Picture Hesitant Fuzzy Normalized Weighted Bonferroni Mean Operator based on Einstein Operations and Its Application

Lihua Yang and Baolin Li

Abstract—**In order to better convey complex decision-making information, we introduce the definition of picture hesitant fuzzy set (PHFS), which is based on the combination of picture fuzzy set (PFS) and hesitant fuzzy set (HFS). The Bonferroni mean (BM) operator has the prominent characteristic of reflecting the correlations between different attributes, which has attracted many attentions in recent years. Nevertheless, the BM operator and its variations cannot cope with picture hesitant fuzzy information. Meanwhile, the existing aggregation operators for picture hesitant fuzzy elements (PHFEs) are all based on Algebraic operations. Motivated by these, we extend the traditional normalized weighted Bonferroni mean (NWBM) operator based on Einstein operations to deal with PHFS in this paper. Firstly, some operational rules of PHFEs based on Einstein operations are defined, and comparative method is also presented. Secondly, the picture hesitant fuzzy normalized weighted Bonferroni mean (PHFNWBM) operator is proposed, and some desirable properties are discussed as well. Finally, based on the proposed operator, an illustrative example of multi-attribute decision-making (MADM) with picture hesitant fuzzy information is given, along with the sensitivity analysis and comparative analysis. The results demonstrate the effectiveness and practicality of the developed method.**

Index Terms—**multi-attribute decision-making, Einstein, NWBM operator, picture hesitant fuzzy set**

I. INTRODUCTION

IN real life, it is commonly hard for decision-makers to convey assessment information by real values. Thus, Zadeh [1] proposed the concept of fuzzy set (FS), which employs a membership function to describe fuzzy information. To overcome the weak of FS, Atanassov [2] defined intuitionistic fuzzy set (IFS), which adds a non-membership function to express uncertain information. In some cases, people may be hesitant and give several different values to depict the membership degree. Thus, Torra [3] introduced the notion of hesitant fuzzy set (HFS). Although a series of achievements regarding FS, IFS, and HFS have been made, it cannot settle some complicated situations in real decision-making problems. Therefore, as an extended form of IFS, Cuong [4] developed the concept of picture fuzzy set (PFS), where an element of the set contains three membership degrees, namely, positive, neutral, and negative. Meanwhile, the sum of all degrees cannot exceed one. PFS can reflect four types of answers of decision-maker, namely, positive, neutral, negative, and refusal, which cannot be expressed by IFS accurately. Thus, the PFS is more practical on dealing with uncertain information than IFS, and many researches relating to PFS have been undertaken in MADM [5-8].

Although the theory of fuzzy set has been developed and generalized, it cannot deal with all types of uncertainties in different real world problems. Hence, motivated by the advantages of PFS and HFS, Wang [9] proposed the theory of picture hesitant fuzzy set (PHFS) to convey complex MADM cognitive information, and applied aggregating operators to fuse picture hesitant fuzzy elements. Furthermore, Yang [10] applied picture hesitant fuzzy information to evaluate end-of-life management alternatives. Jan [11] developed some generalized distance and similarity measures for PHFS. Yang and Li [12] proposed the concept of multiple-valued picture fuzzy linguistic set by merging picture hesitant fuzzy set and linguistic set.

Aggregating operator is an effective tool to fuse cognitive information in MADM problems. Thus, different aggregation operators have been introduced and developed [13-16]. Bonferroni mean (BM) operator was firstly proposed by Bonferroni [17], which can not only reflect the importance of input attribute but also capture the interrelationships between different attributes. Therefore, BM operator has become a research focus. Beliakov et al. [18] defined the generalized Bonferroni mean (GBM) operator. However, the typical BM and GBM do not consider the weight vector of input values. Xu et al. [19] extended the BM operator to cope with intuitionistic fuzzy elements, and then the intuitionistic fuzzy weighted Bonferronia mean (IFWBM) operator reflecting different weight values of all attributes was defined. Nevertheless, the IFWBM operator does not have the property of reducibility. Subsequently, Xia et al. [20] put forward the generalized weighted Bonferroni mean (GWBM) operator. However, the GWBM operator cannot

Manuscript received May 9, 2020; revised July 27, 2020. This work was supported in part by Hubei Technology Innovation Soft Science Research Project under Grant 2019ADC038, and Philosophy and Social Science Research Project of Hubei Provincial Department of Education under Grant 19D066.

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represent the correlations between the individual attribute and the other attributes. Motivated by these issues, Zhou et al. [21] proposed the normalized weighted Bonferronia mean (NWBM) operator and the generalized NWBM operator. BM operator and its extended aggregation operators have been applied to handle different types of fuzzy information, such as, IFS [20], SVN [22], SNLS [23], and MVNLS [24]. Nevertheless, so far, NWBM operator has been unable to aggregate picture hesitant fuzzy elements (PHFEs).

We know different aggregating operators depend on different t-norm (TN) and t-conorm (TCN), for example, Algebraic operations [25], Einstein operations [26], Hamacher operations [27], Dombi operations [28], and so on. At present, there are a few researches concerning aggregating operators under PHFS environment, and the existing aggregation operators for PHFEs are all based on the Algebraic operations. Thus, it is meaningful to explore aggregating operator based on Einstein operations.

Up to now, the existing MADM methods for PHFEs ignore the interrelationships of input arguments, and NWBM operator can solve this issue. Meanwhile, the existing operators fusing PHFEs cannot be calculated based on Einstein operations. Therefore, the main purpose of the paper is to extend NWBM operator based on Einstein operations to deal with PHFEs, that is, the picture hesitant fuzzy normalized weighted Bonferroni mean (PHFNWBM) operator based on Einstein is put forward. Furthermore, some desirable properties of the proposed operator are also discussed.

