Note: Natural Extensions of the Equal Allocation of Non-Separable Costs

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***Abstract*—Usually, participators might take different operational grades (strategies, decisions) in real-world conditions. Thus, we provide several generalized types of the equal allocation of non-separable costs by adopting the participators and its operational grades (strategies, decisions) simultaneously. Based on weights and different types of marginal distinctions respectively, several extensions are defined to differentiate the discrimination among the operational grades. Finally, we also introduce axiomatic justifications which are analogues of Moulin's characterization to analyze the rationality for these principles.**

***Index Terms*—The equal allocation of non-separable costs, axiomatic justification, weight, marginal distinction.**

I. Introduction

In a traditional system, every participator is either to join completely in a condition or not to join at all in participation with some other participators. In real-world conditions, however, participators might take different operational grades (strategies, decisions) to operate. A *multi-choice system* can be deemed as a reasonable extension of a traditional system in which every participator takes different operational grades (strategies, decisions). An application presents in a large organization with many departments, where the earnings-making hinges on its performance. This donates rise to a multi-choice system in which the participators are the departments and the value of a coalition where every department maps at a specific grade is the corresponding earnings caused by the organization.

In this research we apply the notion of the principle concept of the *equal allocation of non-separable costs* (EANSC, Ransmeier [6]). Moulin [5] introduced a type of reduced system on traditional systems and adopted it to present that the EANSC generalizes a fair allocation for distributing usability. By determining overall amounts for a given participators, Cheng et al. [1], Liao [4] and Wei et al. [7] defined several extended power indexes under different conditions respectively.

The above pre-existing statements present one motivation:

- whether different types of the EANSC could be generalized by simultaneously considering the participator and its operational grades (strategies, decisions).

The research is devoted to deal with the motivation. Several main outcomes of this research are as follows.

- In Section 2. we define different types of the EANSC by simultaneously considering the participator and its operational grades (strategies, decisions) on multi-choice systems. Inspired by Moulin's work, we also offer axiomatic justifications to present the rationality for these extended EANSC by adopting a specific reduced system in Section 3.
- Participators might represent constituencies of various sizes; participators might possess different abilities. Furthermore, lack of symmetry might appear when different abilities for different participators are modeled. Based on above statements, we would like to desire that any usability could be allocated among the participators by its weights proportionally. It is reasonable that weights could be appointed to the "grades" of participators to differentiate the discrimination among the operational grades respectively. In Section 4, we apply the *weighted map for grades* to propose several weighted generalizations and related justifications. "Marginal distinctions" instead of "weights", different generalizations and related justifications are also offered in Section 5.

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II. Preliminaries

Let UL be the universe of participators. For $m \in UL$ and $d_m \in \mathbb{N}$, $D_m = \{0, 1, \cdots, d_m\}$ could be treated as the operational grade (strategies, decisions) collection of participator m and $D_m^+ = D_m \setminus \{0\}$, where 0 represents no participation. Let $D^L = \prod_{m \in L} D_m$ be the product collection of the operational grade (strategies, decisions) collections of all participators of L. Denote 0_L the zero vector in $\mathbb{R}^L$.

A **multi-choice system** is a triple (L, d, h), where $L \neq \emptyset$ is a finite collection of participators, $d = (d_m)_{m \in L}$ is the vector that represents the highest operational grades for all participator, and $h : D^L \to \mathbb{R}$ is a characteristic map with $h(0_L) = 0$ which appoints to every $\lambda = (\lambda_m)_{m \in L} \in D^L$ the amount that the participators can get when every participator m takes grade λ_m.

Define the family of total multi-choice systems to be Ω. Given $(L, d, h) \in \Omega$, let $A^L = \{(m, n) \mid m \in L, n \in D_m^+\}$. For $\lambda \in D^L$ and $H \subseteq L$, we denote $\lambda_H \in \mathbb{R}^L$ to be the restriction of λ to H, $|H|$ be the amount of units in H and $\|\lambda\| = \sum_{m \in L} \lambda_m$.

A **principle** on Ω is a map χ assigning to every $(L, d, h) \in \Omega$ a constituent $\chi(L, d, h)$ of $I\!R^{A^L}$. For convenience, we take $\chi_{m,0}(L, d, h) = 0$ for all $m \in L$. Here we provide three generalized EANSC on Ω.

Definition 1: The **regular allocation of non-separable costs (RANSC)** on Ω, $\overline{\tau}$, is the map on Ω which assigns to every $(L, d, h) \in \Omega$, every $m \in L$ and every $n \in D_m^+$ the amount

$$\begin{aligned} & \overline{\tau}_{m,n}(L, d, h) \\ = \; & \tau_{m,n}(L, d, h) + \tfrac{1}{\|d\|} \cdot \Big[h(d) - \sum_{(k,l) \in A^L} \tau_{k,l}(L, d, h)\Big], \end{aligned}$$

where $\tau_{m,n}(L, d, h) = \big[h(d_{L \setminus \{m\}}, n) - h(d_{L \setminus \{m\}}, n-1)\big]$ is the **regular grade-marginal distinction** of participator i

when participator i participates from grade $n-1$ to grade n. By definition of $\overline{\tau}$, all participators get its regular grade-marginal distinctions firstly, and further distribute the rest of usability equally.

The **lower-aggregated allocation of non-separable costs (LANSC)** on Ω, $\overline{\alpha}$, is the map on Ω which assigns to every $(L,d,h)\in\Omega$, every $m\in L$ and every $n\in D_m^+$ the amount

$$\begin{aligned}&\overline{\alpha}_{m,n}(L,d,h)\\ =\;&\alpha_{m,n}(L,d,h)+\tfrac{1}{\|d\|}\big[h(d)-\sum_{(k,l)\in A^L}\alpha_{k,l}(L,d,h)\big],\end{aligned}$$

where $\alpha_{m,n}(L,d,h)=\big[h(d_{L\setminus\{m\}},n)-h(d_{L\setminus\{m\}},0)\big]$ is the **lower-aggregated grade-marginal distinction** of participator i when participator i participates from grade 0 to grade n. By definition of $\overline{\alpha}$, all participators get its lower-aggregated grade-marginal distinctions firstly, and further distribute the rest of usability equally.

