New Family of Chebyshev's Method for Finding Simple Roots of Nonlinear Equations

M. Barrada, R. Benkhouya, Ch. Ziti and A. Rhattoy

Abstract—**In this paper, we present a new family of Chebyshev's method for finding simple roots of nonlinear equations. The proposed schema is represented by a simple and original expression, which depends on a natural integer parameter p, thus generating infinity of methods. The convergence analysis shows that the order of convergence of all methods of the proposed scheme is three. A first study on the global convergence of these methods will performed. The peculiarity and strength of the proposed family lies in the fact that, under certain conditions, the convergence speed of its methods improves by increasing p. In order to show the power of this new family and to support the theory developed in this paper, some numerical tests will performed and some comparisons will make with several other existing third order and higher order methods.**

Index Terms—**Chebyshev's method, root finding, nonlinear equation, third order method, iterative methods, Newton's method, global convergence.**

I. Introduction

In Science, Engineering and Economy [1-4], we find several non-linear problems of the form

$$f(x) = 0 \tag{1}$$

where $f: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ for an open interval I is a scalar function. The Analytic methods for solving such equations are almost non-existent, so the numerical iterative methods are often used to get close to solutions.

The zero α, supposed simple, of equation (1), can be determined as a fixed point of some Iteration Function $(I.F.)$ by means of the one-point iteration method [5-9][50-54]

$$x_{n+1} = F(x_n) \quad \text{for} \quad n = 0, 1, .. \tag{2}$$

where x_0 is starting value. The solution α of the equation (1) is called a fixed-point of F if $F(\alpha)=\alpha$.

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The classical Newton's method [10] given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad n=0, 1, 2\ldots \tag{3}$$

is one of the best known iterative methods for quadratic convergence.

In order to improve the order of convergence of Newton's method, new methods, with third order have been developed at the expense , in general, of an additional evaluation of the second derivative. For example, Halley [1, 10-24], Chebyshev [1, 10, 15, 16, 20, 25-28], Euler [1, 10, 15, 17, 25], super-Halley [15, 29-33], Traub [10], Ostrowski [25], Hansen-Patrick [34], Chun [29], Jiang-Han [31],Sharma [32, 35, 36], Amat [15, 16, 37],Kou, Li and Wang [38] Chun and Neta [39], Liu and Li [22], Barrada and al. [12, 55] have proposed some interesting methods.

On the other note, several researches have been carried out with the aim to creating multi-step iterative methods with improved convergence order. Chun and Ham have proposed a family of sixth-order methods by weight function methods in [44]. Fernandez-Torres and al. in [47] have suggested a method with sixth-order convergence. Noor and al. in [46] have proposed a new predictor–corrector method whit fifth-order convergence. Kou and al. [42], have proposed two sixth-order methods in [42], and have constructed a family of variants of Ostrowski's method with seventh-order convergence in [49]. Wang and al. have proposed two families of sixth-order methods in [48]. Bi W and al. in [45] have constructed some eighth -order methods.

In articles [12, 55], we have proposed an interesting new family of Halley's method. Our aim here is to construct a new iterative scheme, based on Taylor polynomial and Chebyshev's method, for finding simple roots of nonlinear equations with cubical convergence. Unlike the styles of families that have been constructed by deriving from existing methods [29, 31, 32, 34, 36], we present here a new type of family, which has an original form, which generates an infinity of new methods, from a single simple expression, which depends on a natural integer parameter p. The advantage of this family is that, if certain assumptions are met, the convergence speed of these sequences improves by increasing p.

Moreover, in this study, we will obtain new global convergence theorems for these methods. To show the powerful of this family, we're going to run numerical tests on several of his methods. A comparison with many third order and higher order methods and will be realized.

II. Development of the new Chebyshev's family

A. Chebyshev's method

To derive the sequence of Newton, we approximate the given function f by the tangent line to its graph at x_n:

$$y(x) = f(x_n) + f'(x_n)(x - x_n) \quad (4)$$

and, solving $y(x_{n+1}) = 0$ for x_{n+1} yields (3).

