The Wiener-Hopf Factorization for Pricing Options Made Easy

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***Abstract*—The paper suggests a new approach to pricing barrier options under pure non-Gaussian Lévy processes with jumps of finite variation. The key idea behind the method is to represent the process under consideration as a difference between subordinators (increasing Lévy processes). Such splitting rule applied to the process at exponentially distributed randomized time points gives us the possibility to find the option price by analytically solving simple Wiener-Hopf equations.**

***Index Terms*—Wiener-Hopf factorization, numerical methods, option pricing, Lévy processes**

I. Introduction

LAST two decades, researchers give more and more attention to stochastic models of financial markets that depart from the traditional Black-Scholes model. At this moment, a wide range of models is available. We confine ourselves to the class of one-factor non-Gaussian exponential Lévy processes. These models are prevalent in financial markets because Lévy models provide a better fit to empirical asset price distributions that typically have fatter tails than Gaussian ones. Additionally, Lévy models can reproduce the well-known volatility smile phenomena and admit jumps in asset prices. For an introduction to these models applied in finance, we refer to [1].

The problem of pricing exotic options in stochastic models is crucial for mathematical finance. Exotic derivatives include path-dependent options such as lookback, one-touch, or barrier options that differ from the classic American and European call or put options. The value of an exotic derivative depends on the path of the underlying asset and the monitoring policy.

Recall that a barrier option is a contract which pays the specified amount at the terminal date T, provided during its lifetime, the price of the stock does not cross a fixed constant barrier H from above (*down-and-out barrier options*) or from below (*up-and-out barrier options*). When the underlying price crosses the barrier, the option expires worthless, or the option owner is entitled to some *rebate*. We concentrate on continuously monitored barrier options.

From a probabilistic viewpoint, one can express exotic options prices in terms of the conditional expectation of a payoff function that depends on the underlying stochastic process and its extrema. In analytical terms, a barrier derivative's value is the solution to the Kolmogorov backward equation of a specific type subject to appropriate initial and boundary conditions. In the case of Lévy models, one needs to solve complex partial integrodifferential equations. In the financial industry, traders need efficient algorithms for option pricing to perform a fast model calibration to option values. We refer the readers to [1]–[3] for modern approaches to fit model parameters.

By now, there exist several large groups of relatively universal numerical methods for pricing barrier options under exponential Lévy processes. The number of publications is vast, and, therefore, the full list is impossible.

Existing numerical methods in literature can be categorized into three main groups: Monte Carlo simulation (see e.g. [4]–[6]), backward induction methods (see e.g. [7]–[10]), and numerical methods for solving integro-differential equations (see e.g. [11]–[18]). We will consider the last group.

As a theoretical background behind the procedure, methods of the third group use the algorithm of horizontal lines [19], which includes a time discretization while a space variable remains continuous. In [20], Carr suggested a meaningful probabilistic interpretation of this technique, which we call time randomization or Carr's randomization.

After the time discretization, a cascade of stationary boundary problems for integrodifferential equations on a half-line arises. To solve them, one may apply either finite-difference methods like in [12], [21] or Wiener-Hopf factorization method (see, for example, [14], [16], [17]). In the case of continuously monitored options, one can also reduce the initial Kolmogorov backward equation to the Wiener-Hopf one applying the Laplace transform in time variable (see, e.g., [13], [16], [22]). One can show (see, e.g., [15], [24]) that Carr's randomization is equivalent to the Laplace transform inversion in the Post-Widder formula. One can treat a discrete monitoring case analogously using z-transform (see, e.g., [25], [26]).

A Wiener-Hopf method is a universal tool for solving integrodifferential equations with convolution-type kernels on a half-line. In application to finance, researchers widely use the Wiener-Hopf method to solve 2-dimensional initial boundary value problems for pricing path-dependent options under Lévy processes. However, in the case of general Lévy models, the Wiener-Hopf factors are not available in a closed form and should be approximated by using special numerical tricks. In particular, an approximate Wiener-Hopf factorization was suggested in [14] as the main ingredient of the fast, accurate, and universal numerical method for pricing barrier options under Lévy models. We will refer to that method as the "Fast Wiener-Hopf factorization method" (the FWHF-method). The paper [16] generalized the approximate factorization introduced in [14] and suggested an acceleration of the method convergence. Alternative methods that use various complicated approximate techniques for the Wiener-

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Hopf factorization presented in [13], [22], [25], [27], among others. Therefore, pricing exotic options in exponential Lévy models remains a computational challenge.

The goal of the current paper is to suggest a new, easy, and effective algorithm to price barrier options under pure non-Gaussian Lévy processes with jumps of finite variation. The main advantage of the method is applying explicit Wiener-Hopf factorization formulas. The key idea behind our approach is to represent the process under consideration as a difference between two subordinators. Such splitting rule applied to the process at exponentially distributed randomized time points gives us the possibility to find the option price by analytically solving simple Wiener-Hopf equations in sequence.

II. Theoretical background

A. Lévy processes: basic facts

A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see, e.g., [28]). A Lévy model may have a Gaussian component, a pure jump component, or both. The second component is characterized by the Lévy measure, which describes the distribution of jumps. A Lévy process X_t can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}]=e^{-t\psi(\xi)}$ (we confine ourselves to the one-dimensional case).

The Lévy-Khintchine formula gives the characteristic exponent:

$$\psi(\xi)=\frac{\sigma^2}{2}\xi^2-i\mu\xi+\int_{-\infty}^{+\infty}(1-e^{i\xi y}+i\xi y\mathbf{1}_{[-1;1]}(y))F(dy),\tag{1}$$

where $\sigma^2\geq 0$ is the variance of the Gaussian component, $\mathbf{1}_A$ is the indicator function of the set A, and the Lévy measure $F(dy)$ satisfies

$$\int_{\mathbf{R}\setminus\{0\}}\min\{1,y^2\}F(dy)<+\infty.$$

If the jump component is a process of finite variation, that is equivalent to

$$\int_{\mathbf{R}\setminus\{0\}}\min\{1,|y|\}F(dy)<+\infty,\tag{2}$$

then (1) can be simplified

$$\psi(\xi)=\frac{\sigma^2}{2}\xi^2-i\mu\xi+\int_{-\infty}^{+\infty}(1-e^{i\xi y})F(dy),\tag{3}$$

with a different μ, and the new μ is the drift of the Gaussian component.

In the current paper, the method we suggest uses a particular class of Lévy processes named as subordinators. Recall, that a subordinator is a Lévy process with almost surely non-decreasing sample paths. According to [1, Proposition 3.10], a subordinator has no diffusion component, only positive jumps of finite variation and a non-negative drift. It follows that the characteristic exponent of a subordinator has the following form:

$$\psi(\xi)=-i\mu\xi+\int_{0}^{+\infty}(1-e^{i\xi y})F(dy),\tag{4}$$

where $\mu\geq 0$, and its Lévy measure satisfies (2). It is easy to show that if a Lévy process has almost surely non-increasing trajectories, then its characteristic exponent reads

$$\psi(\xi)=-i\mu\xi+\int_{-\infty}^{0}(1-e^{i\xi y})F(dy),\tag{5}$$

with $\mu\leq 0$, and Lévy measure satisfying (2).