The **upper-aggregated allocation of non-separable costs (UANSC)** on Ω, $\overline{\gamma}$, is the map on Ω which assigns to every $(L,d,h)\in\Omega$, every $m\in L$ and every $n\in D_m^+$ the amount

$$\begin{aligned}&\overline{\gamma}_{m,n}(L,d,h)\\ =\;&\gamma_{m,n}(L,d,h)+\tfrac{1}{\|d\|}\big[h(d)-\sum_{(k,l)\in A^L}\gamma_{k,l}(L,d,h)\big],\end{aligned}$$

where $\gamma_{m,n}(L,d,h)=\big[h(d_{L\setminus\{m\}},d_m)-h(d_{L\setminus\{m\}},n-1)\big]$ is the **upper-aggregated grade-marginal distinction** of participator i when participator i participates from grade $n-1$ to grade d_m. By definition of $\overline{\gamma}$, all participators get its upper-aggregated grade-marginal distinctions firstly, and further distribute the rest of usability equally.

The major difference among $\overline{\alpha}$, $\overline{\tau}$ and $\overline{\gamma}$ is the different definition of "marginal distinctions" (α, τ and γ).

Example 1: In the following we introduce a motivating example of multi-choice systems in the notion of "management". This type of condition might be modeled as follows. Let $L=\{1,2,\cdots,p\}$ be a collection of every participators of a management system (L,d,h). The map h could be formed as an usability map which assigns to every grade vector $\lambda=(\lambda_i)_{m\in L}\in D^L$ the value that the participators can get when every participator i participates at operation grade $\lambda_i\in D_m$ in (L,d,h). Modeled by above notion, the management system (L,d,h) could be modeled as a multi-choice system, with h being every characteristic map and D_m being the collection of every operation grades of the participator i. In the following sections, we would like to present that the RANSC, LANSC and UANSC could arise "*optimal allocation mechanisms*" among every participators, in the sense that this tissue can receive worth from every combination of operation grades of every participators under multi-choice consideration.

III. Main Outcomes

Here we axiomatize the RANSC, LANSC and UANSC by means of *bilateral conformance*[1]. Let χ be a principle on Ω. χ matches **completeness (COM)** if for all $(L,d,h)\in\Omega$, $\sum_{(m,n)\in A^L}\chi_{m,n}(L,d,h)=h(d)$. χ matches **regular rule for two-agent systems (RRTAS)** if for all $(L,d,h)\in\Omega$ with $|L|\le 2$, $\chi(L,d,h)=\overline{\tau}(L,d,h)$. χ matches **lower-aggregated rule for two-agent systems (LRTAS)** if for all $(L,d,h)\in\Omega$ with $|L|\le 2$, $\chi(L,d,h)=\overline{\alpha}(L,d,h)$. χ matches **upper-aggregated rule for two-agent systems (URTAS)** if for all $(L,d,h)\in\Omega$ with $|L|\le 2$, $\chi(L,d,h)=\overline{\gamma}(L,d,h)$. COM means that all usability should be entirely allocated. RRTAS, LRTAS and URTAS are extended properties of the standard property of Hart and Mas-Colell [3].

[1]The property has been originally defined by Harsanyi [2] under the notion of bilateral equilibrium.

Lemma 1:

1) The principles $\overline{\tau}$ matches COM and RRTAS.
2) The principles $\overline{\alpha}$ matches COM and LRTAS.
3) The principles $\overline{\gamma}$ matches COM and URTAS.

Proof: Let $(L,d,h)\in\Omega$. Based on Definition 1,

$$\begin{aligned}&\sum_{(m,n)\in A^L}\overline{\tau}_{m,n}(L,d,h)\\
=\;&\sum_{(m,n)\in A^L}\tau_{m,n}(L,d,h)\\
&\quad+\sum_{(m,n)\in A^L}\tfrac{1}{\|d\|}\cdot\big[h(d)-\sum_{(k,l)\in A^L}\tau_{k,l}(L,d,h)\big]\\
=\;&\sum_{(m,n)\in A^L}\tau_{m,n}(L,d,h)\\
&\quad+\tfrac{\|d\|}{\|d\|}\cdot\big[h(d)-\sum_{(k,l)\in A^L}\tau_{k,l}(L,d,h)\big]\\
=\;&\sum_{(m,n)\in A^L}\tau_{m,n}(L,d,h)+h(d)-\sum_{(k,l)\in A^L}\tau_{k,l}(L,d,h)\\
=\;&h(d).\end{aligned}$$

So, $\overline{\tau}$ matches COM. By definitions of $\overline{\tau}$ and RRTAS, the RANSC matches RRTAS absolutely. The proofs of outcomes 2 and 3 are similar. ■

Now we focus on the Moulin reduction. Taken a amount vector appointed by a principle for some system, and taken a sub-coalition of participators, Moulin [5] proposed the reduced system as that in which every category in the sub-coalition could attain amounts to its participators only if they are compatible with the original amounts to "all" the participators outside of the sub-coalition. Here we consider a natural generalized form of Moulin reduction in the context of multi-choice systems.

