 2: Define a principle χ on Ω by for every $(L, h) \in \Omega$ and for every $(m, k_m) \in A^L$, $\chi_{m,k_m}(L, h) = \tau_{m,k_m}(L, h)$. χ matches CTE, SYNC and BCFE, but it violates COM and RTPS.

Example 3: Define a principle χ on Ω by for every $(L, h) \in \Omega$ and for every $(m, k_m) \in A^L$, $\chi_{m,k_m}(L, h) = \frac{h(d)}{|L|}$. χ matches COM, CTE and BCFE, but it violates SYNC.

Example 4: Define a principle χ on Ω by for every $(L, h) \in \Omega$ and for every $(m, k_m) \in A^L$,

$$\begin{aligned} & \chi_{m,k_m}(L, h) \\ = \; & \big[h(d) - h(d_{L\setminus\{n\}}, 0)\big] \\ & + \tfrac{1}{|L|} \cdot \Big[h(d) - \sum_{k \in L} \big[h(d) - h(d_{L\setminus\{k\}}, 0)\big]\Big]. \end{aligned}$$

χ matches COM, SYNC and BCFE, but it violates CTE.

Example 5: Define a principle χ on Ω by for every $(L, h) \in \Omega$ and for every $(m, k_m) \in A^L$,

$$\begin{aligned} & \chi_{m,k_m}(L, h) \\ = \; & \sum_{\substack{K \subseteq L \\ m \in K}} \tfrac{(|K|-1)!(|L|-|K|)!}{|L|!} \cdot \Big[h\big((d_{L\setminus\{m\}}, k_m)_K, 0_{L\setminus K}\big) \\ & - h\big((d_{L\setminus\{m\}}, 0)_K, 0_{L\setminus K}\big)\Big]. \end{aligned}$$

χ matches COM, CTE and SYNC, but it violates BCFE.

IV. Dynamic result

For providing the dynamic results for the MPEANSC, we introduce a representation for the MPEANSC firstly by focusing on the *excess* function. Let $(L, h) \in \Omega$ and $s \in \mathbb{R}^{A^L}$. The **excess** of $\lambda \in D^L$ at s is the real number $ex(\lambda, h, s) = h(\lambda) - s(\lambda)$, where $s(\lambda) = \sum_{m \in J(\lambda)} s_{m,\lambda_m}$. Further, we define that $S(L, h) = \{s \in \mathbb{R}^{A^L} | \sum_{m \in L} s_{m,d_m} = h(d)\}$.

Lemma 8: Let $(L, h) \in \Omega$ and $s \in S(L, h)$. Then

$$\begin{aligned} & ex((d_m, 0_{L\setminus\{m\}}), h, s) = ex((d_n, 0_{L\setminus\{n\}}), h, s) \\ \iff \; & s_{m,d_m} = \overline{\tau}_{m,d_m}(L, h) \; \forall \; m, n \in L. \end{aligned}$$

Proof: Let $(L, h) \in \Omega$ and $s \in S(L, h)$. for every pairs $\{m, n\} \subseteq L$,

$$\begin{aligned} & ex((d_m, 0_{L\setminus\{m\}}), h, s) = ex((d_n, 0_{L\setminus\{n\}}), h, s) \\ \iff & h(d_m, 0_{L\setminus\{m\}}) - s_{m,d_m} = h(d_n, 0_{L\setminus\{n\}}) - s_{n,d_n} \\ \iff & s_{m,d_m} - s_{n,d_n} = h(d_m, 0_{L\setminus\{m\}}) - h(d_n, 0_{L\setminus\{n\}}). \end{aligned} \quad (6)$$

By definition of $\overline{\tau}$,

$$\overline{\tau}_{m,d_m}(L, h) - \overline{\tau}_{n,d_n}(L, h) = h(d_m, 0_{L\setminus\{m\}}) - h(d_n, 0_{L\setminus\{n\}}). \quad (7)$$

By (6) and (7), for every pairs $\{m, n\} \subseteq L$,

$$s_{m,d_m} - s_{n,d_n} = \overline{\tau}_{m,d_m}(L, h) - \overline{\tau}_{n,d_n}(L, h).$$

Hence,

$$\sum_{n \neq m} [s_{m,d_m} - s_{n,d_n}] = \sum_{n \neq m} [\overline{\tau}_{m,d_m}(L, h) - \overline{\tau}_{n,d_n}(L, h)].$$