Assume that the riskless rate r is constant, and, under a risk-neutral measure chosen by the market, the underlying evolves as $S_t=S_0e^{X_t}$, where X_t is a Lévy process. Then we must have $E[e^{X_t}]<+\infty$, and, therefore, ψ must admit the analytic continuation into the strip $\Im\xi\in(-1,0)$ and continuous continuation into the closed strip $\Im\xi\in[-1,0]$.

Further, if $d\geq 0$ is the constant dividend yield on the underlying asset, then the following condition (the EMM-requirement) must hold: $E[e^{X_t}]=e^{(r-d)t}$. Equivalently,

$$r-d+\psi(-i)=0,\tag{6}$$

which can be used to express the drift μ via the other parameters of the Lévy process:

$$\mu=r-d-\frac{\sigma^2}{2}+\int_{-\infty}^{+\infty}(1-e^y+y\mathbf{1}_{[-1;1]}(y))F(dy).\tag{7}$$

In the examples below, we list some popular classes of Lévy processes in empirical studies of financial markets.

Example 1. [Tempered stable Lévy processes] The characteristic exponent of a pure jump KoBoL process of order $\nu\in(0,2),\nu\neq 1$ is given by

$$\begin{aligned}\psi(\xi)=&-i\mu\xi+c\Gamma(-\nu)[\lambda_+^\nu-(\lambda_++i\xi)^\nu]\\&+c\Gamma(-\nu)[(-\lambda_-)^\nu-(-\lambda_--i\xi)^\nu],\end{aligned}\tag{8}$$

where $c>0$, $\mu\in\mathbf{R}$, and $\lambda_-<-1<0<\lambda_+$. The characteristic exponent (8) is derived in [29] from the Lévy-Khintchine formula with the Lévy densities of negative and positive jumps, $F_\mp(dy)$, given by

$$F_\mp(dy)=ce^{\lambda_\pm y}|y|^{-\nu-1}dy;\tag{9}$$

in the first two papers, the name extended Koponen family was used. Later, the same class of processes was used in [30] under the name CGMY-model. The following relations between parameters of KoBoL model and C,G,M,Y parameters of CGMY-model is valid:

$$C=c,\,Y=\nu,\,G=\lambda_+,\,M=-\lambda_-.$$

More general version with $c_\pm$ instead of c, and the different exponents $\nu_\pm$ is known as a Tempered Stable Lévy model [1]. In this case, we have for $\nu_+,\nu_-\in(0,2),\nu_+,\nu_-\neq 1$

$$\begin{aligned}\psi(\xi)=&-i\mu\xi+c_+\Gamma(-\nu_+)[\lambda_+^{\nu_+}-(\lambda_++i\xi)^{\nu_+}]\\&+c_-\Gamma(-\nu_-)[(-\lambda_-)^{\nu_-}-(-\lambda_--i\xi)^{\nu_-}],\end{aligned}\tag{10}$$

where $c_+,c_->0$, $\mu\in\mathbf{R}$, and $\lambda_-<-1<0<\lambda_+$.

Example 2. [Variance Gamma processes] The Lévy density of a Variance Gamma process is of the form (9) with $\nu=0$, and the characteristic exponent is given by (see [31])

$$\psi(\xi)=-i\mu\xi+c[\ln(\lambda_++i\xi)-\ln\lambda_++\ln(-\lambda_--i\xi)-\ln(-\lambda_-)],\tag{11}$$

where $c>0$, $\mu\in\mathbf{R}$, and $\lambda_-<-1<0<\lambda_+$.

Example 3. [Kou model] If $F_{\mp}(dy)$ are given by exponential functions on negative and positive axis, respectively:

$$F_{\mp}(dy) = c_{\pm}(\pm\lambda_{\pm})e^{\lambda_{\pm}y},$$

where $c_{\pm} \geq 0$ and $\lambda_- < 0 < \lambda_+$, then we obtain Kou model. The characteristic exponent of the process is of the form

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi}. \quad (12)$$

The two-sided version was introduced in [32].

B. Wiener-Hopf factorization

There are several forms of the Wiener-Hopf factorization. The Wiener-Hopf factorization formula used in probability reads:

$$E[e^{i\xi X_{T_q}}] = E[e^{i\xi \overline{X}_{T_q}}]E[e^{i\xi \underline{X}_{T_q}}], \quad \forall\ \xi \in \mathbf{R}, \quad (13)$$

where $T_q \sim \operatorname{Exp} q$ is an exponentially distributed random variable independent of X, and $\overline{X}_t = \sup_{0\leq s\leq t} X_s$ and $\underline{X}_t = \inf_{0\leq s\leq t} X_s$ are the supremum and infimum processes. Notice, that for all $t > 0$ the following useful relations hold (see details in [28]):

$$\overline{X}_t \sim X_t - \underline{X}_t \quad (14)$$
$$\underline{X}_t \sim X_t - \overline{X}_t. \quad (15)$$

Introducing the notation

$$\phi_q^+(\xi) = qE\left[\int_0^\infty e^{-qt}e^{i\xi\overline{X}_t}dt\right] = E\left[e^{i\xi\overline{X}_{T_q}}\right], (16)$$
$$\phi_q^-(\xi) = qE\left[\int_0^\infty e^{-qt}e^{i\xi\underline{X}_t}dt\right] = E\left[e^{i\xi\underline{X}_{T_q}}\right], (17)$$

and taking into account that

$$E\left[e^{i\xi X_{T_q}}\right] = qE\left[\int_0^\infty e^{-qt}e^{i\xi X_t}dt\right] = \frac{q}{q+\psi(\xi)}, \quad (18)$$

we can write (13) as

$$\frac{q}{q+\psi(\xi)} = \phi_q^+(\xi)\phi_q^-(\xi), \quad \forall\ \xi \in \mathbf{R}. \quad (19)$$

It follows from (16)-(18) that $\phi_q^+(\xi)$, $\phi_q^-(\xi)$, $\frac{q}{q+\psi(\xi)}$ are the characteristic functions of the random variables $\overline{X}_{T_q}, \underline{X}_{T_q}, X_{T_q}$, respectively.