Definition 2: Given $(L,d,h)\in\Omega$, $S\subseteq L$, $S\neq\emptyset$, and a principle χ, the **reduced system** $(S,d_S,h^{\chi}_{S,d})$ **related to** S **with** χ is defined as for all $\lambda\in D^S$,

$$\begin{aligned}&h^{\chi}_{S,d}(\lambda)\\ =\;&\begin{cases}0 & ,\ \lambda=0_S\\ h(\lambda,d_{L\setminus S})-\sum_{(m,n)\in A^{L\setminus S}}\chi_{m,n}(L,d,h) & ,\ \text{otherwise.}\end{cases}\end{aligned}$$

For any category of two participators under a system, one takes a "reduced system" between them by focusing on the utilities remaining after the rest of the participators are allocated the amounts based on χ. Thus, χ satisfies *bilateral conformance* if, when it is adopted to arbitrary reduced system, it arises the same amounts as in the original system always. More specifically, a principle χ matches **bilateral conformance (BCFE)** if for every $(L,d,h)\in\Omega$, for every $S\subseteq L$ with $|S|=2$ and for every $(m,n)\in A^S$, $\chi_{m,n}(S,d_S,h^{\chi}_{S,d})=\chi_{m,n}(L,d,h)$.

Lemma 2: The principles $\overline{\tau}$, $\overline{\alpha}$ and $\overline{\gamma}$ match BCFE.

Proof: Given $(L,d,h)\in\Omega$ and $S\subseteq L$ with $|S|=2$.

For all $(m,n) \in A^S$,

$$\begin{array}{rl}
& \tau_{m,n}(S, d_S, h^{\overline{\tau}}_{S,d}) \\
= & h^{\overline{\tau}}_{S,d}(d_{S\setminus\{m\}}, n) - h^{\overline{\tau}}_{S,d}(d_{S\setminus\{m\}}, n-1) \\
= & h(d_{L\setminus\{m\}}, n) - \sum\limits_{(k,l)\in A^{L\setminus S}} \overline{\tau}_{k,l}(L,d,h) \\
& -h(d_{L\setminus\{m\}}, n-1) + \sum\limits_{(k,l)\in A^{L\setminus S}} \overline{\tau}_{k,l}(L,d,h) \\
= & h(d_{L\setminus\{m\}}, n) - h(d_{L\setminus\{m\}}, n-1) \\
= & \tau_{m,n}(L,d,h).
\end{array} \quad (1)$$

Hence,

$$\begin{array}{rl}
& \overline{\tau}_{m,n}(S, d_S, h^{\overline{\tau}}_{S,d}) \\
= & \tau_{m,n}(S, d_S, h^{\overline{\tau}}_{S,d}) + \frac{1}{\|d_S\|} \cdot \Big[h^{\overline{\tau}}_{S,d}(d_S) \\
& \qquad - \sum\limits_{(k,l)\in A^S} \tau_{k,l}(S, d_S, h^{\overline{\tau}}_{S,d}) \Big] \\
= & \tau_{m,n}(L,d,h) + \frac{1}{\|d_S\|} \cdot \Big[h^{\overline{\tau}}_{S}(d_S) \\
& \qquad - \sum\limits_{(k,l)\in A^S} \tau_{k,l}(L,d,h) \Big] \\
& \textbf{(by Equation (1))} \\
= & \tau_{m,n}(L,d,h) + \frac{1}{\|d_S\|} \cdot \Big[h(d) \\
& - \sum\limits_{(k,l)\in A^{L\setminus S}} \overline{\tau}_{k,l}(L,d,h) - \sum\limits_{(k,l)\in A^S} \tau_{k,l}(L,d,h) \Big] \\
= & \tau_{m,n}(L,d,h) + \frac{1}{\|d_S\|} \cdot \Big[\sum\limits_{(k,l)\in A^S} \overline{\tau}_{k,l}(L,d,h) \\
& \qquad - \sum\limits_{(k,l)\in A^S} \tau_{k,l}(L,d,h) \Big] \\
& \textbf{(by COM of } \overline{\tau}\textbf{)} \\
= & \tau_{m,n}(L,d,h) + \frac{1}{\|d_S\|} \cdot \Big[\sum\limits_{(k,l)\in A^S} \tau_{k,l}(L,d,h) \\
& \quad + \frac{\|d_S\|}{\|d\|} \cdot \big[h(d) - \sum\limits_{(k,l)\in A^L} \tau_{k,l}(L,d,h) \big] \\
& \qquad - \sum\limits_{(k,l)\in A^S} \tau_{k,l}(L,d,h) \Big] \\
= & \tau_{m,n}(L,d,h) + \frac{1}{\|d\|} \cdot \Big[h(d) - \sum\limits_{(k,l)\in A^L} \tau_{k,l}(L,d,h) \Big] \\
= & \overline{\tau}_{m,n}(L,d,h).
\end{array}$$

The proof of BCFE of $\overline{\tau}$ is completed. Similar to above processes, it is easy to show that $\overline{\alpha}$ and $\overline{\gamma}$ match BCFE. ■

Lemma 3:

1) If a principle χ matches RRTAS and BCFE, then it matches COM.
2) If a principle χ matches LRTAS and BCFE, then it matches COM.
3) If a principle χ matches URTAS and BCFE, then it matches COM.

Proof: Here we prove outcome 1 firstly. Suppose χ matches RRTAS and BCFE. Let $(L,d,h) \in \Omega$. If $|L| \leq 2$, then χ matches COM by RRTAS of χ. Suppose $|L| > 2$, $m,n \in L$ and $S = \{m,n\}$. Since χ matches COM in two-agent systems,

$$\begin{array}{rl}
& \sum\limits_{(k,l)\in A^S} \chi_{k,l}(S, d_S, h^{\chi}_{S,d}) \\
= & h^{\chi}_{S,d}(d) \\
= & h(d) - \sum\limits_{(k,l)\in A^{L\setminus\{m,n\}}} \chi_{k,l}(L,d,h).
\end{array} \quad (2)$$

By BCFE of χ,

$$\chi_{k,l}(S, d_S, h^{\chi}_{S,d}) = \chi_{k,l}(L,d,h) \ \ \forall \ \ (m,n) \in A^S. \quad (3)$$

By (2) and (3), $h(d) = \sum\limits_{(k,l)\in A^L} \chi_{k,l}(L,d,h)$, i.e., χ matches COM. The proofs of outcomes 2, 3 are similar, we omit it. ■

Inspired by the axiomatic outcomes due to Hart and Mas-Colell [3], we axiomatize the RANSC, LANSC and UANSC by means of RRTAS, LRTAS, URTAS and BCFE.