Thus,

$$\begin{aligned} & (|L| - 1) \cdot s_{m,d_m} - \textstyle\sum_{n \neq m} s_{n,d_n} \\ = \; & (|L| - 1) \cdot \overline{\tau}_{m,d_m}(L, h) - \textstyle\sum_{n \neq m} \overline{\tau}_{n,d_n}(L, h). \end{aligned}$$

Since $s \in S(L, h)$ and $\overline{\tau}$ matches COM,

$$|L| \cdot s_{m,d_m} - h(d) = |L| \cdot \overline{\tau}_{m,d_m}(L, h) - h(d).$$

So we have that $s_{m,d_m} = \overline{\tau}_{m,d_m}(L, h)$ for every $m \in L$. ■

Based on the notion of Lemma 8, we define a correction mapping to provide a dynamic result for the MPEANSC.

Definition 2: Let $(L, h) \in \Omega$ with $|L| \geq 2$ and $s \in S(L, h)$. Define the *correction mapping* $g_{m,k_m} : S(L, h) \to \mathbb{R}$ by for every $(m, k_m) \in A^L$,

$$\begin{aligned} & g_{m,k_m}(s) \\ = \; & s_{m,k_m} + \lambda \sum_{n \in L\setminus\{m\}} \Big(ex((d_i, 0_{L\setminus\{m\}}), h, s) \\ & - ex((d_n, 0_{L\setminus\{n\}}), h, s)\Big). \end{aligned}$$

We also define that $g = (g_{m,k_m})_{(m,k_m) \in A^L}$ and $s^0 = s$, $s^1 = g(s^0), \cdots, s^q = g(s^{q-1})$ for every $(L, h) \in \Omega$, for every $s \in S(L, h)$ and for every $q \in \mathbb{N}$.

Lemma 9: Let $(L, h) \in \Omega$. If $s \in S(L, h)$, then $g(s) \in S(L, h)$.

Proof: Let $(L,h)\in\Omega$, $m,n\in L$ and $s\in S(L,h)$.

$$\begin{aligned}
&\sum_{n\in L\setminus\{m\}}\Big(ex((d_m,0_{L\setminus\{m\}}),h,s) - ex((d_n,0_{L\setminus\{n\}}),h,s)\Big)\\
=&\sum_{n\in L\setminus\{m\}}\Big(h(d_m,0_{L\setminus\{m\}})-h(d_n,0_{L\setminus\{n\}}) - s_{m,d_m}+s_{n,d_n}\Big)\\
=&\sum_{n\in L\setminus\{m\}}\Big(\overline{\tau}_{m,d_m}(L,h)-\overline{\tau}_{n,d_n}(L,h)-s_{m,d_m}+s_{n,d_n}\Big)\\
=&\Big((|L|-1)\cdot\big(\overline{\tau}_{m,d_m}(L,h)-s_{m,d_m}\big) - \sum_{n\in L\setminus\{m\}}\overline{\tau}_{n,d_n}(L,h)+\sum_{n\in L\setminus\{m\}}s_{n,d_n}\Big)\\
=&\Big(|L|\cdot\big(\overline{\tau}_{m,d_m}(L,h)-s_{m,d_m}\big)-h(d)+h(d)\Big)\\
=&|L|\cdot\Big(\overline{\tau}_{m,d_m}(L,h)-s_{m,d_m}\Big).
\end{aligned} \tag{8}$$

Further,

$$\begin{aligned}
&\sum_{m\in L}\sum_{n\in L\setminus\{m\}}\Big(ex((d_i,0_{L\setminus\{m\}}),h,s) - ex((d_n,0_{L\setminus\{n\}}),h,s)\Big)\\
=&\sum_{m\in L}|L|\cdot\Big(\overline{\tau}_{m,d_m}(L,h)-s_{m,d_m}\Big)\\
=&|L|\cdot\Big(\sum_{m\in L}\overline{\tau}_{m,d_m}(L,h)-\sum_{m\in L}s_{m,d_m}\Big)\\
=&|L|\cdot\big(h(d)-h(d)\big)\\
=&0.
\end{aligned} \tag{9}$$

So,

$$\begin{aligned}
&\sum_{m\in L}g_{m,d_m}(s)\\
=&\sum_{m\in L}\Big[s_{m,d_m}+\lambda\sum_{n\in L\setminus\{m\}}\Big(ex((d_i,0_{L\setminus\{m\}}),h,s) - ex((d_n,0_{L\setminus\{n\}}),h,s)\Big)\Big]\\
=&h(d).\ \big(\textbf{by equation (9) and } s\in S(L,h)\big)
\end{aligned}$$

Hence, $g(s)\in S(L,h)$ if $s\in S(L,h)$. ■

Theorem 3: Let $(L,h)\in\Omega$ with $|L|\geq 2$. If $0<\lambda<\frac{2}{|L|}$, then $\{s^q_{m,d_m}\}_{q=1}^{\infty}$ converges geometrically to $\overline{\tau}_{m,d_m}(L,h)$ for every $s\in S(L,h)$ and for every $m\in L$.