Recall that a function $\phi(\xi)$ of a real argument ξ defined by the formula

$$\phi(\xi) = E\left[e^{i\xi Y}\right],$$

is called the characteristic function (ch.f.) of the random variable Y. There exists one-to-one correspondence between distributions of random variables and their characteristic functions. Due the definition of ch.f., for a wide class of functions $u(x)$ and random variables with probability density the following useful relation holds

$$E[u(x+Y)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi}\phi(\xi)\hat{u}(\xi)d\xi, \quad (20)$$

where $\hat{u}$ is the Fourier transform of a function u:

$$\hat{u}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}u(x)dx.$$

Introduce the normalized resolvent of X or the expected present value operator (EPV–operator) under X defined for a function $u(x)$ as

$$\mathcal{E}_q u(x) = E[u(x + X_{T_q})]. \quad (21)$$

The name (see e.g. [33]) is due to the observation that, for a stream $u(X_t)$,

$$\mathcal{E}_q g(x) = E\left[\int_0^{+\infty} qe^{-qt}u(X_t)dt \mid X_0 = x\right]. \quad (22)$$

Replacing in (22) process X with the supremum and infimum processes $\overline{X}$ and $\underline{X}$, we obtain the EPV operators $\mathcal{E}_q^{\pm}$ under supremum and infimum process. Equivalently,

$$\mathcal{E}_q^+ u(x) = E[u(x + \overline{X}_{T_q})], \quad (23)$$
$$\mathcal{E}_q^- u(x) = E[u(x + \underline{X}_{T_q})]. \quad (24)$$

The operator form of the Wiener-Hopf factorization is written as follows (see details in [33]):

$$\mathcal{E}_q = \mathcal{E}_q^+\mathcal{E}_q^- = \mathcal{E}_q^-\mathcal{E}_q^+. \quad (25)$$

Note that we understand (25) as equalities for operators in appropriate function spaces, for instance, in the space of semi-bounded Borel functions. Under appropriate conditions on the characteristic exponent, we can define the EPV-operators as operators in spaces of functions of exponential growth at infinity, and (25) will hold in these spaces.

Finally, we note that (19) is the Wiener-Hopf factorization of the symbol of a pseudo-differential operator (PDO). Set $D = -i\frac{d}{dx}$ and recall that a PDO $A = a(D)$ with symbol a acts as follows:

$$Au(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi}a(\xi)\hat{u}(\xi)d\xi = \mathcal{F}^{-1}_{\xi\to x}a(\xi)\mathcal{F}_{x\to\xi}u.$$

Notice that $D = \mathcal{F}^{-1}_{\xi\to x}\xi\mathcal{F}_{x\to\xi}$, where $\mathcal{F}$ is the Fourier transform. It follows that a differential operator is a special case of a PDO with a polynomial symbol. In the case of Lévy processes, the EPV-operator $\mathcal{E}_q$ is the PDO $q(q+\psi(D))^{-1}$ with the symbol $q/(q+\psi(\xi))$.

Taking into account (16), (17), (20), we easily see that $\mathcal{E}_q^{\pm} = \phi_q^{\pm}(D)$. Hence, one can rewrite (23)-(24) as

$$\mathcal{E}_q^+ u = \mathcal{F}^{-1}_{\xi\to x}\phi_q^+(\xi)\mathcal{F}_{x\to\xi}u \quad (26)$$
$$\mathcal{E}_q^- u = \mathcal{F}^{-1}_{\xi\to x}\phi_q^-(\xi)\mathcal{F}_{x\to\xi}u. \quad (27)$$

If we know $\phi_q^{\pm}(\xi)$ explicitly, then we can numerically implement the operators on the rightmost part of (26) and (27) using the Fast Fourier Transform (FFT).

However, in the case of general Lévy models, the characteristic functions $\phi_q^{\pm}(\xi)$ are not available in closed form. Numerical evaluations of the Wiener-Hopf factors $\phi_q^{\pm}(\xi)$ are rather involved. It makes it challenging to implement powerful tools of the Wiener-Hopf method for a wide range of researchers. In the next section, we suggest a new approach that substantially simplifies the factorization technique.

III. Splitting rule and Wiener-Hopf factorization

A. The problem setup and general pricing formulas

Let T, K, H be the maturity, strike and barrier, and the stock price $S_t = S_0e^{X_t}$ under a chosen risk-neutral measure (see (6)) is an exponential Lévy process which has no diffusion component ($\sigma = 0$) and only jumps of finite variation (see (2)). As a basic example to illustrate our method, we consider pricing continuously monitored down-and-out put options without rebate under the Tempered Stable Lévy model with jumps of finite variation (see Example 1). The condition on jumps implies that the parameters ν_+ and ν_-, which characterize the activity of jumps should satisfy the following inequalities:

$$0 < \nu_+ < 1, \, 0 < \nu_- < 1. \tag{28}$$

The riskless rate r, and the dividend rate d are assumed to be constant. Set $h = \ln H/S_0$.

Consider the no-arbitrage price of the barrier option at the beginning of a period under consideration ($t = 0$), see, e.g. [14]:

$$V(T, x) = E^x\left[e^{-rT} G(X_T) \mathbf{1}_{\underline{X}_T > h}\right], \tag{29}$$

where T is the final date, and $G(x) = (K - S_0 e^x)_+$ is the payoff at time T. The short-hand notation $E^x[\cdot]$ means that we take the expectation conditioned on the event $X_0 = \underline{X}_0 = \overline{X}_0 = x$.

Denote by $\hat{V}(q, x)$ the Laplace transform of $V(T, x)$ w.r.t. T. Applying Fubini's theorem, we obtain that $\hat{V}(q, x)$ is the discounted expected value of the payoff function $G(X_t)\mathbf{1}_{\underline{X}_t > h}$ at exponentially distributed time T_{q+r}:

$$\begin{aligned} \hat{V}(q, x) &= \int_0^{+\infty} e^{-qt} E^x\left[e^{-rt} G(X_t) \mathbf{1}_{\underline{X}_t > h}\right] dt \\ &= E^x\left[\int_0^{+\infty} e^{-(q+r)t} G(X_t) \mathbf{1}_{\underline{X}_t > h} dt\right] \qquad (30) \\ &= \frac{1}{q+r} E^x\left[G(X_{T_{q+r}}) \mathbf{1}_{\underline{X}_{T_{q+r}} > h}\right]. \qquad (31) \end{aligned}$$

Once we develop a method for computing the expectations in (31) for appropriate values of q, we may recover $V(T, x)$ by using the Post-Widder formula. If $f(t)$ is a non-negative function of a real variable t, and it's Laplace transform $\tilde{f}(q) = \int_0^\infty e^{-qt} f(t)\, dt$, is known, then the approximate Post-Widder formula for $f(t)$ reads as follows (see e.g. [34])

$$\begin{aligned} f(t) &= \lim_{N \to \infty} f_N(t); \\ f_N(t) &:= \frac{(-1)^N}{N!} \left(\frac{N}{t}\right)^{N+1} \tilde{f}^{(N)}\left(\frac{N}{t}\right), \end{aligned}$$

where $\tilde{f}^{(N)}(q)$ – Nth derivative of $\tilde{f}$ at the point q.