Theorem 1:

1) A principle χ on Ω matches RRTAS and BCFE if and only if $\chi = \overline{\tau}$.
2) A principle χ on Ω matches LRTAS and BCFE if and only if $\chi = \overline{\alpha}$.
3) A principle χ on Ω matches URTAS and BCFE if and only if $\chi = \overline{\gamma}$.

Proof: Based on Lemma 1, $\overline{\tau}$, $\overline{\alpha}$, $\overline{\gamma}$ match RRTAS, LRTAS and URTAS respectively. By Lemma 2, $\overline{\tau}, \overline{\alpha}, \overline{\gamma}$ match BCFE.

To present the uniqueness of outcome 1, assume χ matches RRTAS and BCFE. Since χ matches RRTAS and BCFE, χ matches COM by Lemma 3. Let $(L,d,h) \in \Omega$. If $|L| \leq 2$, then by RRTAS of χ, $\chi(L,d,h) = \overline{\tau}(L,d,h)$. The case $|L| > 2$: For every $(m,n),(k,l) \in A^L$ satisfying $m \neq k$, let $S = \{m,k\}$, one can derive that

$$\begin{array}{rl}
& \chi_{m,n}(L,d,h) - \chi_{k,l}(L,d,h) \\
= & \chi_{m,n}(S, d_S, h^{\chi}_{S,d}) - \chi_{k,l}(S, d_S, h^{\chi}_{S,d}) \\
& \textbf{(by BCFE of } \chi\textbf{)} \\
= & \overline{\tau}_{m,n}(S, d_S, h^{\chi}_{S,d}) - \overline{\tau}_{k,l}(S, d_S, h^{\chi}_{S,d}) \\
& \textbf{(by RRTAS of } \chi\textbf{)} \\
= & \tau_{m,n}(S, d_S, h^{\chi}_{S,d}) - \tau_{k,l}(S, d_S, h^{\chi}_{S,d}) \\
= & \Big[h^{\chi}_{S,d}(d_k, n) - h^{\chi}_{S,d}(d_k, n-1) \Big] \\
& - \big[h^{\chi}_{S,d}(d_m, l) - h^{\chi}_{S,d}(d_m, l-1) \big] \\
= & \Big[h(d_{L\setminus\{m\}}, n) - h(d_{L\setminus\{m\}}, n-1) \Big] \\
& - \Big[h(d_{L\setminus\{k\}}, l) - h(d_{L\setminus\{k\}}, l-1) \Big]
\end{array} \quad (4)$$

Similarly, $\overline{\tau}$ instead of χ in (4), one can have that

$$\begin{array}{rl}
& \overline{\tau}_{m,n}(L,d,h) - \overline{\tau}_{k,l}(L,d,h) \\
= & \Big[h(d_{L\setminus\{m\}}, n) - h(d_{L\setminus\{m\}}, n-1) \Big] \\
& - \Big[h(d_{L\setminus\{k\}}, l) - h(d_{L\setminus\{k\}}, l-1) \Big]
\end{array} \quad (5)$$

By (4) and (5),

$$\chi_{m,n}(L,d,h) - \chi_{k,l}(L,d,h) = \overline{\tau}_{m,n}(L,d,h) - \overline{\tau}_{k,l}(L,d,h).$$

This implies that $\chi_{m,n}(L,d,h) - \overline{\tau}_{m,n}(L,d,h) = c$ for every (m,n). It remains to prove that $c = 0$. By COM of χ and $\overline{\tau}$,

$$0 = \sum_{m\in L} \sum_{n=1}^{d_m} [\chi_{m,n}(L,d,h) - \overline{\tau}_{m,n}(L,d,h)] = \|d\| \cdot c.$$

Hence, $c = 0$. By similar processes, the uniqueness of outcomes 2 and 3 could be finished. ■

In the following we present that every of the properties applied in Theorem 1 is independent of the rest of properties.

Example 2: For every $(L,d,h) \in \Omega$ and for every $(m,n) \in A^L$, we define the principle χ by

$$\chi_{m,n}(L,d,h) = \begin{cases} \overline{\tau}_{m,n}(L,d,h) & \text{if } |L| \leq 2, \\ 0 & \text{o.w.} \end{cases}$$

χ matches RRTAS, but it infringes BCFE.

Example 3: For every $(L,d,h) \in \Omega$ and for every $(m,n) \in A^L$, we define the principle χ by

$$\chi_{m,n}(L,d,h) = \begin{cases} \overline{\alpha}_{m,n}(L,d,h) & \text{if } |L| \leq 2, \\ 0 & \text{o.w.} \end{cases}$$

χ matches LRTAS, but it infringes BCFE.

Example 4: For every $(L,d,h) \in \Omega$ and for every $(m,n) \in A^L$, we define the principle χ by

$$\chi_{m,n}(L,d,h) = \begin{cases} \overline{\gamma}_{m,n}(L,d,h) & \text{if } |L| \leq 2, \\ 0 & \text{o.w.} \end{cases}$$

χ matches URTAS, but it infringes BCFE.

Example 5: We define the principle χ by $\chi_{m,n}(L,d,h) = \frac{h(d)}{\|d\|}$ for every $(L,d,h) \in \Omega$ and for every $(m,n) \in A^L$. χ matches BCFE, but it infringes RRTAS, LRTAS and URTAS.