Now, let's use a second degree Taylor polynomial:

$$y(x) = f(x_n) + f'(x_n)(x - x_n) + \frac{f''(x_n)}{2}(x - x_n)^2 \quad (5)$$

Where x_n is again an approximate solution of (1).The goal is to calculate the next iterate $x_{n+1}$where the graph of y intersect the abscissa axis, that is, to solve the following equation for x_{n+1} :

$$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{f''(x_n)}{2}(x_{n+1} - x_n)^2 \quad (6)$$

Approximating the quantity $x_{n+1} - x_n$ on the last term of the right-hand side of (6) by Newton's correction given in (3), we obtain

$$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{f''(x_n)}{2}\left(\frac{f(x_n)}{f'(x_n)}\right)^2$$

From which it follows that

$$x_{n+1}^0 = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + \frac{L_n}{2}\right) \quad (7)$$

Where $\quad L_n = L_f(x_n) = \frac{f(x_n)f''(x_n)}{f'(x_n)^2} \quad (8)$

widely known as Chebyshev's method. It is a third-order method for simple roots and it admits a geometric derivation, from a parabola in the form $ay^2 + y + bx + c = 0$. The Chebyshev's I.F. for f is thus given by

$$C_f(x) = x - \frac{f(x)}{f'(x)}\left(1 + \frac{f(x)f''(x)}{2(f'(x))^2}\right) \quad (9)$$

B. Derivation of the new iterative scheme

Factoring $x_{n+1} - x_n$ from the last two term of (6) [5, 18], we obtain

$$0 = f(x_n) + (x_{n+1} - x_n)\left(f'(x_n) + \frac{f''(x_n)}{2}(x_{n+1} - x_n)\right),$$

From which it follows that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) + \frac{f''(x_n)}{2}(x_{n+1} - x_n)}$$

This schema is implicit because it does not directly find x_{n+1}as a function of x_n. It can be modified to make it explicit, using a progressive prediction and replacing x_{n+1} of the right-hand side by x_{n+1}^p and x_{n+1} of the left-hand side by x_{n+1}^{p+1}. We obtain:

$$x_{n+1}^{p+1} = x_n - \frac{f(x_n)}{f'(x_n) + \frac{f''(x_n)}{2}(x_{n+1}^p - x_n)} \quad (10)$$

where x_{n+1}^0 is given by (7), and p is a non-zero natural integer parameter.

Therfore, by approximating the quantity $x_{n+1}^p - x_n$ remaining in the denominator of the right-hand side of (10) by Chebyshev's correction $x_{n+1}^0 - x_n$ given in (7), we obtain

$$x_{n+1}^1 = x_n - \frac{4f(x_n)}{f'(x_n)}\left(\frac{1}{-L_n^2 - 2L_n + 4}\right) \quad (11)$$

Similarly, by approximating $x_{n+1}^p - x_n$ placed to right of (10) by the new correction $x_{n+1}^1 - x_n$ given by (11), we obtain

$$x_{n+1}^2 = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{L_n^2 + 2L_n - 4}{L_n^2 + 4L_n - 4}\right) \quad (12)$$

The sequences given by (10) represents a general family of Chebyshev's method (C_p) for finding simple roots.

Lemma 1. *Let p be a natural integer parameter and f a real function sufficiently smooth in some neighborhood of zero, α, say. The family of Chebyshev's method (C_p) defined by the sequences (x_n^p) given by the recurrence formula (10) can be expressed, in the following explicit form:*

$$x_{n+1}^{p+1} = x_n - B_{p+1}(L_n)\frac{f(x_n)}{f'(x_n)} \quad (13)$$

$$\text{Where} \begin{cases} B_{p+1}(x) = \frac{H_p(x)}{H_{p+1}(x)} \\ H_p(x) = \sum_{k=0}^{\left[\frac{p+3}{2}\right]} G_k^p x^k \\ G_k^p = \frac{(-1)^k (p-k+2)!(p-3k+3)}{2^k (p-2k+3)!k!} \end{cases}$$

and $[x]$ is integer part of number x, $L_n$is given by (8), and p is an integer parameter.

The schema (13) is very original, powerful and generates an infinity of new and interesting methods via the parameter p. It is a special case of (2) with the following iteration function (I.F) :

$$F_{p+1}(x) = x - \frac{f(x)}{f'(x)}.B_{p+1}(L_f(x)) \quad (14)$$

Proof. Let (w_p) be defined by the sequence (x_{n+1}^{p+1}) given by (13) for a given integer n. We must demonstrate by induction that, for all $p \in \mathbb{N}$, the formula w_p given in (13) is the same to that defined by (10).

If $p = 0$, the formula (10) leads to the expression (11) given by: $w_0 = x_{n+1}^1$.

On the other hand, according (13), we have:

$x_{n+1}^1 = x_n - B_1(L_n)\frac{f(x_n)}{f'(x_n)}$ where $B_1(x) = \frac{H_0(x)}{H_1(x)}$.

As $H_0(L_n) = G_0^0 + G_1^0 L_n = 1$

and $H_1(L_n) = G_0^1 + G_1^1 L_n + G_2^1 L_n^2 = 1 - \frac{L_n}{2} - \frac{L_n^2}{4}$,

so, the sequence (13) leads to the same expression (11) for w_0.