To do so, the structure of the paper is arranged as follows. In section II, we give the definition of PHFS, PHFE, comparative method and operational rules based on Einstein operations. In section III, we propose the concept of PHFNWBM operator, and discuss its promising properties. In section IV, an illustrative example of MADM approach depending on the proposed operator is established, together with sensitivity analysis and comparative analysis. In section V, the conclusions are summarized.

II. PHFS AND EINSTEIN OPERATIONS

In this section, we introduce the concepts of PHFS and PHFE. Meanwhile, the comparative method and the operational rules based on Einstein operations are also proposed.

A. PHFS and Comparative Method

Definition 1 [9]. Let X be a set of points, with a generic element in X denoted by x, an PHFS A in X is characterized by:

$$A = \left\{ \left\langle x, \tilde{\mu}_A(x), \tilde{\eta}_A(x), \tilde{\nu}_A(x) \right\rangle \middle| x \in X \right\},$$

where

$\tilde{\mu}_A(x) = \left\{ \mu \middle| \mu \in \tilde{\mu}_A(x) \right\}$, $\tilde{\eta}_A(x) = \left\{ \eta \middle| \eta \in \tilde{\eta}_A(x) \right\}$, $\tilde{\nu}_A(x) = \left\{ \nu \middle| \nu \in \tilde{\nu}_A(x) \right\}$, $\tilde{\mu}_A(x)$, $\tilde{\eta}_A(x)$ and $\tilde{\nu}_A(x)$ are three sets of several values in $[0,1]$, representing the positive, neutral, and negative membership degrees of x in X, satisfying $0 \le \mu, \eta, \nu \le 1$, $0 \le \mu^+ + \eta^+ + \nu^+ \le 1$, $\mu^+ = \max\{\mu\}$, $\eta^+ = \max\{\eta\}$, and $\nu^+ = \max\{\nu\}$.

If there is only one element in X, then $A = \left\langle \tilde{\mu}_A, \tilde{\eta}_A, \tilde{\nu}_A \right\rangle$ is called a picture hesitant fuzzy element (PHFE).

Definition 2. Let $A = \left\langle \tilde{\mu}_A, \tilde{\eta}_A, \tilde{\nu}_A \right\rangle$ be an PHFE, the score and accuracy functions are defined below.

$$(1) s(A) = \frac{1}{3}\left(\frac{1}{\ell_1} \sum_{\mu \in \tilde{\mu}_A} \mu + \frac{1}{\ell_2} \sum_{\eta \in \tilde{\eta}_A} (1-\eta) + \frac{1}{\ell_3} \sum_{\gamma \in \tilde{\gamma}_A} (1-\gamma) \right)$$

$$(2) a(A) = \frac{1}{3}\left(\frac{1}{\ell_1} \sum_{\mu \in \tilde{\mu}_A} \mu + \frac{1}{\ell_2} \sum_{\eta \in \tilde{\eta}_A} \eta + \frac{1}{\ell_3} \sum_{\gamma \in \tilde{\gamma}_A} \gamma \right)$$

Where ℓ_1, ℓ_2, and ℓ_3 are the numbers in $\tilde{\mu}_A, \tilde{\eta}_A$ and $\tilde{\nu}_A$, respectively.

For an PHFE A, if the positive membership degree $\tilde{\mu}_A$ is bigger, the neutral membership degree $\tilde{\eta}_A$ and the neutral membership degree $\tilde{\eta}_A$ are smaller, then the PHFE is higher.

According to the score function and accuracy function in Definition 2, we can rank the PHFEs, and the comparative method is provided as follows.

Definition 3. Let A_1 and A_2 be two PHFEs, and then the comparative method are defined below.

(1) If $s(A_1) > s(A_2)$, then $A_1 > A_2$;

(2) If $s(A_1) = s(A_2)$, and $a(A_1) > a(A_2)$, then $A_1 > A_2$;

(3) If $s(A_1) = s(A_2)$, and $a(A_1) = a(A_2)$, then $A_1 = A_2$;

B. Einstein Operations

We know Einstein operations contain Einstein TN and Einstein TCN, namely,

$$TN(x, y) = \frac{xy}{1 + (1-x)(1-y)} \tag{1}$$

$$TCN(x, y) = \frac{x + y}{1 + xy} \tag{2}$$

Then, the operational laws of PHFEs based on Einstein operations are defined in the following.

Definition 4.

Let $A = \left\langle \tilde{\mu}_A, \tilde{\eta}_A, \tilde{\nu}_A \right\rangle$ and $B = \left\langle \tilde{\mu}_B, \tilde{\eta}_B, \tilde{\nu}_B \right\rangle$ be two PHFEs, and $\lambda > 0$, the operations of PHFEs based on Einstein operations are represented as follows.

$$(1)\ A \oplus B = \left\langle \bigcup_{\mu_A \in \tilde{\mu}_A, \mu_B \in \tilde{\mu}_B} \left\{ \frac{\mu_A + \mu_B}{1 + \mu_A \mu_B} \right\}, \bigcup_{\eta_A \in \tilde{\eta}_A, \eta_B \in \tilde{\eta}_B} \left\{ \frac{\eta_A \eta_B}{1 + (1-\eta_A)(1-\eta_B)} \right\}, \bigcup_{\nu_A \in \tilde{\nu}_A, \nu_B \in \tilde{\nu}_B} \left\{ \frac{\nu_A \nu_B}{1 + (1-\nu_A)(1-\nu_B)} \right\} \right\rangle;$$