IV. Weighted generalizations

As mentioned in Section 1, weights turn up naturally under the context of usability distribution. For instance, one might be dealing with a situation of usability distribution among investment plans. Then the weights could be appointed to the profitability of different options of all plans. Weights are also contained in contracts concluded by the holders of a condominium and adopted to portion the cost of maintaining or building common facilities. Another application is patent or data pooling among companies where the scale of the companies, evaluated for example by its market shares, might be natural weights. Therefore, it is reasonable that weights could be appointed to the "grades" of participators to differentiate the difference. If $w : D^U \to \mathbb{R}^+$ be a positive map, then w is said to be a **weight map for grades**. By the weight map for grades, several weighted generalizations could be defined.

Definition 3: The **weighted regular allocation of non-separable costs (WRANSC)** on Ω, $\overline{\tau^w}$, is the map on Ω which assigns to every $(L,d,h) \in \Omega$, every $m \in L$ and every $n \in D_m^+$ the amount

$$\begin{aligned} & \overline{\tau^w}_{m,n}(L,d,h) \\ = \; & \tau_{m,n}(L,d,h) + \tfrac{w(n)}{\|d\|_w}\big[h(d) - \sum_{(k,l)\in A^L} \tau_{k,l}(L,d,h)\big], \end{aligned}$$

where $\|d\|_w = \sum_{i=1}^{N} \sum_{j=1}^{d_m} w(n)$. Based on definition of $\overline{\tau^w}$, all participators get its regular grade-marginal distinctions firstly, and further distribute the rest of usability proportionally by weights.

The **weighted lower-aggregated allocation of non-separable costs (WLANSC)** on Ω, $\overline{\alpha^w}$, is the map on Ω which assigns to every $(L,d,h) \in \Omega$, every $m \in L$ and every $n \in D_m^+$ the amount

$$\begin{aligned} & \overline{\alpha^w}_{m,n}(L,d,h) \\ = \; & \alpha_{m,n}(L,d,h) + \tfrac{w(n)}{\|d\|_w}\big[h(d) - \sum_{(k,l)\in A^L} \alpha_{k,l}(L,d,h)\big]. \end{aligned}$$

By definition of $\overline{\alpha^w}$, all participators get its lower-aggregated grade-marginal distinctions firstly, and further distribute the rest of usability proportionally by weights.

The **weighted upper-aggregated allocation of non-separable costs (WUANSC)** on Ω, $\overline{\gamma^w}$, is the map on Ω which assigns to every $(L,d,h) \in \Omega$, every $m \in L$ and every $n \in D_m^+$ the amount

$$\begin{aligned} & \overline{\gamma^w}_{m,n}(L,d,h) \\ = \; & \gamma_{m,n}(L,d,h) + \tfrac{w(n)}{\|d\|_w}\big[h(d) - \sum_{(k,l)\in A^L} \gamma_{k,l}(L,d,h)\big]. \end{aligned}$$

By definition of $\overline{\gamma^w}$, all participators get its upper-aggregated grade-marginal distinctions firstly, and further distribute the rest of usability proportionally by weights.

Similar to Theorem 1, several axiomatic outcomes of $\overline{\tau^w}$, $\overline{\alpha^w}$ and $\overline{\gamma^w}$ could be provided as follows. A principle χ matches **weighted regular rule for two-agent systems (WRRTAS)** if for all $(L,d,h) \in \Omega$ with $|L| \leq 2$, $\chi(L,d,h) = \overline{\tau^w}(L,d,h)$. A principle χ matches **weighted lower-aggregated rule for two-agent systems (WLRTAS)** if for all $(L,d,h) \in \Omega$ with $|L| \leq 2$, $\chi(L,d,h) = \overline{\alpha^w}(L,d,h)$. A principle χ matches **weighted upper-aggregated rule for two-agent systems (WURTAS)** if for all $(L,d,h) \in \Omega$ with $|L| \leq 2$, $\chi(L,d,h) = \overline{\gamma^w}(L,d,h)$.

Lemma 4:

1) The principles $\overline{\tau^w}$ matches COM and WRRTAS.
2) The principles $\overline{\alpha^w}$ matches COM and WLRTAS.
3) The principles $\overline{\gamma^w}$ matches COM and WURTAS.

Proof: The proof is similar to Lemma 1, we omit it. ■

Lemma 5: The principles $\overline{\tau^w}$, $\overline{\alpha^w}$ and $\overline{\gamma^w}$ match BCFE.

Proof: Given $(L,d,h) \in \Omega$ and $S \subseteq L$ with $|S| = 2$. For all $(m,n) \in A^S$,

$$\begin{aligned} & \overline{\tau^w}_{m,n}(S, m_S, v_{S,m}^{\overline{\tau^w}}) \\ = \; & \tau_{m,n}(S, m_S, v_{S,m}^{\overline{\tau^w}}) + \tfrac{w(i)}{\|d_S\|_w} \cdot \Big[v_{S,m}^{\overline{\tau^w}}(m_S) \\ & \qquad - \sum_{(k,l)\in A^S} \tau_{k,l}(S, m_S, v_{S,m}^{\overline{\tau^w}}) \Big] \\ = \; & \tau_{m,n}(L,d,h) + \tfrac{w(i)}{\|d_S\|_w} \cdot \Big[v_S^{\overline{\tau^w}}(m_S) \\ & \qquad - \sum_{(k,l)\in A^S} \tau_{k,l}(L,d,h) \Big] \\ & \textbf{(by Equation (1))} \\ = \; & \tau_{m,n}(L,d,h) + \tfrac{w(i)}{\|d_S\|_w} \cdot \Big[h(d) \\ & - \sum_{(k,l)\in A^{L\setminus S}} \overline{\tau^w}_{k,l}(L,d,h) - \sum_{(k,l)\in A^S} \tau_{k,l}(L,d,h) \Big] \\ = \; & \tau_{m,n}(L,d,h) + \tfrac{w(i)}{\|d_S\|_w} \cdot \Big[\sum_{(k,l)\in A^S} \overline{\tau^w}_{k,l}(L,d,h) \\ & \qquad - \sum_{(k,l)\in A^S} \tau_{k,l}(L,d,h) \Big] \\ & \textbf{(by COM of } \overline{\tau^w}\textbf{)} \\ = \; & \tau_{m,n}(L,d,h) + \tfrac{w(i)}{\|d_S\|_w} \cdot \Big[\sum_{(k,l)\in A^S} \tau_{k,l}(L,d,h) \\ & + \tfrac{\|d_S\|_w}{\|d\|_w} \cdot \big[h(d) - \sum_{(k,l)\in A^L} \tau_{k,l}(L,d,h) \big] \\ & \qquad - \sum_{(k,l)\in A^S} \tau_{k,l}(L,d,h) \Big] \\ = \; & \tau_{m,n}(L,d,h) + \tfrac{w(i)}{\|d\|_w} \cdot \Big[h(d) - \sum_{(k,l)\in A^L} \tau_{k,l}(L,d,h) \Big] \\ = \; & \overline{\tau^w}_{m,n}(L,d,h). \end{aligned}$$