Now, we suppose that, for a given p, the expression of $w_p = x_{n+1}^{p+1}$ given by (10) is equal to that defined by (13). Let's show that this is true for w_{p+1}.

The expression (10) becomes:

$$w_{p+1} = x_{n+1}^{p+2} = x_n - \frac{f(x_n)}{f'(x_n) + \frac{f''(x_n)}{2}(x_{n+1}^{p+1} - x_n)}$$

$$= x_n - \frac{f(x_n)}{f'(x_n) + \frac{f''(x_n)}{2}\left(-B_{p+1}(L_n)\frac{f(x_n)}{f'(x_n)}\right)}$$

which leads to:

$$w_{p+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{H_{p+1}(L_n)}{H_{p+1}(L_n) - \frac{L_n}{2}.H_p(L_n)}\right) \quad (15)$$

On the other hand, according (13), we have:

$$w_{p+1} = x_{n+1}^{p+2} = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{H_{p+1}(L_n)}{H_{p+2}(L_n)}\right)$$

So, to show that the expression of w_{p+1} given by (10) is the same to the one defined by (13), it is enough to prove that :

$$H_{p+1}(x) - \frac{x}{2}.H_p(x) = H_{p+2}(x) \qquad (16)$$

We have:

$$H_{p+1}(x) - H_{p+2}(x) = \sum_{k=0}^{[\frac{p+4}{2}]} G_k^{p+1} x^k - \sum_{k=0}^{[\frac{p+5}{2}]} G_k^{p+2} x^k$$

If p is even, we have: $\left[\frac{p+4}{2}\right] = \left[\frac{p+5}{2}\right]$ and as $G_0^{p+1} = G_0^{p+2}$, we obtain:

$$H_{p+1}(x) - H_{p+2}(x) = \sum_{k=1}^{[\frac{p+4}{2}]} \left(G_k^{p+1} - G_k^{p+2}\right) x^k = x \sum_{k=0}^{[\frac{p+2}{2}]} \left(G_{k+1}^{p+1} - G_{k+1}^{p+2}\right) x^k$$

Since $G_{k+1}^{p+1} - G_{k+1}^{p+2} = \frac{1}{2}\, G_k^p$, and $\left[\frac{p+2}{2}\right] = \left[\frac{p+3}{2}\right]$, we deduce that:

$$H_{p+1}(x) - H_{p+2}(x) = \frac{x}{2}.H_p(x).$$

Analogously, we can prove that equality (16) is satisfied for the case where p is odd. By induction, we then obtain that (10) can be explicitly expressed by (13) for all $p \in \mathrm{N}$. This completes the demonstration.

III. Order of convergence

Theorem 1. *Suppose that the function f(x) has at least two continuous derivatives in the neighborhood of a zero, α, say. Further, assume that $f'(\alpha) \neq 0$ and x_0 is sufficiently close to α. Then, the methods (C_p) defined by (13) converges cubically and satisfies the error equation*

$$For\ all\ p \in \mathrm{IN} \quad e_{n+1} = -c_3 e_n^3 + O(e_n^4) \qquad (17)$$

where $e_n = x_n - \alpha$ is the error at nth iteration and

$$c_i = \frac{f^{(i)}(\alpha)}{i!\, f'(\alpha)}\ , \ \ i = 2, 3 \ldots \qquad (18)$$

Proof. Let α be a simple root of f and $e_n = x_n - \alpha$ be the error in approximating α by the nth iterate x_n. Using the Taylor series expansion about α and taking into account that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, one can obtain [36]:

$$\begin{aligned} f(x_n) &= f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5)], \\ f'(x_n) &= f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + O(e_n^4)], \qquad (19) \\ f''(x_n) &= f'(\alpha)[2c_2 + 6c_3e_n + 12c_4e_n^2 + O(e_n^3)], \end{aligned}$$

Using (19) we obtain

$$[f'(x_n)]^2 = [f'(\alpha)]^2[1 + 4c_2e_n + 2(2c_2^2 + 3c_3)e_n^2 + 4(3c_2c_3 + 2c_4)e_n^3 + O(e_n^4)],$$

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4), \qquad (20)$$

and

$$L_n = \frac{f(x_n)f''(x_n)}{[f'(x_n)]^2} = 2c_2e_n - 6(c_2^2 - c_3)e_n^2 + 4(4c_2^3 - 7c_2c_3 +$$
3c4en3+0en4 (21)