$$(2)\ A \otimes B = \left\langle \bigcup_{\mu_A \in \tilde{\mu}_A, \mu_B \in \tilde{\mu}_B} \left\{ \frac{\mu_A \mu_B}{1 + (1 - \mu_A)(1 - \mu_B)} \right\}, \bigcup_{\eta_A \in \tilde{\eta}_A, \eta_B \in \tilde{\eta}_B} \left\{ \frac{\eta_A + \eta_B}{1 + \eta_A \eta_B} \right\}, \bigcup_{\nu_A \in \tilde{\nu}_A, \nu_B \in \tilde{\nu}_B} \left\{ \frac{\nu_A + \nu_B}{1 + \nu_A \nu_B} \right\} \right\rangle;$$

$$(3)\ \lambda A = \left\langle \bigcup_{\mu_A \in \tilde{\mu}_A} \left\{ \frac{(1 + \mu_A)^\lambda - (1 - \mu_A)^\lambda}{(1 + \mu_A)^\lambda + (1 - \mu_A)^\lambda} \right\}, \bigcup_{\eta_A \in \tilde{\eta}_A} \left\{ \frac{2(\eta_A)^\lambda}{(2 - \eta_A)^\lambda + (\eta_A)^\lambda} \right\}, \bigcup_{\nu_A \in \tilde{\nu}_A} \left\{ \frac{2(\nu_A)^\lambda}{(2 - \nu_A)^\lambda + (\nu_A)^\lambda} \right\} \right\rangle;$$

$$(4)\ A^\lambda = \left\langle \bigcup_{\mu_A \in \tilde{\mu}_A} \left\{ \frac{2(\mu_A)^\lambda}{(2 - \mu_A)^\lambda + (\mu_A)^\lambda} \right\}, \bigcup_{\eta_A \in \tilde{\eta}_A} \left\{ \frac{(1 + \eta_A)^\lambda - (1 - \eta_A)^\lambda}{(1 + \eta_A)^\lambda + (1 - \eta_A)^\lambda} \right\}, \bigcup_{\nu_A \in \tilde{\nu}_A} \left\{ \frac{(1 + \nu_A)^\lambda - (1 - \nu_A)^\lambda}{(1 + \nu_A)^\lambda + (1 - \nu_A)^\lambda} \right\} \right\rangle;$$

$$(5)\ A^c = \left\langle \bigcup_{\nu_A \in \tilde{\nu}_A} \{\nu_A\}, \bigcup_{\eta_A \in \tilde{\eta}_A} \{\eta_A\}, \bigcup_{\mu_A \in \tilde{\mu}_A} \{\mu_A\} \right\rangle.$$

If $\tilde{\mu}_A, \tilde{\eta}_A$ and $\tilde{\nu}_A$ has only one number, respectively, then the operations in Definition 4 are reduced to the operations of picture fuzzy elements (PFEs) in the following.

$$(6)\ A \oplus B = \left\langle \frac{\mu_A + \mu_B}{1 + \mu_A \mu_B}, \frac{\eta_A \eta_B}{1 + (1 - \eta_A)(1 - \eta_B)}, \frac{\nu_A \nu_B}{1 + (1 - \nu_A)(1 - \nu_B)} \right\rangle;$$

$$(7)\ A \otimes B = \left\langle \frac{\mu_A \mu_B}{1 + (1 - \mu_A)(1 - \mu_B)}, \frac{\eta_A + \eta_B}{1 + \eta_A \eta_B}, \frac{\nu_A + \nu_B}{1 + \nu_A \nu_B} \right\rangle;$$

$$(8)\ \lambda A = \left\langle \frac{(1 + \mu_A)^\lambda - (1 - \mu_A)^\lambda}{(1 + \mu_A)^\lambda + (1 - \mu_A)^\lambda}, \frac{2(\eta_A)^\lambda}{(2 - \eta_A)^\lambda + (\eta_A)^\lambda}, \frac{2(\nu_A)^\lambda}{(2 - \nu_A)^\lambda + (\nu_A)^\lambda} \right\rangle;$$

$$(9)\ A^\lambda = \left\langle \frac{2(\mu_A)^\lambda}{(2 - \mu_A)^\lambda + (\mu_A)^\lambda}, \frac{(1 + \eta_A)^\lambda - (1 - \eta_A)^\lambda}{(1 + \eta_A)^\lambda + (1 - \eta_A)^\lambda}, \frac{(1 + \nu_A)^\lambda - (1 - \nu_A)^\lambda}{(1 + \nu_A)^\lambda + (1 - \nu_A)^\lambda} \right\rangle;$$

$$(10)\ A^c = \langle \nu_A, \eta_A, \mu_A \rangle.$$

III. Novel Operations

In this section, the classical NWBM operator is extended to PHFS environment, in which the input values take the form of PHFEs. Furthermore, some interesting properties are discussed as well.

A. PHFNWBM Operator

Definition 5 [21]. Let $p, q \geq 0$, and $a_j (j = 1, 2, \cdots, n)$ be a set of nonnegative values. $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ is the corresponding weight vector of $a_j (j = 1, 2, \cdots, n)$, satisfying $\omega_j \geq 0, \sum_{j=1}^{n} \omega_j = 1$. ω_j denotes the importance of $a_j (j = 1, 2, \cdots, n)$. Then, the NWBM operator is defined as

$$NWBM^{p,q}(a_1, \cdots, a_n) = \left(\sum_{\substack{i,j=1 \\ j \neq i}}^{n} \frac{\omega_i \omega_j}{1 - \omega_i} a_i^{\,p} a_j^{\,q} \right)^{\frac{1}{p+q}} \tag{3}$$