Differentiating $n - 1$ times the expression (30) w.r.t q and multiplying it by $\frac{(-1)^{n-1} q^n}{(n-1)!}$, we obtain

$$\begin{aligned} v_n(q, x) &:= \frac{(-1)^{n-1} q^n}{(n-1)!} \partial_q^{n-1} \hat{V}(q, x) \qquad (32) \\ &= \frac{q^n}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{-(q+r)t} E^x\left[G(X_t) \mathbf{1}_{\underline{X}_t > h}\right] dt \\ &= \frac{1}{(1 + r/q)^n} E^x[G(X_{\Gamma(n, q+r)}) \mathbf{1}_{\underline{X}_{\Gamma(n, q+r)} > h}], \qquad (33) \end{aligned}$$

where $\Gamma(n, q)$ is a Gamma random variable with the shape parameter $n > 0$ and the rate parameter $q > 0$.

Using the relation $\Gamma(n, q) \sim \Gamma(n - 1, q) + T_q$, and taking into account that

$$\mathbf{1}_{\underline{X}_{\Gamma(n, q+r)} > h} = \mathbf{1}_{X_{T_{q+r}} + \underline{X}_{\Gamma(n-1, q+r)} > h} \mathbf{1}_{\underline{X}_{T_{q+r}} > h},$$

we conclude that for $n = 1, 2, \ldots$

$$v_n(q, x) = \frac{1}{(1 + r/q)} E^x[v_{n-1}(q, X_{T_{q+r}}) \mathbf{1}_{\underline{X}_{T_{q+r}} > h}], \tag{34}$$

where $v_0(q, x) = G(x)$.

Hence, we see that due the Post-Widder formula, for a fixed x, $v_N(N/T, x)$ converges to $V(T, x)$ as $N \to +\infty$.

The procedure (34) with $q = N/T$ is equivalent to Carr's randomization, introduced in [20] for the case of American call options. Later it was generalized for a wider class of stochastic control problems in the paper [35].

Recall that Carr's randomization means assuming that the maturity time is random. If we suppose that the maturity date of the barrier option is Gamma distributed with the shape parameter N and the rate parameter N/T, then we obtain a randomized analog of (29):

$$V'(T, x) \approx E^x\left[e^{-r\Gamma(N, q)} G(X_{\Gamma(N, q)}) \mathbf{1}_{\underline{X}_{\Gamma(N, q)} > h}\right], \tag{35}$$

where $q = N/T$. Notice that the choice of the randomized time is due the fact that $\Gamma(N, q)$ converges in quadratic mean to T as $N \to +\infty$, since $E[\Gamma(N, q)] = T$ and $Var[\Gamma(N, q)] = T^2/N$.

It is easy to show that $v_N(q, x)$ defined by (33) is exactly the right hand side of (35). Hence, we see that the randomized option price $V'(T, x)$ converges to $V(T, x)$ as $N \to +\infty$ as well.

The methods developed in [14], [17] demonstrated the successful use of the time randomization for pricing barrier options in Lévy models. Proofs of the Carr's randomization convergence in the case of similar problems for Lévy processes are presented in [15], [24]. Notice that in the current paper, we applied a different technique.

B. Splitting procedure

The state-of-art implementation of the Wiener-Hopf method in option pricing (see e.g. [13], [14], [17], [22]) leads to the factorization (19) of $(q+r)/(q+r+\psi(\xi))$, where $\psi(\xi)$ is the characteristic exponent of the Lévy process X_t. Then using (14)-(15) and (21)-(24) one can calculate the sequence (34) with $q = N/T$ as follows: for $n = 1, \ldots, N$

$$\begin{aligned} & v_n(q, x) = \frac{1}{(1 + r/q)} \cdot \\ & E^x[v_{n-1}(q, (X_{T_{q+r}} - \underline{X}_{T_{q+r}}) + \underline{X}_{T_{q+r}}) \mathbf{1}_{\underline{X}_{T_{q+r}} > h}] \\ & = \frac{1}{(1 + r/q)} E^x[\mathcal{E}^+_{q+r} v_{n-1}(q, \underline{X}_{T_{q+r}}) \mathbf{1}_{\underline{X}_{T_{q+r}} > h}] \\ & = \frac{1}{(1 + r/q)} \cdot \\ & E[\mathcal{E}^+_{q+r} v_{n-1}(q, x + \underline{X}_{T_{q+r}}) \mathbf{1}_{(h, +\infty)}(x + \underline{X}_{T_{q+r}})] \\ & = \frac{1}{(1 + r/q)} \mathcal{E}^-_{q+r} \mathbf{1}_{(h, +\infty)} \mathcal{E}^+_{q+r} v_{n-1}(q, x). \qquad (36) \end{aligned}$$

Recall that according to (14) and (23),

$$\begin{aligned} E^x[v_{n-1}(q, (X_{T_{q+r}} - \underline{X}_{T_{q+r}}) &= E^x[v_{n-1}(q, \overline{X}_{T_{q+r}})] \\ = E[v_{n-1}(q, x + \overline{X}_{T_{q+r}})] &= \mathcal{E}^+_{q+r} v_{n-1}(q, x). \end{aligned}$$

Alternatively, we can write the Laplace transform $\hat{V}(q, x)$ by the formula (31), and calculate it similar to (36) at a number of points q specified by a numerical algorithm of the Laplace transform inversion:

$$\hat{V}(q,x) = \frac{1}{(q+r)}\mathcal{E}^-_{q+r}\mathbf{1}_{(h,+\infty)}\mathcal{E}^+_{q+r}G(x). \quad (37)$$

Then we can recover $V(T,x)$ from (37) using the chosen algorithm of the inversion.

As we mentioned above, the main computational challenge in (36) is to implement the Wiener-Hopf operators numerically. The most straightforward approach involves the Fourier integral transform technique (26)-(27) provided the factors $\phi^{\pm}(\xi)$ are known, see also (20). Most Lévy models, including Tempered Stable Lévy processes, do not admit explicit Wiener-Hopf factorization (19). The numerical tricks for approximating the functions $\phi^{\pm}(\xi)$ can be found in [14]–[17], [22], [23].

Before describing the proposed method, we consider particular cases of Lévy models, which allow a trivial Wiener-Hopf factorization. At first, we suppose that the log-price X_t moves by upward jumps of finite variation or positive linear drift only. It follows that X_t is a subordinator with the characteristic exponent of the form (4), $\overline{X}_t = X_t$ and $\underline{X}_t = X_0$. Then we can rewrite the formula (34) as follows:

$$\begin{aligned} v_n(q,x) &= \frac{\mathbf{1}_{(h,+\infty)}(x)}{(1+r/q)}E[v_{n-1}(q,x+\overline{X}_{T_{q+r}})] \\ &= \frac{\mathbf{1}_{(h,+\infty)}(x)}{(1+r/q)}\mathcal{E}^+_{q+r}v_{n-1}(q,x). \end{aligned}$$

Notice that in this particular case, according to (16)-(18), the characteristic functions of $\overline{X}_{T_{q+r}}$ and $\underline{X}_{T_{q+r}}$ are

$$\phi^+_{q+r}(\xi) = (q+r)/(q+r+\psi(\xi)) \text{ and } \phi^-_{q+r}(\xi) = 1, \quad (38)$$

respectively. Then we take into account (20), and we obtain

$$\begin{aligned} v_n(q,x) &= \frac{\mathbf{1}_{(h,+\infty)}(x)}{(1+r/q)} \times \quad (39) \\ &(2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}(q+r)/(q+r+\psi(\xi))\hat{v}_{n-1}(q,\xi)d\xi, \end{aligned}$$

where $\hat{v}_{n-1}(q,\xi)$ is the Fourier transform of the function $v_{n-1}(q,x)$ w.r.t. x.