Proof: Let $(L,h)\in\Omega$ with $|L|\geq 2$, $m\in L$ and $s\in S(L,h)$. By definition of g and equation (8),

$$\begin{aligned}
&g_{m,d_m}(s)-s_{m,d_m}\\
=&\lambda\sum_{n\in L\setminus\{m\}}\Big(ex((d_i,0_{L\setminus\{m\}}),h,s) - ex((d_n,0_{L\setminus\{n\}}),h,s)\Big)\\
=&\lambda\cdot|L|\cdot\Big(\overline{\tau}_{m,d_m}(L,h)-s_{m,d_m}\Big).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\overline{\tau}_{m,d_m}(L,h)-g_{m,d_m}(s)\\
=&\overline{\tau}_{m,d_m}(L,h)-s_{m,d_m}+s_{m,d_m}-g_{m,d_m}(s)\\
=&\overline{\tau}_{m,d_m}(L,h)-s_{m,d_m}-\lambda\cdot|L|\cdot\Big[\overline{\tau}_{m,d_m}(L,h)-x_{m,d_m}\Big]\\
=&\Big(1-\lambda\cdot|L|\Big)\Big[\overline{\tau}_{m,d_m}(L,h)-s_{m,d_m}\Big].
\end{aligned}$$

So, for every $q\in\mathbb{N}$,

$$\overline{\tau}_{m,d_m}(L,h)-s^q_{m,d_m}=\Big(1-\lambda\cdot|L|\Big)^q\Big[\overline{\tau}_{m,d_m}(L,h)-s_{m,d_m}\Big].$$

If $0<\lambda<\frac{2}{|L|}$, then $-1<\Big(1-\lambda\cdot|L|\Big)<1$ and $\{s^q_{m,d_m}\}_{q=1}^{\infty}$ converges to $\overline{\tau}_{m,d_m}(L,h)$. ■

Similar to Liao [7], a different definition of completeness in the framework of multi-choice systems is defined as follows. Let $(L,h)\in\Omega$. A payoff vector s matches **plurality-completeness (PCOM)** in (L,h) if for every $(m,k_m)\in A^L$,

$$s_{m,k_m}+\sum_{n\in L\setminus\{m\}}s_{n,d_n}=h\big(d_{L\setminus\{m\}},k_m\big).$$

Clearly, if there exists (L,h) such that a payoff vector s matches PCOM in (L,h), then $s\in S(L,h)$.

Theorem 4: Let $(L,h)\in\Omega$ with $|L|\geq 2$. If $0<\lambda<\frac{2}{|L|}$, then $\{s^q\}_{q=1}^{\infty}$ converges geometrically to $\overline{\tau}(L,h)$ for every a payoff vector s which matches PCOM in (L,h).

Proof: Similar to Theorem 3, the proof could be completed. ■

V. Sports management systems

Due to the constant renovation of the trend of sports resource management systems in real-world situations, management skills related to combinations of different theoretical concepts have become the main conception. Here we would like to adopt several system-theoretical results to improve the management completeness no matter for training of sports skills or management of sports industry. In a sports tissue, every branch of the sports tissue may take several operational decisions to operate. Besides competing in sports matches, all branches should develop to increase total utilities of whole the sports tissue also, such as affiliated products, box office and so on. This type of condition could be modeled as follows. Let $L=\{1,2,\cdots,q\}$ be a collection of all branches of the sports tissue that could be formed jointly by some coalitions and let $h(\lambda)$ be the profit of offering the operational vector $\lambda=(\lambda_m)_{m\in L}$ in L jointly. For every branch m in this sports tissue, the operational level λ_m could be treated as one of the operational decisions of the branch m. The mapping h might be treated as resource mapping which assigns to every operational vector λ the amount that the branches can get when every branch m takes the operational decision $\lambda_m\in D_m$. Modeled by above approach, the sports resource management system of a sports tissue could be formed as a multi-choice system, with h being its characteristic mapping.