Definition6. Let $p, q \geq 0$, and $a_i = \langle \tilde{\mu}_{a_i}, \tilde{\eta}_{a_i}, \tilde{\nu}_{a_i} \rangle (i = 1, 2, \cdots, n)$ be a set of PHFEs. $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ is the corresponding weight vector of $a_j (j = 1, 2, \cdots, n)$, satisfying $\omega_j \geq 0, \sum_{j=1}^{n} \omega_j = 1$. ω_j denotes the importance of $a_j (j = 1, 2, \cdots, n)$. Then, the picture hesitant fuzzy normalized weighted Bonferroni mean (PHFNWBM) operator based on Einstein operations is defined as follows, and the aggregating result is still an PHFE.

$$PHFNWBM^{p,q}(a_1, a_2, \cdots, a_n) = \left(\mathop{\oplus}_{\substack{i,j=1 \\ j \neq i}}^{n} \frac{\omega_i \omega_j}{1 - \omega_i} \left(a_i^{\,p} \otimes a_j^{\,q} \right) \right)^{\frac{1}{p+q}}$$

$$= \left\langle \bigcup_{\substack{\mu_i \in \tilde{\mu}_{a_i}, \\ \mu_j \in \tilde{\mu}_{a_j}}} \left\{ \frac{2\left(\prod_{\substack{i,j=1 \\ j \neq i}}^{n} u_{ij}^{\varepsilon_{ij}} - \prod_{\substack{i,j=1 \\ j \neq i}}^{n} m_{ij}^{\varepsilon_{ij}} \right)^{\frac{1}{p+q}}}{\left(\prod_{\substack{i,j=1 \\ j \neq i}}^{n} u_{ij}^{\varepsilon_{ij}} + 3\prod_{\substack{i,j=1 \\ j \neq i}}^{n} m_{ij}^{\varepsilon_{ij}} \right)^{\frac{1}{p+q}} + \left(\prod_{\substack{i,j=1 \\ j \neq i}}^{n} u_{ij}^{\varepsilon_{ij}} - \prod_{\substack{i,j=1 \\ j \neq i}}^{n} m_{ij}^{\varepsilon_{ij}} \right)^{\frac{1}{p+q}}} \right\}, \right.$$

$$\bigcup_{\substack{\eta_i \in \tilde{\eta}_{a_i}, \\ \eta_j \in \tilde{\eta}_{a_j}}} \left\{ \frac{\left(\prod_{\substack{i,j=1 \\ j \neq i}}^{n} y_{ij}^{\varepsilon_{ij}} + 3\prod_{\substack{i,j=1 \\ j \neq i}}^{n} x_{ij}^{\varepsilon_{ij}} \right)^{\frac{1}{p+q}} - \left(\prod_{\substack{i,j=1 \\ j \neq i}}^{n} y_{ij}^{\varepsilon_{ij}} - \prod_{\substack{i,j=1 \\ j \neq i}}^{n} x_{ij}^{\varepsilon_{ij}} \right)^{\frac{1}{p+q}}}{\left(\prod_{\substack{i,j=1 \\ j \neq i}}^{n} y_{ij}^{\varepsilon_{ij}} + 3\prod_{\substack{i,j=1 \\ j \neq i}}^{n} x_{ij}^{\varepsilon_{ij}} \right)^{\frac{1}{p+q}} + \left(\prod_{\substack{i,j=1 \\ j \neq i}}^{n} y_{ij}^{\varepsilon_{ij}} - \prod_{\substack{i,j=1 \\ j \neq i}}^{n} x_{ij}^{\varepsilon_{ij}} \right)^{\frac{1}{p+q}}} \right\},$$

$$\left. \bigcup_{\substack{\nu_i \in \tilde{\nu}_{a_i}, \\ \nu_j \in \tilde{\nu}_{a_j}}} \left\{ \frac{\left(\prod_{\substack{i,j=1 \\ j \neq i}}^{n} z_{ij}^{\varepsilon_{ij}} + 3\prod_{\substack{i,j=1 \\ j \neq i}}^{n} t_{ij}^{\varepsilon_{ij}} \right)^{\frac{1}{p+q}} - \left(\prod_{\substack{i,j=1 \\ j \neq i}}^{n} z_{ij}^{\varepsilon_{ij}} - \prod_{\substack{i,j=1 \\ j \neq i}}^{n} t_{ij}^{\varepsilon_{ij}} \right)^{\frac{1}{p+q}}}{\left(\prod_{\substack{i,j=1 \\ j \neq i}}^{n} z_{ij}^{\varepsilon_{ij}} + 3\prod_{\substack{i,j=1 \\ j \neq i}}^{n} t_{ij}^{\varepsilon_{ij}} \right)^{\frac{1}{p+q}} + \left(\prod_{\substack{i,j=1 \\ j \neq i}}^{n} z_{ij}^{\varepsilon_{ij}} - \prod_{\substack{i,j=1 \\ j \neq i}}^{n} t_{ij}^{\varepsilon_{ij}} \right)^{\frac{1}{p+q}}} \right\} \right\rangle \tag{4}$$

Where $\varepsilon_{ij} = \frac{\omega_i \omega_j}{1 - \omega_i}$

$$u_{ij} = \left(2-\mu_i\right)^p \left(2-\mu_j\right)^q + 3\mu_i^{\ p}\mu_j^{\ q};$$

$$m_{ij} = \left(2-\mu_i\right)^p \left(2-\mu_j\right)^q - \mu_i^{\ p}\mu_j^{\ q};$$

$$x_{ij} = \left(1+\eta_i\right)^p \left(1+\eta_j\right)^q - \left(1-\eta_i\right)^p \left(1-\eta_j\right)^q;$$

$$y_{ij} = \left(1+\eta_i\right)^p \left(1+\eta_j\right)^q + 3\left(1-\eta_i\right)^p \left(1-\eta_j\right)^q;$$

$$t_{ij} = \left(1+\nu_i\right)^p \left(1+\nu_j\right)^q - \left(1-\nu_i\right)^p \left(1-\nu_j\right)^q;$$

$$z_{ij} = \left(1+\nu_i\right)^p \left(1+\nu_j\right)^q + 3\left(1-\nu_i\right)^p \left(1-\nu_j\right)^q.$$