We use the following Taylor's expansion of $B_{p+1}(L_n)$ about $L(\alpha)$ [36]:

$$\begin{aligned} B_{p+1}(L_n) = B_{p+1}(L(\alpha)) &+ (L_n - L(\alpha))B'_{p+1}(L(\alpha)) \\ &+ \frac{1}{2}(L_n - L(\alpha))^2 B''_{p+1}(L(\alpha)) \\ &+ O\left((L_n - L(\alpha))^3\right). \end{aligned}$$

Taking into account that $L(\alpha) = 0$, we obtain

$$B_{p+1}(L_n) = B_{p+1}(0) + L_nB'_{p+1}(0) + \tfrac{1}{2}L_n^2B''_{p+1}(0) + O(L_n^3)$$

By a straightforward calculation, the formula (13) gives

$For\ all\ p \in \mathrm{IN} \quad H_p(0) = 1\,,\ \ H'_p(0) = -\frac{p}{2}$

and $H''_p(0) = \frac{p(p-3)}{4}$

This leads to

$$\begin{aligned} &For\ all\ p \in \mathrm{IN}\ \ B_{p+1}(0) = 1\,, B'_{p+1}(0) = \frac{1}{2} \\ &and\ B''_{p+1}(0) = 1 \end{aligned} \qquad (22)$$

Thus the Formula (20) becomes

$$B_{p+1}(L_n) = 1 + \tfrac{1}{2}L_n + \tfrac{1}{2}L_n^2 + O(L_n^3) \qquad (23)$$

Taking into account that L_n is given by (21), we get

$$B_{p+1}(L_n) = 1 + c_2e_n + [-c_2^2 + 3c_3]e_n^2 + O(e_n^3) (24)$$

Substituting (20) and (24) in formula (13), we obtain the error equation

$$For\ all\ \ p \in \mathrm{IN}\ \ e_{n+1} = -c_3e_n^3 + O(e_n^4)$$

Which establishes the three-order convergence of (13). This completes the proof of the theorem.

IV. Study of global convergence of new methods

We are going to make a first study of the global convergence of the methods of the proposed family, in case they converge to the root in a monotonous way [20, 22, 33]. But before we do, we're going to introduce some important lemmas. Burden and Faires in [6] gave the following theorem:

Theorem 2. *Let $F \in C[a, b]$ be such that $F(x) \in [a, b]$, for all x in $[a, b]$. Suppose, in addition, that F' exists on (a, b) and that a constant $0 < k < 1$ exists with: $|F'(x)| \le k$, for all $x \in (a, b)$. Then for any number x_0 in $[a, b]$, the sequence defined by (2) converges to the unique fixed point α in $[a, b]$.*

Lemma 2. *Let p be a non-zero natural integer. Then:*
(1)The polynomial H_p, defined in (13), admits at least one positive real root,
(2)The function H_p is strictly positive over the interval $[0, a_p)$, where a_p is the smallest positive root of H_p, and
(3)The sequence $(\boldsymbol{a_p})$, constituted by the smallest roots of all the polynomials H_p, is strictly decreasing.

Proof of (2). Let p be a non-zero natural integer. We Assume that the polynomial H_p, defined in (13), admits some positives reels roots. Let us call a_p the smallest positive real root of the polynomial H_p.

We have $H_p(a_p) = 0, H_p(0) = 1$ and the function H_p is continuous on interval $[0, a_p]$. We conclude that $H_p(x) > 0$, for every real $x \in [0, a_p)$.

Proof of (1) and (3) by induction:

We have $H_1(x) = 1 - \frac{x}{2} - \frac{x^2}{4}$ then $a_1 = \sqrt{5} - 1 > 0$, *and* $H_2(x) = 1 - x - \frac{x^2}{4}$, then $a_2 = 2(\sqrt{2} - 1) > 0$. *So* a_1 *and* a_2 *exist and* $a_2 < a_1$.

Let p a non-zero natural integer. we assume that a_k exist for $k \leq p+1$ and $0 < a_{k+1} < a_k$ for every $k \leq p$. We will prove that a_{p+2} exist and that $0 < a_{p+2} < a_{p+1}$.

For all $x \in (0, a_p)$, we have $H_p(x) > 0$ and $H_{p+1}(x) - H_{p+2}(x) = \frac{x}{2} H_p(x)$. We deduce that $H_{p+2}(x) < H_{p+1}(x)$.

As $a_{p+1} \in (0, a_p)$ then $H_{p+2}(a_{p+1}) < H_{p+1}(a_{p+1})$, so $H_{p+2}(a_{p+1}) < 0$.