As the second example of a trivial factorization identity in (19), we consider the log-price X_t having only downward jumps of finite variation under negative drift. Hence, X_t has the characteristic exponent of the form (5), $\overline{X}_t = X_0$ and $\underline{X}_t = X_t$. Then we represent $v_n(q,x)$ in (34) as

$$\begin{aligned} &\frac{1}{(1+r/q)}E[v_{n-1}(q,x+\underline{X}_{T_{q+r}})\mathbf{1}_{(h,+\infty)}(x+\underline{X}_{T_{q+r}})] \\ = &\frac{1}{(1+r/q)}\mathcal{E}^-_{q+r}\mathbf{1}_{(h,+\infty)}v_{n-1}(q,x). \end{aligned}$$

Analogously to the previous case, the characteristic functions of $\overline{X}_{T_{q+r}}$ and $\underline{X}_{T_{q+r}}$ are trivial:

$$\phi^+_{q+r}(\xi) = 1 \text{ and } \phi^-_{q+r}(\xi) = (q+r)/(q+r+\psi(\xi)). \quad (40)$$

Then we have

$$\begin{aligned} v_n(q,x) &= \frac{1}{(1+r/q)} \times \quad (41) \\ &(2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}(q+r)/(q+r+\psi(\xi))\hat{v}^+_{n-1}(q,\xi)d\xi, \end{aligned}$$

where $\hat{v}^+_{n-1}(q,\xi)$ is the Fourier transform of the function $v^+_{n-1}(q,\xi)(= \mathbf{1}_{(h,+\infty)}(x)\cdot v_{n-1}(q,x))$ w.r.t. x. It follows from the definition of $\mathcal{E}^-_{q+r}$ (see (24)) that $v_n(q,x) = 0$ as $x \le h$. Hence, $v^+_n(q,\xi) = v_n(q,x)$ for $n > 0$.

Thus, in the case when X_t or $-X_t$ is subordinator, the implementation of the Wiener-Hopf factorization method is rather simple. We are going to use these considerations in the construction of our method.

Let the characteristic exponent $\psi(\xi)$ of the Tempered Stable Lévy model X_t is defined by (10) and the parameters $\nu_{\pm}$ satisfy (28). The new approach to calculating (34) requires the following steps. First, we represent X_t as a difference between two subordinators X^+_t and $-X^-_t$:

$$X_t = X^+_t - (-X^-_t).$$

Let X^+_t and X^-_t be Lévy processes with the characteristic exponent $\psi_+(\xi)$ and $\psi_-(\xi)$, respectively. If $\mu \ge 0$ we define $\psi_+(\xi)$ and $\psi_-(\xi)$ as follows

$$\begin{aligned} \psi_+(\xi) &= -i\mu\xi + c_+\Gamma(-\nu_+)[\lambda_+^{\nu_+} - (\lambda_+ + i\xi)^{\nu_+}], \\ \psi_-(\xi) &= c_-\Gamma(-\nu_-)[(-\lambda_-)^{\nu_-} - (-\lambda_- - i\xi)^{\nu_-}], \end{aligned}$$

otherwise

$$\begin{aligned} \psi_+(\xi) &= c_+\Gamma(-\nu_+)[\lambda_+^{\nu_+} - (\lambda_+ + i\xi)^{\nu_+}], \\ \psi_-(\xi) &= -i\mu\xi + c_-\Gamma(-\nu_-)[(-\lambda_-)^{\nu_-} - (-\lambda_- - i\xi)^{\nu_-}]. \end{aligned}$$

Notice that X^-_t has almost surely non-increasing sample paths. Hence we have that

$$\overline{X}^+_t = X^+_t, \quad (42)$$
$$\underline{X}^-_t = X^-_t. \quad (43)$$

Let $X^{+,1}_t$ and $X^{+,2}_t$ be Lévy processes with the same characteristic exponent $\psi_+(\xi)$, i.e. $X^{+,1}_t \sim X^+_t$ and $X^{+,2}_t \sim X^+_t$. Due the property of increments of a Lévy process to be stationary independent, we conclude that X_t and $X^{+,1}_{t/2} + X^-_t + X^{+,2}_{t/2}$ are identically distributed. It means that for a fixed $t > 0$ the current position of X_t with starting point x has the same distribution as the final position of the process $(Y^t_s, 0 \le s \le 2t)$ with the following dynamics:

- $Y^t_0 = x$;
- $Y^t_s = x + X^{+,1}_s$ – an upward movement as the time s varies from $s = 0$ to $s = t/2$;
- $Y^t_s = Y^t_{t/2} + X^-_{s-t/2}$ – a downward movement as the time s varies from $s = t/2$ to $s = 3t/2$;
- $Y^t_s = Y^t_{3t/2} + X^{+,2}_{s-3t/2}$ – an upward movement as the time s varies from $s = 3t/2$ till $s = 2t$.

Further, in the paper, we omit the upper index of Y^t_{2t} for ease of notation.

For a short time period $[0,t]$, we may approximate the value of $\overline{X}$ and $\underline{X}$ at a given time t by the correspondent supremum and infimum processes of Y^t at the time $s = 2t$. Notice that extrema of Y^t can be easily defined as follows:

$$\begin{aligned} \underline{Y}_{2t} &= \min\{x, x + X^{+,1}_{t/2} + X^-_t\}, \quad (44) \\ \overline{Y}_{2t} &= \max\{x, x + X^{+,1}_{t/2}, x + X^{+,1}_{t/2} + X^-_t + X^{+,2}_{t/2}\}. \end{aligned}$$

Let a natural number N is sufficiently large and $q = N/T$. As explained in Subsection A of Section III, $v_N(N/T,x)$ defined iteratively by (34) gives an approximate price of the barrier option $V(T,x)$ (see (29)). Since the randomized time T_{q+r} converges in quadratic mean to 0 as $N \to +\infty$, we may approximate $X_{T_{q+r}}$ in (34) with $Y_{2T_{q+r}}$. Notice that in this case, the following relations hold

$$\begin{aligned} Y_{2T_{q+r}} &= x + X^{+,1}_{T_{q+r}/2} + X^{-}_{T_{q+r}} + X^{+,2}_{T_{q+r}/2}; && (45)\\ \mathbf{1}_{\underline{Y}_{2T_{q+r}}>h} &= \mathbf{1}_{(h,+\infty)}(x)\times && (46)\\ &\quad \mathbf{1}_{(h,+\infty)}(x + X^{+,1}_{T_{q+r}/2} + X^{-}_{T_{q+r}}). \end{aligned}$$

In some sense, we may consider such approximation as an analog to the operator splitting method suggested in [7], where the backward jump-diffusion PIDE for option prices is solved by splitting the related operator into the differential, positive and negative jump parts. Recall that the fundamental idea behind the approach in [7] involves representing a jump operator as a PDO with subsequent transforming into the correspondent matrix exponential.