Proof. According to the operational rules of PHFEs given in Definition 4, we get

$$a_i^{\ p} = \left\langle \bigcup_{\mu_i \in \tilde{\mu}_{a_i}} \left\{ \frac{2\left(\mu_i\right)^p}{\left(2-\mu_i\right)^p + \left(\mu_i\right)^p} \right\}, \bigcup_{\eta_i \in \tilde{\eta}_{a_i}} \left\{ \frac{\left(1+\eta_i\right)^p - \left(1-\eta_i\right)^p}{\left(1+\eta_i\right)^p + \left(1-\eta_i\right)^p} \right\}, \bigcup_{\nu_i \in \tilde{\nu}_{a_i}} \left\{ \frac{\left(1+\nu_i\right)^p - \left(1-\nu_i\right)^p}{\left(1+\nu_i\right)^p + \left(1-\nu_i\right)^p} \right\} \right\rangle;$$

$$a_j^{\ q} = \left\langle \bigcup_{\mu_j \in \tilde{\mu}_{a_j}} \left\{ \frac{2\left(\mu_j\right)^q}{\left(2-\mu_j\right)^q + \left(\mu_j\right)^q} \right\}, \bigcup_{\eta_j \in \tilde{\eta}_{a_j}} \left\{ \frac{\left(1+\eta_j\right)^q - \left(1-\eta_j\right)^q}{\left(1+\eta_j\right)^q + \left(1-\eta_j\right)^q} \right\}, \bigcup_{\nu_j \in \tilde{\nu}_{a_j}} \left\{ \frac{\left(1+\nu_j\right)^q - \left(1-\nu_j\right)^q}{\left(1+\nu_j\right)^q + \left(1-\nu_j\right)^q} \right\} \right\rangle;$$

And,

$$a_i^{\ p} \otimes a_j^{\ q} = \left\langle \bigcup_{\substack{\mu_i \in \tilde{\mu}_{a_i}, \\ \mu_j \in \tilde{\mu}_{a_j}}} \left\{ \frac{2\left(\mu_i\right)^p \left(\mu_j\right)^q}{\left(2-\mu_i\right)^p \left(2-\mu_j\right)^q + \left(\mu_i\right)^p \left(\mu_j\right)^q} \right\}, \bigcup_{\substack{\eta_i \in \tilde{\eta}_{a_i}, \\ \eta_j \in \tilde{\eta}_{a_j}}} \left\{ \frac{\left(1+\eta_i\right)^p \left(1+\eta_j\right)^q - \left(1-\eta_i\right)^p \left(1-\eta_j\right)^q}{\left(1+\eta_i\right)^p \left(1+\eta_j\right)^q + \left(1-\eta_i\right)^p \left(1-\eta_j\right)^q} \right\}, \bigcup_{\substack{\nu_i \in \tilde{\nu}_{a_i}, \\ \nu_j \in \tilde{\nu}_{a_j}}} \left\{ \frac{\left(1+\nu_i\right)^p \left(1+\nu_j\right)^q - \left(1-\nu_i\right)^p \left(1-\nu_j\right)^q}{\left(1+\nu_i\right)^p \left(1+\nu_j\right)^q + \left(1-\nu_i\right)^p \left(1-\nu_j\right)^q} \right\} \right\rangle;$$

Then,

$$\frac{\omega_i \omega_j}{1-\omega_i}\left(a_i^{\ p} \otimes a_j^{\ q}\right) = \left\langle \bigcup_{\substack{\mu_i \in \tilde{\mu}_{a_i}, \\ \mu_j \in \tilde{\mu}_{a_j}}} \left\{ \frac{u_{ij}^{\ \varepsilon_{ij}} - m_{ij}^{\ \varepsilon_{ij}}}{u_{ij}^{\ \varepsilon_{ij}} + m_{ij}^{\ \varepsilon_{ij}}} \right\}, \bigcup_{\substack{\eta_i \in \tilde{\eta}_{a_i}, \\ \eta_j \in \tilde{\eta}_{a_j}}} \left\{ \frac{2x_{ij}^{\ \varepsilon_{ij}}}{y_{ij}^{\ \varepsilon_{ij}} + x_{ij}^{\ \varepsilon_{ij}}} \right\}, \bigcup_{\substack{\nu_i \in \tilde{\nu}_{a_i}, \\ \nu_j \in \tilde{\nu}_{a_j}}} \left\{ \frac{2t_{ij}^{\ \varepsilon_{ij}}}{z_{ij}^{\ \varepsilon_{ij}} + t_{ij}^{\ \varepsilon_{ij}}} \right\} \right\rangle; \quad (5)$$