Furthermore, we have $H_{p+2}(0) = 1 > 0$, and the function H_p is continuous on $[0, a_{p+1}]$. So from Intermediate value theorem, there exist a real $c \in (0, a_{p+1})$ such as $H_{p+2}(c) = 0$.

Let a_{p+2} the smallest positive real root of polynomial H_{p+2}, then $H_{p+2}(a_{p+2}) = 0$ and $a_{p+2} \in (0, a_{p+1})$. *So* a_{p+2} exist and $0 < a_{p+2} < a_{p+1}$. This completes the demonstration of the Lemma 2.

Theorem 3. *Let* $f \in C^m[a,b], m \geq 4, f' \neq 0, f'' \neq 0$, $0 \leq L_f < a_{p+1}$ *and the iterative function* F_{p+1} *of* f, *defined by (14) for a chosen natural integer p, is increasing function on an interval [a,b] containing the root* α *of* f. *Then the sequences* (C_p) *given by (13) are decreasing (resp. increasing) and converges to* α *from any point* $x_0 \in [a,b]$ *checking* $f(x_0)f'(x_0) > 0$ *(resp.* $f(x_0)f'(x_0) < 0$*).*

Proof. Let's choose an x_0 such that $f(x_0)f'(x_0) > 0$, then $x_0 > \alpha$. Applying the Mean Value Theorem to the function F_{p+1}, where p is a natural integer, we obtain:

$$x_1^{p+1} - \alpha = F_{p+1}(x_0) - F_{p+1}(\alpha) = F'_{p+1}(\eta)(x_0 - \alpha)$$

for some $\eta \in (\alpha, x_0)$. As F_{p+1} is increasing function on $[a,b]$, then its derivative checks $F'_{p+1}(x) \geq 0$ in $[\alpha, b]$, so $x_1^{p+1} \geq \alpha$.

By induction, we obtain $x_n^{p+1} \geq \alpha$ for all $n \in \mathbb{N}$.

Furthermore, from (13) we have

$$x_1^{p+1} - x_0 = -B_{p+1}(L_0)\frac{f(x_0)}{f'(x_0)}.$$

As, from Lemma 2, the functions H_p and H_{p+1} are strictly positives over interval $[0, a_{p+1})$ *and* $0 \leq L_0 < a_{p+1}$ then we have $B_{p+1}(L_0) = \frac{H_p(L_0)}{H_{p+1}(L_0)} > 0$ for $p \in \mathbb{N}$. Like, on top of that, we've got $\frac{f(x_0)}{f'(x_0)} > 0$, we deduce that $x_1^{p+1} \leq x_0$. Now it is easy to prove by induction that $x_{n+1}^{p+1} \leq x_n^{p+1}$ for all $n \in \mathbb{N}$.

Consequently, the sequences (C_p), given by (13) are decreasing and converges to a limit $\mathrm{r} \in [a,b]$ where $\mathrm{r} \geq \alpha$. So, by making the limit of (13) we get:

$$\mathrm{r} = \mathrm{r} - \frac{f(\mathrm{r})}{f'(\mathrm{r})} B_{p+1}(\mathrm{L}_f(\mathrm{r}))$$

We have $B_{p+1}(\mathrm{L}_f) = \frac{H_p(\mathrm{L}_f)}{H_{p+1}(\mathrm{L}_f)} > 0$ for all $p \in \mathbb{N}$ and for every real $\mathrm{L}_f \in [0, a_{p+1})$, so $B_{p+1}(\mathrm{L}_f) \neq 0$ and consequently $f(\mathrm{r}) = 0$. As α is the unique zero of f in $[a,b]$, therefore $\mathrm{r} = \alpha$. This ends the proof of theorem.

Similarly, we can prove that the sequence (13) is increasing and converges to α under the same assumptions of the Theorem 3, but for $f(x_0)f'(x_0) < 0$.

V. Principal Characteristic of the Chebyshev's Family

In order to prove the powerful new Chebyshev's family (C_p), we will make an analytical comparison of the convergence speeds of its methods with each other. We first give the following elementary lemma.