Thus, $\overline{\tau^w}$ matches BCFE. By similar processes, it is easy to show that $\overline{\alpha^w}$ and $\overline{\gamma^w}$ match BCFE. ■

Lemma 6:

1) If a principle χ matches WRRTAS and BCFE, then it matches COM.
2) If a principle χ matches WLRTAS and BCFE, then it matches COM.

3) If a principle χ matches WURTAS and BCFE, then it matches COM.

Proof: Here we prove outcome 1 firstly. Suppose χ matches WRRTAS and BCFE. Let $(L,d,h)\in\Omega$. If $|L|\leq 2$, then χ matches COM by WRRTAS of χ. Suppose $|L|>2$, $i,n\in L$ and $S=\{m,n\}$. Since χ matches COM in two-agent systems,

$$\begin{aligned}
& \sum_{(k,l)\in A^S}\chi_{k,l}(S,m_S,h^{\chi}_{S,d}) \\
= \;& h^{\chi}_{S,d}(m) \\
= \;& h(d)-\sum_{(k,l)\in A^{N\setminus\{m,n\}}}\chi_{k,l}(L,d,h).
\end{aligned}\tag{6}$$

By BCFE of χ,

$$\chi_{k,l}(S,m_S,h^{\chi}_{S,d})=\chi_{k,l}(L,d,h)\;\;\forall\;\;(m,n)\in A^S.\tag{7}$$

By (6) and (7), $h(d)=\sum_{(k,l)\in A^L}\chi_{k,l}(L,d,h)$, i.e., χ matches COM. The proofs of outcomes 2, 3 are similar, we omit it. ■

Theorem 2:

1) A principle χ on Ω matches WRRTAS and BCFE if and only if $\chi=\overline{\tau^w}$.
2) A principle χ on Ω matches WLRTAS and BCFE if and only if $\chi=\overline{\alpha^w}$.
3) A principle χ on Ω matches WURTAS and BCFE if and only if $\chi=\overline{\gamma^w}$.

Proof: Based on Lemma 5, the principles $\overline{\tau^w}$, $\overline{\alpha^w}$ and $\overline{\gamma^w}$ also match BCFE simultaneously. By Lemma 4, $\overline{\tau^w}$, $\overline{\alpha^w}$ and $\overline{\gamma^w}$ match WRRTAS, WLRTAS and WURTAS respectively.

To present the uniqueness of outcome 1, suppose χ matches WRRTAS and BCFE. Based on Lemma 6, χ also matches COM. Let $(L,d,h)\in\Omega$ and w be weight map for grades. By WRRTAS of χ, $\chi(L,d,h)=\overline{\tau^w}(L,d,h)$ if $|L|\leq 2$. The condition $|L|>2$: Let $(m,n)\in A^L$ and $S=\{m,k\}$ for $k\in L\setminus\{m\}$.

$$\begin{aligned}
& \chi_{m,n}(L,d,h)-\overline{\tau^w}_{m,n}(L,d,h) \\
= \;& \chi_{m,n}(S,d_S,h^{\chi}_{S,d})-\overline{\tau^w}_{m,n}(S,d_S,h^{\overline{\tau^w}}_{S,d}) \\
& \textbf{(by BCFE of } \overline{\tau^w} \textbf{ and } \chi\textbf{)} \\
= \;& \overline{\tau^w}_{m,n}(S,d_S,h^{\chi}_{S,d})-\overline{\tau^w}_{m,n}(S,d_S,h^{\overline{\tau^w}}_{S,m}). \\
& \textbf{(by WRRTAS of } \chi\textbf{)}
\end{aligned}\tag{8}$$

Similar to equation (1)

$$\tau_{m,n}(S,d_S,h^{\chi}_{S,d})=\tau_{m,n}(L,d,h)=\tau_{m,n}(S,d_S,h^{\overline{\tau^w}}_{S,d}).\tag{9}$$

By (8) and (9),

$$\begin{aligned}
& \chi_{m,n}(L,d,h)-\overline{\tau^w}_{m,n}(L,d,h) \\
= \;& \chi_{m,n}(S,d_S,h^{\chi}_{S,d})-\overline{\tau^w}_{m,n}(S,d_S,h^{\overline{\tau^w}}_{S,d}) \\
= \;& \tfrac{w(n)}{\|d_S\|_w}\cdot\big[h^{\chi}_{S,d}(d_S)-h^{\overline{\tau^w}}_{S,d}(d_S)\big].
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \chi_{k,l}(L,d,h)-\overline{\tau^w}_{k,l}(L,d,h) \\
= \;& \tfrac{w(l)}{\|d_S\|_w}\cdot\big[h^{\chi}_{S,d}(d_S)-h^{\overline{\tau^w}}_{S,d}(d_S)\big].
\end{aligned}$$