Thus,

$$\bigoplus_{\substack{i,j=1 \\ j \neq i}}^{n} \frac{\omega_i \omega_j}{1-\omega_i}\left(a_i^{\ p} \otimes a_j^{\ q}\right) = \left\langle \bigcup_{\substack{\mu_i \in \tilde{\mu}_{a_i}, \\ \mu_j \in \tilde{\mu}_{a_j}}} \left\{ \frac{\prod_{\substack{i,j=1 \\ j \neq i}}^{n} u_{ij}^{\ \varepsilon_{ij}} - \prod_{\substack{i,j=1 \\ j \neq i}}^{n} m_{ij}^{\ \varepsilon_{ij}}}{\prod_{\substack{i,j=1 \\ j \neq i}}^{n} u_{ij}^{\ \varepsilon_{ij}} + \prod_{\substack{i,j=1 \\ j \neq i}}^{n} m_{ij}^{\ \varepsilon_{ij}}} \right\}, \bigcup_{\substack{\eta_i \in \tilde{\eta}_{a_i}, \\ \eta_j \in \tilde{\eta}_{a_j}}} \left\{ \frac{2\prod_{\substack{i,j=1 \\ j \neq i}}^{n} x_{ij}^{\ \varepsilon_{ij}}}{\prod_{\substack{i,j=1 \\ j \neq i}}^{n} y_{ij}^{\ \varepsilon_{ij}} + \prod_{\substack{i,j=1 \\ j \neq i}}^{n} x_{ij}^{\ \varepsilon_{ij}}} \right\}, \bigcup_{\substack{\nu_i \in \tilde{\nu}_{a_i}, \\ \nu_j \in \tilde{\nu}_{a_j}}} \left\{ \frac{2\prod_{\substack{i,j=1 \\ j \neq i}}^{n} t_{ij}^{\ \varepsilon_{ij}}}{\prod_{\substack{i,j=1 \\ j \neq i}}^{n} z_{ij}^{\ \varepsilon_{ij}} + \prod_{\substack{i,j=1 \\ j \neq i}}^{n} t_{ij}^{\ \varepsilon_{ij}}} \right\} \right\rangle;$$

So,

$$PHFNWBM^{p,q}(a_1,a_2,\cdots,a_n)=\left(\underset{\substack{i,j=1\\ j\neq i}}{\overset{n}{\oplus}}\frac{\omega_i\omega_j}{1-\omega_i}\left(a_i^{\,p}\otimes a_j^{\,q}\right)\right)^{\frac{1}{p+q}}$$

$$=\left\langle \bigcup_{\substack{\mu_i\in\tilde{\mu}_{a_i},\\ \mu_j\in\tilde{\mu}_{a_j}}}\left\{\frac{2\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}u_{ij}^{\ \varepsilon_{ij}}-\prod_{\substack{i,j=1\\ j\neq i}}^{n}m_{ij}^{\ \varepsilon_{ij}}\right)^{\frac{1}{p+q}}}{\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}u_{ij}^{\ \varepsilon_{ij}}+3\prod_{\substack{i,j=1\\ j\neq i}}^{n}m_{ij}^{\ \varepsilon_{ij}}\right)^{\frac{1}{p+q}}+\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}u_{ij}^{\ \varepsilon_{ij}}-\prod_{\substack{i,j=1\\ j\neq i}}^{n}m_{ij}^{\ \varepsilon_{ij}}\right)^{\frac{1}{p+q}}}\right\},\right.$$

$$\bigcup_{\substack{\eta_i\in\tilde{\eta}_{a_i},\\ \eta_j\in\tilde{\eta}_{a_j}}}\left\{\frac{\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}y_{ij}^{\ \varepsilon_{ij}}+3\prod_{\substack{i,j=1\\ j\neq i}}^{n}x_{ij}^{\ \varepsilon_{ij}}\right)^{\frac{1}{p+q}}-\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}y_{ij}^{\ \varepsilon_{ij}}-\prod_{\substack{i,j=1\\ j\neq i}}^{n}x_{ij}^{\ \varepsilon_{ij}}\right)^{\frac{1}{p+q}}}{\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}y_{ij}^{\ \varepsilon_{ij}}+3\prod_{\substack{i,j=1\\ j\neq i}}^{n}x_{ij}^{\ \varepsilon_{ij}}\right)^{\frac{1}{p+q}}+\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}y_{ij}^{\ \varepsilon_{ij}}-\prod_{\substack{i,j=1\\ j\neq i}}^{n}x_{ij}^{\ \varepsilon_{ij}}\right)^{\frac{1}{p+q}}}\right\},$$

$$\left.\bigcup_{\substack{\nu_i\in\tilde{\nu}_{a_i},\\ \nu_j\in\tilde{\nu}_{a_j}}}\left\{\frac{\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}z_{ij}^{\ \varepsilon_{ij}}+3\prod_{\substack{i,j=1\\ j\neq i}}^{n}t_{ij}^{\ \varepsilon_{ij}}\right)^{\frac{1}{p+q}}-\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}z_{ij}^{\ \varepsilon_{ij}}-\prod_{\substack{i,j=1\\ j\neq i}}^{n}t_{ij}^{\ \varepsilon_{ij}}\right)^{\frac{1}{p+q}}}{\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}z_{ij}^{\ \varepsilon_{ij}}+3\prod_{\substack{i,j=1\\ j\neq i}}^{n}t_{ij}^{\ \varepsilon_{ij}}\right)^{\frac{1}{p+q}}+\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}z_{ij}^{\ \varepsilon_{ij}}-\prod_{\substack{i,j=1\\ j\neq i}}^{n}t_{ij}^{\ \varepsilon_{ij}}\right)^{\frac{1}{p+q}}}\right\}\right\rangle$$

The proof is completed.

B. Properties of PHFNWBM Operator

The proposed PHFNWBM operator has the following properties.