Introduce the following operators:

$$\begin{aligned} \mathcal{E}^{+}u(x) &= E[u(x+\overline{X}^{+}_{T_{q+r}/2})], && (47)\\ \mathcal{E}^{-}u(x) &= E[u(x+\underline{X}^{-}_{T_{q+r}})]. && (48) \end{aligned}$$

Notice that $T_{q+r}/2$ is also an exponentially distributed random variable but with the intensity parameter equal to $2(q+r)$.

Set

$$\begin{aligned} \phi_{+}(\xi) &= E[e^{i\xi\underline{X}^{+}_{T_{2(q+r)}}}],\\ \phi_{-}(\xi) &= E[e^{i\xi\overline{X}^{-}_{T_{q+r}}}]. \end{aligned}$$

Taking into account (42)-(43), we obtain that $X^{+}_{T_{2(q+r)}}$ and $-X^{-}_{T_{q+r}}$ admit trivial factorizations described above in the subsection. Hence, we have (see (38) and (40) for clarifications):

$$\begin{aligned} \phi_{+}(\xi) &= E[e^{i\xi X^{+}_{T_{2(q+r)}}}]\\ &= \frac{2(q+r)}{2(q+r)+\psi_{+}(\xi)}; && (49)\\ \phi_{-}(\xi) &= E[e^{i\xi X^{-}_{T_{q+r}}}]\\ &= \frac{q+r}{q+r+\psi_{-}(\xi)}. && (50) \end{aligned}$$

Thus, we can rewrite the operators $\mathcal{E}^{+}$ and $\mathcal{E}^{-}$ in (47)-(48) as PDOs with symbols (49) and (50) as follows

$$\begin{aligned} \mathcal{E}^{+}u(x) &= E[u(x+X^{+}_{T_{2(q+r)}})]\\ &= \mathcal{F}^{-1}_{\xi\to x}\phi_{+}(\xi)\mathcal{F}_{x\to\xi}u(x)\\ &= \mathcal{F}^{-1}_{\xi\to x}\frac{2(q+r)}{2(q+r)+\psi_{+}(\xi)}\mathcal{F}_{x\to\xi}u(x);\\ \mathcal{E}^{-}u(x) &= E[u(x+X^{-}_{T_{q+r}})]\\ &= \mathcal{F}^{-1}_{\xi\to x}\phi_{-}(\xi)\mathcal{F}_{x\to\xi}u(x)\\ &= \mathcal{F}^{-1}_{\xi\to x}\frac{q+r}{q+r+\psi_{-}(\xi)}\mathcal{F}_{x\to\xi}u(x). \end{aligned}$$

Let Z and W be short notations for $X^{+,1}_{T_{q+r}/2} + X^{-}_{T_{q+r}}$ and $X^{+,1}_{T_{q+r}/2}$, respectively. Now, using the relations (45) and (46) we may approximate $v_n(q,x)$ in (34) as follows:

$$\begin{aligned} v_n(q,x) &\approx \frac{1}{(1+r/q)}E^x[v_{n-1}(q,Y_{2T_{q+r}})\mathbf{1}_{\underline{Y}_{2T_{q+r}}>h}]\\ &= \frac{\mathbf{1}_{(h,+\infty)}(x)}{(1+r/q)}E^x[v_{n-1}(q,Z+X^{+,2}_{T_{q+r}/2})\mathbf{1}_{(h,+\infty)}(Z)]\\ &= \frac{\mathbf{1}_{(h,+\infty)}(x)}{(1+r/q)}E^x[v_{n-1}(q,Z+X^{+}_{2T_{q+r}})\mathbf{1}_{(h,+\infty)}(Z)]\\ &= \frac{\mathbf{1}_{(h,+\infty)}(x)}{(1+r/q)}\times\\ &\quad E^x[\mathbf{1}_{(h,+\infty)}(W+X^{-}_{T_{q+r}})\mathcal{E}^{+}v_{n-1}(q,W+X^{-}_{T_{q+r}})]\\ &= \frac{\mathbf{1}_{(h,+\infty)}(x)}{(1+r/q)}\times\\ &\quad E^x[\mathcal{E}^{-}\mathbf{1}_{(h,+\infty)}(X^{+,1}_{T_{q+r}/2})\mathcal{E}^{+}v_{n-1}(q,X^{+,1}_{T_{q+r}/2})]\\ &= \frac{\mathbf{1}_{(h,+\infty)}(x)}{(1+r/q)}\times\\ &\quad E[\mathcal{E}^{-}\mathbf{1}_{(h,+\infty)}(x+X^{+}_{2T_{q+r}})\mathcal{E}^{+}v_{n-1}(q,x+X^{+}_{2T_{q+r}})]\\ &= \frac{\mathbf{1}_{(h,+\infty)}(x)}{(1+r/q)}\mathcal{E}^{+}\mathcal{E}^{-}\mathbf{1}_{(h,+\infty)}\mathcal{E}^{+}v_{n-1}(q,x). && (51) \end{aligned}$$

Summing up, we conclude that the formula (51) leads to consecutive evaluations of simple Fourier integrals similar to (39) or (41).

The EPV-operators $\mathcal{E}^{-}$ and $\mathcal{E}^{+}$ can be efficiently implemented by using the Fast Fourier Transform (FFT) for real-valued functions. Recall that the discrete Fourier transform (DFT) is defined by

$$F_l = DFT[f](l) = \sum_{k=0}^{M-1} f_k e^{2\pi ikl/M}, l = 0, ..., M-1.$$

The inverse DFT recovers the set of f_k's exactly from G_l's. The correspondent formula reads:

$$f_k = iDFT[F](k) = \frac{1}{M}\sum_{l=0}^{M-1} F_l e^{-2\pi ikl/M}, k = 0, ..., M-1.$$

If the data consist of a real-valued array $\{f_k\}_{k=0}^{M}$, then the resulting transform satisfies $F_{M-l} = \bar{F}_l$. Since F_0 and $F_{M/2}$ are real, the transformed complex-valued array has the same "degrees of freedom" as the original real data set $\{f_l\}$. In this case, it is efficient to use FFT algorithm for real-valued functions (see [36] for technical details). To distinguish the DFT of real functions, we will use notation RDFT.