Lemma 3. Let $p \in \mathbb{N}, (s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ be defined respectively by the sequences (x_n^{p+2}) and (x_n^{p+1}) given by sequences (13). Then we have:

$$s_{n+1} - t_{n+1} = -\frac{f(x_n)}{f'(x_n)}\left(\frac{\left(\frac{L_n}{2}\right)^{\mathrm{p}+3}\left(2 + \frac{L_n}{2}\right)}{H_{p+1}(L_n) \times H_{p+2}(L_n)}\right) \quad (25)$$

Proof. We have:

$$s_{n+1} - t_{n+1} = x_{n+1}^{p+2} - x_{n+1}^{p+1} = F_{p+2}(x_n) - F_{p+1}(x_n)$$
$$= \frac{f(x_n)}{f'(x_n)}\left(\frac{H_{p+2}(L_n) \times H_p(L_n) - \left(H_{p+1}(L_n)\right)^2}{H_{p+1}(L_n) \times H_{p+2}(L_n)}\right)$$

Knowing that:

$H_{p+2}(L_n) = H_{p+1}(L_n) - \frac{L_n}{2} \times H_p(L_n)$ (see (16))

it follow that:

$$H_{p+2}(L_n) \times H_p(L_n) - \left(H_{p+1}(L_n)\right)^2$$
$$= -\frac{L_n}{2}\left(H_p(L_n)\right)^2 + H_{p+1}(L_n)\left[H_p(L_n) - H_{p+1}(L_n)\right]$$
$$= \frac{L_n}{2}\left[H_{p+1}(L_n) \times H_{p-1}(L_n) - \left(H_p(L_n)\right)^2\right]$$
$$= \left(\frac{L_n}{2}\right)^p \left[H_2(L_n) \times H_0(L_n) - \left(H_1(L_n)\right)^2\right]$$

As $H_0(L_n) = 1$, $H_1(L_n) = 1 - \frac{L_n}{2} - \frac{L_n^2}{4}$

and $H_2(L_n) = 1 - L_n - \frac{L_n^2}{4}$

We deduce that

$$T_{p+2}(L_n).T_p(L_n) - \left(T_{p+1}(L_n)\right)^2 = -\left(\frac{L_n}{2}\right)^{\mathrm{p}+3}\left(2 + \frac{L_n}{2}\right)$$

and (25) is completed.

Theorem 4. *Let* $f \in C^m[a,b]$, $m \geq 4$, $f' \neq 0$, $f'' \neq 0$, $0 \leq L_f < a_{p+2}$ *and the iterative functions* F_{p+1} *and* F_{p+2} *of* f, *defined by (14) for a natural integer p, are increasing functions an interval* $[a,b]$ *containing the root* α *of* f. *Starting from the same initial point* $x_0 \in [a,b]$, *the rate of convergence of sequence* (x_n^{p+2}) *given by (13) is higher than one of sequence* (x_n^{p+1}).

Proof. Suppose that the initial value x_0 checkes $f(x_0)f'(x_0) > 0$, so $x_0 > \alpha$. From Theorem 3, we know that if $f \in C^m[a,b]$, $m \geq 4$, $f' \neq 0$, $f'' \neq 0$, $0 \leq L_f < a_{p+2}$ and F_{p+1} and F_{p+2} are increasing functions an interval $[a,b]$, then the sequences (x_n^{p+1}) and (x_n^{p+2}) given by (13), are decreasing and converges to α from any point $x_0 \in [a,b]$.

Let (s_n) and (t_n) be defined by (x_n^{p+2}) and (x_n^{p+1}) respectively.

Since $s_0 = t_0 = x_0$ and the two sequences are decreasing, we expect that $s_n \leq t_n$ for all $n \in \mathbb{N}$. This can be proved by induction. Let $n = 1$, then, from (25):

$$s_1 - t_1 = x_1^{p+2} - x_1^{p+2} = -\frac{f(x_0)}{f'(x_0)}\left(\frac{\left(\frac{L_0}{2}\right)^{p+3}\left(2+\frac{L_0}{2}\right)}{H_{p+1}(L_0)\times H_{p+2}(L_0)}\right)$$

As $0 \leq L_0 = L_f(x_0) < a_{p+2}$ then, from Lemma 2, $H_{p+1}(L_0) > 0$ and $H_{p+2}(L_0) > 0$, so $\frac{\left(\frac{L_0}{2}\right)^{p+3}\left(2+\frac{L_0}{2}\right)}{H_{p+1}(L_0)\times H_{p+2}(L_0)} > 0$.

Consequently: $s_1 \leq t_1$.

Now we assume that $s_n \leq t_n$. Since, under the above hypotheses, F_{p+2} is increasing function in $[a, b]$, we obtain $F_{p+2}(s_n) \leq F_{p+2}(t_n)$.

On the other hand, we have:

$$F_{p+2}(t_n) - F_{p+1}(t_n) = -\frac{f(t_n)}{f'(t_n)}\left(\frac{\left(\frac{L_n}{2}\right)^{p+3}\left(2+\frac{L_n}{2}\right)}{H_{p+1}(L_n)\times H_{p+2}(L_n)}\right)$$

where $L_n = L_f(t_n)$.

$$so \quad F_{p+2}(t_n) - F_{p+1}(t_n) \leq 0$$

We deduce that $F_{p+2}(s_n) \leq F_{p+1}(t_n))$. So $s_{n+1} \leq t_{n+1}$ and the induction is completed.