Theorem1. (Reducibility): Let $\omega=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)$, then

$$PHFNWBM^{p,q}(a_1,a_2,\cdots,a_n)=PHFBM^{p,q}(a_1,a_2,\cdots,a_n).$$

Theorem2. (Idempotency): Let $a_i = a(i = 1,2,\cdots,n)$ be a set of PHFEs, then

$$PHFNWBM^{p,q}(a_1,a_2,\cdots,a_n)=a.$$

Theorem3. (Commutativity): Let a_i^* be any permutation of $a_i(i = 1,2,\cdots,n)$, then

$$PHFNWBM^{p,q}(a_1,a_2,\cdots,a_n)=PHFNWBM^{p,q}(a_1^*,a_2^*,\cdots,a_n^*).$$

Theorem4. (Monotonicity): Let $a_i=\left\langle\tilde{\mu}_{a_i},\tilde{\eta}_{a_i},\tilde{\nu}_{a_i}\right\rangle$ and $a_i'=\left\langle\tilde{\mu}_{a_i'},\tilde{\eta}_{a_i'},\tilde{\nu}_{a_i'}\right\rangle(i = 1,2,\cdots,n)$ be two sets of PHFEs. If $a_i\le a_i'(i=1,2,\cdots n)$, that is, $\tilde{\mu}_{a_i}\le\tilde{\mu}_{a_i'},\tilde{\eta}_{a_i}\ge\tilde{\eta}_{a_i'}$, and $\tilde{\nu}_{a_i}\ge\tilde{\nu}_{a_i'}$, then

$$PHFNWBM^{p,q}(a_1,a_2,\cdots,a_n)\le PHFNWBM^{p,q}(a_1',a_2',\cdots,a_n').$$

Theorem5. (Boundedness): Let $a_i=\left\langle\tilde{\mu}_{a_i},\tilde{\eta}_{a_i},\tilde{\nu}_{a_i}\right\rangle(i=1,2,\cdots n)$ be a set of PHFEs, and $a^-=\left\langle\tilde{\mu}_{a^-},\tilde{\eta}_{a^-},\tilde{\nu}_{a^-}\right\rangle$, $a^+=\left\langle\tilde{\mu}_{a^+},\tilde{\eta}_{a^+},\tilde{\nu}_{a^+}\right\rangle$ .If for each i, $\tilde{\mu}_{a^-}\le\tilde{\mu}_{a_i}\le\tilde{\mu}_{a^+},\tilde{\eta}_{a^+}\le\tilde{\eta}_{a_i}\le\tilde{\eta}_{a^-}$, $\tilde{\nu}_{a^+}\le\tilde{\nu}_{a_i}\le\tilde{\nu}_{a^-}$, then

$$a^-=PHFNWBM^{p,q}(a^-,a^-,\cdots,a^-)\le PHFNWBM^{p,q}(a_1,a_2,\cdots,a_n)$$
$$\le PHFNWBM^{p,q}(a^+,a^+,\cdots,a^+)=a^+.$$

That is,

$$a^-\le PHFNWBM^{p,q}(a_1,a_2,\cdots,a_n)\le a^+.$$

IV. Example and Result Analysis

In this section, to illustrate the effectiveness and application of the proposed operator, an example [9] of MADM problem is performed, along with sensitivity analysis and comparative analysis.

A. Decision-making Process

Suppose an enterprise want to purchase an appropriate enterprise resource planning (ERP) system from different vendors. Here $A_i(i=1,2,3,4,5)$ indicates five potential vendors, $C_j(j=1,2,3,4)$ indicates four evaluation attributes, where C_1 is function and technology, C_2 is strategic adaptability, C_3 is vendor's ability, C_4 is vendor's reputation. And $\omega=(0.2,0.1,0.3,0.4)$ is the weighting vector of four attributes. Considering the hesitancy of decision-makers, the assessment value of each alternative $A_i(i=1,2,3,4,5)$ regarding criteria $C_j(j=1,2,3,4)$ takes the form of PHFE. The corresponding original PHFE decision matrix $R=\left[a_{ij}\right]_{5\times4}$ can be shown as below.

$$R=\left[a_{ij}\right]_{5\times4}$$
$$=\begin{bmatrix}
\left\langle\begin{matrix}\{0.43,0.53\},\\ \{0.33\},\\ \{0.06,0.09\}\end{matrix}\right\rangle & \left\langle\begin{matrix}\{0.76,0.89\},\\ \{0.05,0.08\},\\ \{0.03\}\end{matrix}\right\rangle & \left\langle\begin{matrix}\{0.42\},\\ \{0.35\},\\ \{0.12,0.18\}\end{matrix}\right\rangle & \left\langle\begin{matrix}\{0.08\},\\ \{0.75,0.89\},\\ \{0.02\}\end{matrix}\right\rangle\\
\left\langle\begin{matrix}\{0.53,0.65,0.73\},\\ \{0.10,0.12\},\\ \{0.08\}\end{matrix}\right\rangle & \left\langle\begin{matrix}\{0.13\},\\ \{0.53,0.64\},\\ \{0.12,0.21\}\end{matrix}\right\rangle & \left\langle\begin{matrix}\{0.03\},\\ \{0.77,0.82\},\\ \{0.10,0.13\}\end{matrix}\right\rangle & \left\langle\begin{matrix}\{0.58,0.73\},\\ \{0.15\},\\ \{0.08\}\end{matrix}\right\rangle\\
\left\langle\begin{matrix}\{0.72,0.86,0.91\},\\ \{0.03\},\\ \{0.02\}\end{matrix}\right\rangle & \left\langle\begin{matrix}\{0.07\},\\ \{0.05,0.09\},\\ \{0.05\}\end{matrix}\right\rangle & \left\langle\begin{matrix}\{0.04\},\\ \{0.65,0.72,0.85\},\\ \{0.05,0.10\}\end{matrix}\right\rangle & \left\langle\begin{matrix}\{0.45,0.68\},\\ \{0.18,0.26\},\\ \{0.06\}\end{matrix}\right\rangle\\
\left\langle\begin{matrix}\{0.77,0.85\},\\ \{0.09\},\\ \{0.05\}\end{matrix}\right\rangle & \left\langle\begin{matrix}\{0.65,0.74\},\\ \{0.10,0.16\},\\ \{0.1\}\end{matrix}\right\rangle & \left\langle\begin{matrix}\{0.02\},\\ \{0.78,0.89\},\\ \{0.05\}\end{matrix}\right\rangle & \left\langle\begin{matrix}\{0.08\},\\ \{0.65,0.84\},\\ \{0.06\}\end{matrix}\right\rangle\\
\left\langle\begin{matrix}\{0.70,0.81,0.90\},\\ \{0.05\},\\ \{0.02\}\end{matrix}\right\rangle & \left\langle\begin{matrix}\{0.68\},\\ \{0.08\},\\ \{0.13,0.21\}\end{matrix}\right\rangle & \left\langle\begin{matrix}\{0.05\},\\ \{0.77,0.87\},\\ \{0.06\}\end{matrix}\right\rangle & \left\langle\begin{matrix}\{0.13\},\\ \{0.65,0.75\},\\ \{0.09\}\end{matrix}\right\rangle
\end{bmatrix}$$