Fix the spatial step $h > 0$ and the number of the space points $M = 2^m$. Define the partitions of the normalized log-price domain $[-\frac{Mh}{2}; \frac{Mh}{2})$ by points $x_k = -\frac{Mh}{2} + kh$, $k = 0, ..., M-1$, and the frequency domain $[-\frac{\pi}{h}; \frac{\pi}{h}]$ by points $\xi_l = \frac{2\pi l}{hM}$, $l = -M/2, ..., M/2$. Then the Fourier transform of a function u on the real line can be approximated as follows:

$$\hat{u}(\xi_l) \approx h(-1)^l\overline{RDFT[u](l)}, \quad l = 0, ..., M/2.$$

Here and below, $\overline{z}$ denotes the complex conjugate of z. Using the notation (49)-(50), we can approximate $\mathcal{E}^{\pm}$ as follows:

$$(\mathcal{E}^{\pm}u)(x_k) = iRDFT[\overline{\phi_{\pm}}.*RDFT[u]](k),\ k = 0, ..., M-1,$$

where $.*$ is the element-wise multiplication of arrays that represent the functions.

The main parameters of the algorithm for evaluating the function $V(T, x)$ are the number of time steps N, the step h of the mesh, and the localization interval $(-L\ln(2); L\ln(2))$, where L is the scaling factor (for more details about the choice of the corresponding parameters of the algorithm, see [16], [14]).

Thus, our approach uses constructions from trivial factorizations only. In this context, we will refer to the developed method as the "Simple Wiener-Hopf factorization method" (the SWHF-method).

IV. Numerical experiments

In this section, we compare the performance of the suggested SWHF-method and other numerical pricing methods. As a basic example, we consider the down-and-out put option with strike K, barrier H and time to expiry T. To illustrate our method, we take the KoBoL (CGMY) model of order $\nu \in (0, 1)$ (see Example 1), with the parameters $\nu = 0.5$, $\lambda_+ = 9$, $\lambda_- = -8$, $c = 1$. We choose the instantaneous interest rate $r = 0.072310$, the time to expiry $T = 0.5$ year, the strike price $K = 100$ and the barrier $H = 90$. In this case, the drift parameter μ is approximately zero.

In numerical examples, we implemented the algorithm of the SFWH–method described in Section III. For comparison, we choose a Monte Carlo method (MC-method), a finite difference scheme (FDS-method), and an approximate Wiener-Hopf factorization method (FWHF-method). The clue parameters of the Monte Carlo algorithm are the number of monitoring dates N (for observing barrier crossing events) and the number of sample paths Z. Then the method's complexity is $O(NZ)$. The FDS-method and the FWHF-method have the same parameters as our new SFWH–method. For a fixed number of the time steps N and space step h, the numerical complexity of all three algorithms is similar and equal to $O(NM \log M)$, where M is the number of spatial points. Notice that the parameter h fixes M for a given localization domain.

In Table I, we present the option prices obtained by the SFWH–method at 5 spot levels ($S = 91, 101, 111, 121, 131$)) and by those methods as reported in Table 1, [14].

We use the prices calculated by the MC-method with 500, 000 paths simulations as the benchmark. In Table II, the sample mean values are compared with the prices computed by FDS, FWHF and SWHF-methods. The results show a general agreement between the Monte Carlo simulation results and those computed by the other methods, including the new one.

The SFWH-prices converge very fast, and the relative errors reported in Table II are less than the Monte Carlo errors (MC errors). Notice that the MC errors indicate the ratio between the half-width of the 95% confidence interval and the sample mean.

Numerical experiments show that the same algorithm's parameters for the FWHF and SWHF-methods lead to close results. It follows that both methods have similar computational efficiency. However, the SWHF-method suggests a much more simple and straightforward construction of an approximate Wiener-Hopf factorization then its competitor.

TABLE I
Prices: down-and-out put in KoBoL (CGMY) model, $\nu_+ = \nu_- = 0.5$

	MC	FDS	FWHF	SWHF
h		10^{-4}	10^{-3}	10^{-3}
N	10^5	1600	800	800
$S = 91$	0.23650	0.23587	0.23649	0.23652
$S = 101$	0.56997	0.56691	0.56777	0.56571
$S = 111$	0.38399	0.38498	0.38556	0.38388
$S = 121$	0.20949	0.20809	0.20841	0.20974
$S = 131$	0.10836	0.10726	0.10752	0.10805

KoBoL parameters: $\nu = 0.5$, $\lambda_+ = 9$, $\lambda_- = -8$, $c = 1$, $\mu \approx 0$.
Option parameters: $K = 100$, $H = 90$, $r = 0.072310$, $d = 0$, $T = 0.5$.
Algorithm parameters: h – space step, N – number of time steps, S – spot price.
Option prices are calculated by using MC, FDS, FWHF and SWHF methods.

TABLE II
Relative errors w.r.t. MC: down-and-out put in KoBoL (CGMY) model, $\nu_+ = \nu_- = 0.5$

	MC	FDS	FWHF	SWHF
h		10^{-4}	10^{-3}	10^{-3}
N	10^5	1600	800	800
$S = 91$	1.3%	-0.27%	0.00%	0.008%
$S = 101$	0.8%	-0.54%	-0.39%	-0.747%
$S = 111$	1.0%	0.26%	0.41%	-0.029%
$S = 121$	1.4%	-0.67%	-0.52%	0.119%
$S = 131$	1.9%	-1.01%	-0.78%	-0.286%

KoBoL parameters: $\nu = 0.5$, $\lambda_+ = 9$, $\lambda_- = -8$, $c = 1$, $\mu \approx 0$.
Option parameters: $K = 100$, $H = 90$, $r = 0.072310$, $d = 0$, $T = 0.5$.
Algorithm parameters: h – space step, N – number of time steps, S – spot price.
Option prices are calculated by using MC, FDS, FWHF and SWHF methods.

V. Conclusion

In the paper, we suggested a new approach for pricing options whose payoff depends on the infimum or supremum of Lévy processes at expiry. The method suggested makes it easy to implement such a complicated tool as the Wiener-Hopf factorization for general Lévy models with jumps of finite variation. In future research, we plan to develop our approach in the following directions. The efficiency of the method can be improved by increasing the number or changing the order of terms in the splitting rule. We also consider extending our approach to the problems of pricing double barrier options. Finally, the Wiener-Hopf factorization procedure can be generalized for Lévy models with jumps of infinite variation.

References

[1] R. Cont and P. Tankov, *Financial Modelling with Jump Processes*, 2nd ed. Chapman & Hall/CRC, USA: Boca Raton, FL, 2008.
[2] Z. Xu and X. Jia, "The calibration of volatility for option pricing models with jump diffusion processes," *Applicable Analysis*, vol. 98, no. 4, pp. 810-827, 2019.
[3] C. Wei, "Parameter Estimation for Discretely Observed Vasicek Model Driven by Small Levy Noises," *IAENG International Journal of Applied Mathematics*, vol. 48, no.2, pp. 118-122, 2018.
[4] A. Kuznetsov, A. E. Kyprianou, J. C. Pardo and K. van Schaik, "A Wiener-Hopf Monte Carlo Simulation Technique for Lévy Processes," *Ann. Appl. Probab.*, vol. 21, no. 6, pp. 2171-2190. 2011.