As a consequence, the power of the present family is illustrated analytically by justifying that, under certain conditions, the convergence speed of its methods increases with the parameter p.

VI. Numerical Results

In this section, we will present the results of some numerical tests to show the robustness of the newly proposed family and to support the theoretical results.

Numerical computations reported here have been carried out in MATLAB R2015b and the stopping criterion has been taken as $|x_{n+1} - x_n| \leq 10^{-15}$ and $|f(x_n)| \leq 10^{-15}$. We give the number of iterations (N) required to satisfy the stopping criterion. V denotes that method converges to undesired root and D denotes for divergence.

The tests functions, used in Table II, III and IV, and their roots α, are displayed in Table I.

A. Numerical comparison of the methods of the new family

Practically, in order to illustrate Theorem 4, we will give an example, which states that, under certain conditions, the higher the p parameter, the faster the convergence of methods (C_p) becomes.

Example. Given the function $f_6(x) = x^2 + 3x - 4$ in the interval $I = [1,\ 50]$. $f \in C^m[1,50]$, $m \geq 4$, $f'(x) \neq 0$ and $f''(x) \neq 0$ in I.

We have $L_f(x) = \frac{2(x^2+3x-4)}{(2x+3)^2}$. By tacking $x_0 = 30$, we have $f(x_0)f'(x_0) > 0$.

We will make a comparison between the four methods of our family C_0, C_2, C_5 and C_{11} obtained from formula (13), giving p the values 0, 2, 5 and 11.

We note in Table II, that:

- All the sequences $(C_0), (C_2), (C_5)$ and (C_{11}) are decreasing and converges to the root $\alpha = 1$ of function f in I;
- In the case of monotonic convergence, the convergence speed of its methods increases with the parameter p.

Therefore, the power of the proposed family is manifested in the improvement of the convergence speed of its sequences with the increase of p, provided when certain assumptions are met. Thus, since the integer p can take high values, then the convergence speed can always be improved with p. As the Chebyshev's method is obtained for $p = 0$. then its convergence rate will be lower than that of the other methods of our family.

TABLE I
TEST FUNCTIONS AND THEIR ROOTS.

Test function	Root (**α**)
$f_1(x) = x^2 + 3x - 4$	1.0000000000000000
$f_2(x) = 8x^3 - 3 - (2x-1)^3$	0.7287135538781691
	-0.2287135538781691
$f_3(x) = x^2 - 6x + 9 - \ln(x-1)$	4.057103549994738
$f_4(x) = (\sin x)^2 - x^2 + 1$	1.404491648215341
$f_5(x) = e^{x-1} - 4x^2 + 8x - 4$	1.71480591236277
$f_6(x) = -\ln(x+3) + (x+1)^2$	0.057103549994738
$f_7(x) = (x-1)^3 + 4(x-1)^2 - 10$	2.36523001341409
$f_8(x) = e^x + e^{-x} - 6$	1.76274717403908
$f_9(x) = e^x \sin x + \ln(1+x^2)$	0.0000000000000
$f_{10}(x) = -x + 1 + \ln(x^2 + x + 2)$	4.152590736757158

TABLE II
COMPARISON BETWEEN SOME METHODS OF CHEBYSHEV'S FAMILY (C_p) IN THE CASE OF MONOTONIC CONVERGENCE.

Methods	C_0	C_2	C_5	C_{11}
x_0	30	30	30	30
Iteration 1	7.313073283587293	4.417142259963999	2.683740790683693	1.53709634920714
Iteration 2	1.663953087755009	1.029901639194437	1.000030593295682	1
Iteration 3	1.00161985266443	1.000000000000437	1	
Iteration 4	1.00000000000011	1		
Iteration 5	1			

B. Comparison with other third order methods

In Table III, we will present the numerical results obtained by comparing the present new family with some well-known third-order methods in addition to Newton's method. Compared are the Newton's method (N) defined by formula (3), Chebyshev's method (C) defined here by (7), Chun's method (CH) defined by (23) with $a_n = 1$ in [29], Sharma's method (S) defined by (20) with $a_n = 1$ in [32], Jiang-Han's method (JH) [31] defined by (19) with parameter $\alpha = 1$ in [36], Ostrowski's method (O) defined by (26) in [36], Halley's method (H) defined by (25) in [36]. To represent new family $\left(C_{\mathrm{p}}\right)$ given by (13), we choose the five methods designated as $(C_5), (C_8), (C_{12}),$ (C_{18}) and (C_{21}).