Thus,

$$\begin{aligned}
& w(l)\cdot\big[\chi_{m,n}(L,d,h)-\overline{\tau^w}_{m,n}(L,d,h)\big] \\
= \;& w(n)\cdot\big[\chi_{k,l}(L,d,h)-\overline{\tau^w}_{k,l}(L,d,h)\big].
\end{aligned}$$

By COM of $\overline{\tau^w}$ and χ,

$$\begin{aligned}
& \big[\chi_{m,n}(L,d,h)-\overline{\tau^w}_{m,n}(L,d,h)\big]\cdot\sum_{(k,l)\in A^L}w(l) \\
= \;& w(n)\cdot\sum_{(k,l)\in A^L}\big[\chi_{k,l}(L,d,h)-\overline{\tau^w}_{k,l}(L,d,h)\big] \\
= \;& w(n)\cdot\big[h(d)-h(d)\big] \\
= \;& 0.
\end{aligned}$$

Hence, $\chi_{m,n}(L,d,h)=\overline{\tau^w}_{m,n}(L,d,h)$ for all $(m,n)\in A^L$. The proofs of outcomes 2, 3 are similar, we omit it. ■

V. Other Generalizations and Revised Conformance

In Section 4, several weighted generalizations are defined by the weight map for grades. However, the weight map for grades are assigned artificially. It is reasonable that the weights could be replaced by marginal distinctions naturally.

"Marginal distinctions" instead of "weights", three generalizations are defined as follows.

Definition 4: The **interior regular allocation of non-separable costs (IRANSC)** on Ω, $\overline{\tau^I}$, is the map on Ω which assigns to every $(L,d,h)\in\Omega$, every $m\in L$ and every $n\in D^+_m$ the amount

$$\begin{aligned}
& \overline{\tau^I}_{m,n}(L,d,h) \\
= \;& \tau_{m,n}(L,d,h)+\tfrac{\tau_{m,n}(L,d,h)}{\|d\|_\tau}\big[h(d)-\|d\|_\tau\big],
\end{aligned}$$

where $\|d\|_\tau=\sum_{(k,l)\in A^L}\tau_{k,l}(L,d,h)$. By definition of $\overline{\tau^I}$, all participators get its regular grade-marginal distinctions, and further distribute the rest of usability proportionally by regular grade-marginal distinctions.

The **interior lower-aggregated allocation of non-separable costs (ILANSC)** on Ω, $\overline{\alpha^I}$, is the map on Ω which assigns to every $(L,d,h)\in\Omega$, every $m\in L$ and every $n\in D^+_m$ the amount

$$\begin{aligned}
& \overline{\alpha^I}_{m,n}(L,d,h) \\
= \;& \alpha_{m,n}(L,d,h)+\tfrac{\alpha_{m,n}(L,d,h)}{\|d\|_\alpha}\big[h(d)-\|d\|_\alpha\big],
\end{aligned}$$

where $\|d\|_\alpha=\sum_{(k,l)\in A^L}\alpha_{k,l}(L,d,h)$. By definition of $\overline{\alpha^I}$, all participators get its lower-aggregated grade-marginal distinctions, and further distribute the rest of usability proportionally by lower-aggregated grade-marginal distinctions.

The **interior upper-aggregated allocation of non-separable costs (IUANSC)** on Ω, $\overline{\gamma^I}$, is the map on Ω which assigns to every $(L,d,h)\in\Omega$, every $m\in L$ and every $n\in D^+_m$ the amount

$$\begin{aligned}
& \overline{\gamma^I}_{m,n}(L,d,h) \\
= \;& \gamma_{m,n}(L,d,h)+\tfrac{\gamma_{m,n}(L,d,h)}{\|d\|_\gamma}\big[h(d)-\|d\|_\gamma\big],
\end{aligned}$$

where $\|d\|_\gamma=\sum_{(k,l)\in A^L}\gamma_{k,l}(L,d,h)$. By definition of $\overline{\gamma^I}$, all participators get its upper-aggregated grade-marginal distinctions, and further distribute the rest of usability proportionally by upper-aggregated grade-marginal distinctions.

Similar to Theorems 1 and 2, several axiomatic outcomes of $\overline{\tau^I}$, $\overline{\alpha^I}$ and $\overline{\gamma^I}$ could be provided also. A principle χ matches **interior regular rule for two-agent systems (IRRTAS)** if for all $(L,d,h)\in\Omega$ satisfying $|L|\leq 2$,

$\chi(L,d,h) = \overline{\tau^I}(L,d,h)$. A principle χ matches **interior lower-aggregated rule for two-agent systems (ILRTAS)** if for all $(L,d,h) \in \Omega$ satisfying $|L| \leq 2$, $\chi(L,d,h) = \overline{\alpha^I}(L,d,h)$. A principle χ matches **interior upper-aggregated rule for two-agent systems (IURTAS)** if for all $(L,d,h) \in \Omega$ satisfying $|L| \leq 2$, $\chi(L,d,h) = \overline{\gamma^I}(L,d,h)$.

It is trivial to verify that $\sum_{(k,l)\in A^S} \tau_k(L,d,h) = 0$ (or $\sum_{(k,l)\in A^S} \alpha_k(L,d,h) = 0$, $\sum_{(k,l)\in A^S} \gamma_k(L,d,h) = 0$) for some $(L,d,h) \in \Omega$ and for some $S \subseteq L$, i.e., $\overline{\tau^I}(S,d_S,h^{\overline{\tau^I}}_{S,d})$ (or $\overline{\alpha^I}(S,d_S,h^{\overline{\alpha^I}}_{S,d})$, $\overline{\gamma^I}(S,d_S,h^{\overline{\gamma^I}}_{S,d})$) doesn't exist for some $(L,d,h) \in \Omega$ and for some $S \subseteq L$. So, we focus on the *revised conformance* as follows. A principle χ matches **revised-conformance (RCFE)** if $(S,d_S,h^{\chi}_{S,d})$ and $\chi(S,d_S,h^{\chi}_{S,d})$ exist for some $(L,d,h) \in \Omega$ and for some $S \subseteq L$, it holds that $\chi_{m,n}(S,d_S,h^{\chi}_{S,d}) = \chi_{m,n}(L,d,h)$ for all $(m,n) \in A^S$

Lemma 7:

1) The principles $\overline{\tau^I}$ matches COM and IRRTAS.
2) The principles $\overline{\alpha^I}$ matches COM and ILRTAS.
3) The principles $\overline{\gamma^I}$ matches COM and IURTAS.