In the following, we offer an sports application as follows.

Example 6: Let $(L,h)\in\Omega$ and L be a collection of departments of a sports management, such as the MLB, the NBA and so on. Assume that the budget of every $m\in L$ is D_m. Under this condition the budget of a participator could be non-positive; surely, some participators might be in need of budget (under this condition the giving of a negative budget could be a financing process). For every $\lambda\in D^L$, λ might be treated as a multi-choice coalition. A multi-choice coalition λ might be seen as a sports tissue meant to realize some aims, which are coincident to its participators. The utility of a multi-choice coalition λ with the budget it needs for its activities is completed by the participators and the level of relationship of participator $m\in L$ to multi-choice coalition λ is determined by the level of budget D_m participator m puts in the multi-choice coalition λ. Notice that this notion of determining the level of relationship is different from the more general one in which the level of relationship is determined by the share of coalitional budget a participator takes. It better incarnates the risks participators are ready to operate over when releasing in a sports tissue and also its personal interest in achieving the aims the sports tissue is meant to realize: if a participator with a budget of $30000 and another participator with a budget of $3000000 put the same value of $30000 in sports tissue λ, it asserts that the first participator is much more interested in λ and, therefore, more individually involved and supposing a greater risk than the second participator for the realization of the aims of λ. In that trails we translate the relationship level of a participator to a multi-choice coalition as a measure of the risk the participator supposes by passing a unit of its budget to the coalition treated as a gathered strategy maker.

One would like to expect that the MPEANSC could offer "optimal outcomes" from every combination of decisions of all branches in sports resource management systems. In above sections, it is shown that the MPEANSC really exists and to arise payoff for a specific participator taking a specific level. In order to display how the MPEANSC could be applied and to rise its implication clearer, we firstly offer some relations among the system-theoretical axioms and sports management systems as follows.

1) **Completeness**: In order to promote the completeness, total resources within sports resource management systems should be entirely allocated.
2) **Rule for two-person systems**: Humans' interaction patterns usually lead to the behavior patterns of two-person coalitions, which will also affect the whole grand coalition. Thus, the allocation or the coordination behaviors of two-person coalitions usually affect the whole sports resource management system.
3) **Bilateral conformance**: A decision is unavoidable to appear unsatisfying under sports resource management systems. If the result of unsatisfying coalition's re-decision-making coincides with that before the re-decision-making under an allocation rule, it means such allocation rule is steady and consistent.
4) **Coincident treatment effect**: A good decision should not only promote common interests, but also consider the fair symmetry like the point of view of "**equal treatment for equal need**" under sports resource management systems.
5) **Synchronization**: When the interests of grand coalition alter, the whole under sports management system should be modified parallel with the change of resources.

By above examples, one could see that a sports resource management system might be formulated as a multi-choice system. In Section 3, it is shown that the MPEANSC is the unique principle matching completeness, coincident treatment effect, synchronization, conformance and rule for two-person systems simultaneously. Based on previous argument and Theorems 1, 2, the MPEANSC could be applied to be an useful allocation rule in the context of sports resource management systems.

VI. Concluding Remarks

1) By both taking the participators and its levels (strategies), Hwang and Liao [5] analyzed an extended Shapley value [11] on fuzzy systems. One could compare our outcomes with the outcomes due to Hwang and Liao [5]. There are several major differences:
   - Hwang and Liao [5] analyzed the framework of fuzzy systems. Here we focus on the context of multi-choice systems.
   - By both taking the participators and its levels (strategies), Hwang and Liao [5] extended the Shapley value [11], and the reduction (w.r.t. Hart and Mas-Colell [2]) to axiomatize the extended Shapley value on fuzzy systems. Differing from the outcomes of Hwang and Liao [5], we investigate the PEANSC and the reduction defined by Hsieh and Liao [3].
   - The system-theoretical results of this paper are applied to analyze resource allocation rules under sports resource management systems. These applications do not appear in Hwang and Liao [5].
2) By taking bilateral conformance axiom, we introduce axiomatic outcomes of the MPEANSC. By taking completeness axiom, the MPEANSC could not be arisen by dynamic methods. We would like to offer axiomatic outcomes by neglecting conformance axiom and offer dynamic methods by neglecting completeness axiom.
3) Here we extend the PEANSC of Hsieh and Liao [3] to multi-choice systems by both considering the participators and its activity levels (strategies). It is reasonable that more traditional principles could be generalized to multi-choice systems.

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