The presented results in Table III, indicate that, for the most of the examples considered, the new family's chosen methods appear more efficient and perform better than the other used third-order methods, as its converge to the root much faster and take lesser number of iterations.

C. Comparison with higher order methods

In Table IV, we compare four selected methods of the new family $(C_8), (C_{16})\ et\ (C_{20})$ with some higher order methods. CH represents the method of Chun and Ham [44] (formulas (10), (11), (12)), K denotes for Kou's method [42] (formula (30) in [43]), T denotes for Thukral's method [43] (formula (27)), FT denotes for Fernandez-Torres and al. (formulas (14) and (15) in [47]), four sixth order methods, S denotes for Subaihi's method [41], fourth-order iterative method.

The efficiency of the Chebyshev's new family is also confirmed by the Table IV which shows that, for the examples chosen, ours methods require a smaller or similar number of function evaluations than several well-known methods of higher order.

TABLE III
COMPARISON WITH OTHER THIRD ORDER METHODS

Test functions	N : number of iterations												
	x_0	N	C	S	CH	JH	O	H	C_5	C_8	C_{12}	C_{18}	C_{21}
f_1	-1	7	V	8	D	8	4	5	3	2	2	2	2
f_1	15	7	5	4	5	5	4	5	3	2	2	2	2
f_2	10	10	7	5	6	7	5	6	3	3	3	2	2
f_2	-10	9	6	5	6	8	6	6	3	3	3	2	2
f_3	3.4	7	D	8	6	8	4	5	3	3	3	3	3
f_3	10	8	5	5	5	6	5	5	4	3	3	3	3
f_4	0.9	6	D	17	5	5	3	4	3	3	3	3	3
f_5	1.4	6	5	4	5	5	3	4	3	3	3	3	3
f_6	4	7	5	4	5	5	4	5	3	3	3	3	3
f_7	1.55	6	12	6	6	6	3	4	3	3	3	3	3
f_8	1	6	5	5	5	5	3	4	3	3	3	3	3
f_9	0.5	6	4	4	4	5	4	4	3	3	3	3	3
f_{10}	1.5	5	5	4	4	5	4	4	3	3	3	3	3

TABLE IV
COMPARISON WITH HIGHER ORDER METHODS

Test	x_0	N : number of iterations								NOFE : number of functions evaluations							
		K	CH	S	T	FT	C_8	C_{16}	C_{20}	K	CH	S	T	FT	C_8	C_{16}	C_{20}
f_1	-1	3	7	V	9	19	2	2	2	12	28	V	36	76	6	6	6
f_1	-21	3	3	4	4	V	2	2	2	12	12	12	16	V	6	6	6
f_2	-10	4	4	5	4	18	3	3	2	16	16	15	16	72	9	9	6
f_2	10	4	4	6	5	10	3	3	2	16	16	18	20	40	9	9	6
f_3	10	3	3	4	4	D	3	3	3	12	12	12	16	D	9	9	9
f_3	3.6	3	3	3	4	5	3	3	3	12	12	9	16	20	9	9	9
f_4	0.7	3	8	7	7	10	4	4	4	12	32	21	28	40	12	12	12
f_4	1	3	3	3	4	5	3	3	3	12	12	9	16	20	9	9	9
f_5	1	3	6	V	8	V	3	3	3	12	24	V	32	V	9	9	9
f_6	4	3	3	4	3	D	3	3	3	12	12	12	12	D	9	9	9
f_7	1.6	3	4	28	6	4	3	3	3	12	16	84	24	16	9	9	9
f_7	1.55	3	4	10	8	4	3	3	3	12	16	30	32	16	9	9	9
f_8	1	3	3	4	6	D	3	3	3	12	12	12	24	D	9	9	9

VII. CONCLUSION

In this paper, we have proposed a new family of methods to solve nonlinear equations with simple roots. The construction of this family is based on Chebyshev's formula and on the second degree Taylor polynomial. The convergence analysis has shown that all the proposed methods are at least cubically convergent for single roots. A first study of the overall convergence of these techniques has been carried out.

The originality of this family is manifested, on the one hand, in the fact that these sequences are derived from a single formula dependent on a natural integer parameter p and, on the other hand, in the improvement of the convergence speed of its sequences with the increase in p, provided that certain hypotheses are satisfied.

In order to reveal the quality of the new family, several digital examples are produced. The performance of our methods is compared with several methods of third order and much higher order. The numerical results clearly illustrated the speed and power of the techniques of the new family built in this article.

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