Proof: Similar to Lemma 1, this lemma could be finished. ■

Lemma 8: The principles $\overline{\tau^I}$, $\overline{\alpha^I}$ and $\overline{\gamma^I}$ match RCFE.

Proof: Given $(L,d,h) \in \Omega$ and $S \subseteq L$ satisfying $|S| = 2$. For every $(m,n) \in A^S$,

$$
\begin{aligned}
& \overline{\tau^I}_{m,n}(S,d_S,h^{\overline{\tau^I}}_{S,d}) \\
= \;& \tau_{m,n}(S,d_S,h^{\overline{\tau^I}}_{S,d}) + \frac{\tau_{m,n}(S,d_S,h^{\overline{\tau^I}}_{S,d})}{\sum\limits_{(k,l)\in A^S} \tau_{k,l}(S,d_S,h^{\overline{\tau^I}}_{S,d})} \cdot \Big[\\
& \qquad h^{\overline{\tau^I}}_{S,d}(d_S) - \sum_{(k,l)\in A^S} \tau_{k,l}(S,d_S,h^{\overline{\tau^I}}_{S,d})\Big] \\
= \;& \tau_{m,n}(L,d,h) + \frac{\tau_{m,n}(L,d,h)}{\sum\limits_{(k,l)\in A^S} \tau_{k,l}(L,d,h)} \cdot \Big[h^{\overline{\tau^I}}_{S,d}(d_S) \\
& \qquad - \sum_{(k,l)\in A^S} \tau_{k,l}(L,d,h)\Big] \\
& \textbf{(by Equation (1))} \\
= \;& \tau_{m,n}(L,d,h) + \frac{\tau_{m,n}(L,d,h)}{\sum\limits_{(k,l)\in A^S} \tau_{k,l}(L,d,h)} \cdot \Big[h(d) \\
& - \sum_{(k,l)\in A^{L\setminus S}} \overline{\tau^I}_{k,l}(L,d,h) - \sum_{(k,l)\in A^S} \tau_{k,l}(L,d,h)\Big] \\
= \;& \tau_{m,n}(L,d,h) + \frac{\tau_{m,n}(L,d,h)}{\sum\limits_{(k,l)\in A^S} \tau_{k,l}(L,d,h)} \cdot \Big[\\
& \sum_{(k,l)\in A^S} \overline{\tau^I}_{k,l}(L,d,h) - \sum_{(k,l)\in A^S} \tau_{k,l}(L,d,h)\Big] \\
& \textbf{(by COM of } \overline{\tau^I}\textbf{)} \\
= \;& \tau_{m,n}(L,d,h) + \frac{\tau_{m,n}(L,d,h)}{\sum\limits_{(k,l)\in A^S} \tau_{k,l}(L,d,h)} \cdot \Big[\\
& \frac{\sum\limits_{(k,l)\in A^S} \tau_{k,l}(L,d,h)}{\sum\limits_{(k,l)\in A^L} \tau_{k,l}(L,d,h)} \cdot \big[h(d) - \sum_{(k,l)\in A^L} \tau_{k,l}(L,d,h)\big]\Big] \\
= \;& \tau_{m,n}(L,d,h) + \frac{\tau_{m,n}(L,d,h)}{\sum\limits_{(k,l)\in A^L} \tau_{k,l}(L,d,h)} \cdot \Big[h(d) \\
& \qquad - \sum_{(k,l)\in A^L} \tau_{k,l}(L,d,h)\Big] \\
= \;& \overline{\tau^I}_{m,n}(L,d,h).
\end{aligned}
$$

Thus, $\overline{\tau^I}$ matches RCFE. By similar processes, it is easy to show that $\overline{\alpha^I}$ and $\overline{\gamma^I}$ match RCFE. ■

Lemma 9:

1) If a principle χ matches IRRTAS and RCFE, then it matches COM.
2) If a principle χ matches ILRTAS and RCFE, then it matches COM.
3) If a principle χ matches IURTAS and RCFE, then it matches COM.

Proof: Similar to Lemma 6, this lemma could be finished. ■

Theorem 3:

1) A principle χ on Ω matches IRRTAS and RCFE if and only if $\chi = \overline{\tau^I}$.
2) A principle χ on Ω matches ILRTAS and RCFE if and only if $\chi = \overline{\alpha^I}$.
3) A principle χ on Ω matches IURTAS and RCFE if and only if $\chi = \overline{\gamma^I}$.

Proof: Similar to Theorem 2, this lemma could finished. ■

VI. Conclusions

1) Here we analyze several generalizations of the equal allocation of non-separable costs. By adopting reduced system, axiomatic outcomes for these generalizations are proposed. By adopting weights and different types of marginal distinctions respectively, some more generalizations and related characterizations are also introduced. One could compare our outcomes with related pre-existing outcomes:
   - These generalizations are initially introduced throughout the contexts of traditional systems and multi-choice systems.
   - The major difference is that these generalizations proposed in this research are considered by simultaneously applying the participator and its operational grades, and the pre-existing generalizations of the EANSC are considered by respectively determining overall amounts for a given participators under different situations.
2) These mentioned above generalize one motivation:
   - Whether some more principles, its extensions and related outcomes could be investigated in the context of multi-choice systems.

   This is left to the researches.

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