[5] O. E. Kudryavtsev, “Approximate Wiener–Hopf Factorization and Monte Carlo Methods for Lev́y Processes,” *Theory of Probability & Its Applications*, vol. 64, no. 2, pp. 186-208, 2019.
[6] Y. Yang, J. Ma, and Y. Liang, “The Research on the Calculation of Barrier Options under Stochastic Volatility Models Based on the Exact Simulation,” *IAENG International Journal of Applied Mathematics*, vol. 48, no.3, pp. 349-361, 2018.
[7] A. Itkin, *Pricing Derivatives Under Levy Models*, Basel: Birkhauser, 2017.
[8] K. R. Jackson, S. Jaimungal and V. Surkov, “Fourier Space Time-Stepping for Option Pricing with Lévy Models,” *Journal of Computational Finance*, vol. 12, no. 2, pp. 1-28, 2008
[9] J. L. Kirkby, “Robust Option Pricing with Characteristic Functions and the B-spline Order of Density Projection”, *Journal of Computational Finance*, vol. 21, no. 2, pp. 61-100, 2017.
[10] M. J. Ruijter and C. W. Oosterlee, “Numerical Fourier Method and Second-order Taylor Scheme for Backward SDEs in Finance,” *Applied Numerical Mathematics*, vol. 103, no. C, pp. 1-26, May. 2016.
[11] R. Brummelhuis and R. T. L. Chan, “A Radial Basis Function Scheme for Option Pricing in Exponential Lévy Models,” *Applied Mathematical Finance*, vol. 21, no. 3, pp. 238-269, 2014.
[12] R. Cont and E. Voltchkova, “A Finite Difference Scheme for Option Pricing in Jump Diffusion and Exponential Lévy Models” *SIAM Journal on Numerical Analysis*, vol. 43, no. 4, pp. 1596-1626, 2005.
[13] C. E. Phelan, D. Marazzina, G. Fusai, G. Germano, “Fluctuation Identities with Continuous Monitoring and Their Application to Price Barrier Options”, *European Journal of Operational Research*, vol. 271, no. 1, pp. 210-223, 2018.
[14] O. Kudryavtsev and S. Levendorskiĭ, “Fast and Accurate Pricing of Barrier Options under Lévy Processes,” *Finance and Stochastics*, vol. 13, no. 4, pp. 531-562, 2009.
[15] O. Ye. Kudryavtsev, “An Efficient Numerical Method to Solve a Special Class of Integrodifferential Equations Relating to the Levy Models,” *Mathematical Models and Computer Simulations*, vol. 3, no.6, pp. 706-711, 2011.
[16] O. Kudryavtsev, “Advantages of the Laplace Transform Approach in Pricing First Touch Digital Options in Lévy-driven Models,” *Boletín de la Sociedad Matemática Mexicana*, vol. 22, no. 2, pp. 711-731, 2016.
[17] M. Boyarchenko and S. Levendorskiĭ, “Prices and Sensitivities of Barrier and First-Touch Digital Options in Levy-Driven Models,” *International Journal of Theoretical and Applied Finance*, vol. 12, no. 8, pp. 1125-1170, 2009.
[18] A. T. P. Najafabadi, “A Mixture-Exponential Approximation to Finite and Infinite-time Ruin Probabilities of Compound Poisson Processes,” *IAENG International Journal of Applied Mathematics*, vol. 48, no.1, pp. 105-110, 2018.
[19] E. Rothe, “Zweidimensionale Parabolische Randwertaufgaben als Grenzfall Eindimensinaler Randwerttaufgaben,” *Math. Ann.*, vol. 102, 1930.
[20] P. Carr, “Randomization and the American Put,” *Review of Financial Studies*, vol. 11, pp. 597-626, 1998.
[21] O. Kudryavtsev, “Finite Difference Methods for Option Pricing under Levy Processes: Wiener-Hopf Factorization Approach,” *The Scientific World Journal*, Article ID 963625, 12 p., 2013.
[22] S. Z. Levendorskiĭ, “Method of Paired Contours and Pricing Barrier Options and CDS of Long Maturities,” *International Journal of Theoretical and Applied Finance*, vol. 17, no. 5, pp. 1-58, 2014.
[23] S. Z. Levendorskiĭ, “Ultra-fast pricing barrier options and CDSs,” *International Journal of Theoretical and Applied Finance*, vol.20, no. 5, 1750033, 2017.
[24] S. Z. Levendorskiĭ, “Convergence of Carr’s Randomization Approximation Near Barrier,” *SIAM FM*, vol. 2, no. 1, pp. 79-111, 2011.
[25] G. Fusai, G. Germano, D. Marazzina, “Spitzer identity, Wiener-Hopf factorization and Pricing of Discretely Monitored Exotic Options,” *European Journal of Operational Research*, vol. 251, no. 1, pp. 124-134, 2016.
[26] C. E. Phelan, D. Marazzina, G. Fusai, G. Germano, “Hilbert Transform, Spectral Filters and Option Pricing,” *Annals of Operations Research*, vol. 282, pp. 1-26, 2018.
[27] M. Boyarchenko and S. Z. Levendorskiĭ, “Ghost Calibration and Pricing Barrier Options and Credit Default Swaps in Spectrally One-sided Lévy Models: The Parabolic Laplace Inversion Method,” *Quantitative Finance*, vol. 15, no. 3, pp. 421-441, 2015.
[28] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, 2nd ed. Cambridge, England: Cambridge University Press, 2013.
[29] S.I. Boyarchenko and S.Z. Levendorskiĭ, *Non-Gaussian Merton-Black-Scholes Theory*, River Edge, NJ: World Scientific Publishing Co., 2002.
[30] P. Carr, H. Geman, D. B. Madan and M. Yor, “The Fine Structure of Asset Returns: an Empirical Investigation,” *Journal of Business*, vol. 75, pp. 305-332, 2002
[31] D. Madan and E. Seneta, “The Variance Gamma (V.G.) Model for Share Market Returns,” *Journal of Business*, vol. 63, no. 4, pp. 511-524, 1990.
[32] S. G. Kou, “A jump-diffusion model for option pricing,” *Management Science*, vol 48, pp. 1086-1101, 2002.
[33] S.I. Boyarchenko and S.Z. Levendorskiĭ, “American Options: the EPV Pricing Model,” *Annals of Finance*, vol. 1, no. 3, pp. 267-292, 2005.
[34] J. Abate and W. Whitt, “A Unified Framework for Numerically Inverting Laplace Transforms,” *INFORMS Journal on Computing*, vol. 18, no. 4, pp. 408-421, 2006.
[35] B. Bouchard, N. El. Karoui, N. Touzi, “Maturity Randomization for Stochastic Control Problems,” *Annals of Applied Probability*, vol. 15, no. 4, pp. 2575-2605, 2005.
[36] W. Press, B. Flannery, S. Teukolsky and W. Vetterling, *Numerical recipes in C: The Art of Scientific Computing*, 3rd ed. Cambridge : Cambridge Univ. Press, 2007.