Step1. Normalize the decision matrix.

In MADM problems, there commonly two types of attributes, namely, the benefit attribute and cost attribute. The cost attribute should be transformed into the benefit attribute based on the following equation.

$$a_{ij}'=\begin{cases}\left\langle\tilde{\mu}_{a_{ij}},\tilde{\eta}_{a_{ij}},\tilde{\nu}_{a_{ij}}\right\rangle & for\ \ benefit\ \ attribute\\ \left\langle\tilde{\nu}_{a_{ij}},\tilde{\eta}_{a_{ij}},\tilde{\mu}_{a_{ij}}\right\rangle & for\ \ \cos t\ \ attribute\end{cases}\quad(6)$$

Since all attributes are of the benefit type, there is no need to normalize the original decision matrix.

Step2. Calculate the collective assessment value of each alternative.

Depending on the PHFNWBM operator presented in Definition 6, we can compute the collective assessment value a_i of each alternative $A_i(i=1,2,3,4,5)$, and then the aggregating values are shown below. Here, we assign $p=q=1$.

$$a_1 = \left\langle \begin{array}{l} \{0.3328, 0.3534, 0.3543, 0.3749\}, \{0.4474, 0.5034, 0.4528, 0.5091\}, \\ \{0.0564, 0.0697, 0.0632, 0.0774\} \end{array} \right\rangle;$$

$$a_2 = \left\langle \begin{array}{l} \{0.3318, 0.3770, 0.3662, 0.4148, 0.3892, 0.4403\}, \\ \{0.3504, 0.3651, 0.3618, 0.3768, 0.3578, 0.3726, 0.3693, 0.3846\}, \\ \{0.0904, 0.0992, 0.0987, 0.1079\} \end{array} \right\rangle;$$

$$a_3 = \left\langle \begin{array}{l} \{0.3381, 0.4168, 0.3758, 0.4618, 0.3897, 0.4789\}, \\ \left\{ \begin{array}{l} 0.2377, 0.2709, 0.2540, 0.2881, 0.2854, 0.3208, 0.2467, 0.2801, \\ 0.2635, 0.2976, 0.2957, 0.3312 \end{array} \right\}, \\ \{0.0462, 0.0598\} \end{array} \right\rangle;$$

$$a_4 = \left\langle \begin{array}{l} \{0.2581, 0.2730, 0.2742, 0.2901\}, \\ \{0.4801, 0.5550, 0.5123, 0.5895, 0.4917, 0.5677, 0.5244, 0.6029\}, \\ \{0.0587\} \end{array} \right\rangle;$$

$$a_5 = \left\langle \begin{array}{l} \{0.2860, 0.3092, 0.3295\}, \{0.4598, 0.4976, 0.4884, 0.5270\}, \\ \{0.0672, 0.0732\} \end{array} \right\rangle.$$

Step3. Calculate the comparative function values.

According to Definition 2, the score function value $s(a_i)$ can be gained.

$S(a_1) = 0.6030$; $S(a_2) = 0.6401$; $S(a_3) = 0.6921$; $S(a_4) = 0.5582$; $S(a_5) = 0.5816$.

Step4. Rank all the alternatives.

Based on Definition 3, the ranking order of five alternatives is $A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4$. Apparently, the best ERP vendor is A_3, and the worst vendor is A_4.

B. Sensitivity Analysis

To reflect the impact of different parameters p and q on the ranking orders, we perform a sensitivity analysis. The ranking results with different parameters are listed in TABLE I.

From TABLE I, we find that the ranking results are different with different parameters p and q. When parameter p or q is assigned to a fixed value, the worst vendor changes from A_4 to A_2 with the increasing of parameter q or p. However, the optimal vendor is always A_3.

When p or q is assigned to a fixed value, the variation trends of score values for all the alternatives are presented in Figure 1 and Figure 2.

TABLE I
RANKING RESULTS WITH DIFFERENT PARAMETERS

p, q	Ranking
$p = 0.01, q = 0$	$A_3 \succ A_2 \succ A_5 \succ A_1 \succ A_4$
$p = 0.1, q = 0$	$A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4$
$p = 1, q = 0$	$A_3 \succ A_2 \succ A_5 \succ A_1 \succ A_4$
$p = 2, q = 0$	$A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_1$
$p = 5, q = 0$	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$
$p = 10, q = 0$	$A_3 \succ A_5 \succ A_4 \succ A_1 \succ A_2$
$p = 0.01, q = 1$	$A_3 \succ A_2 \succ A_5 \succ A_1 \succ A_4$
$p = 0.1, q = 1$	$A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4$
$p = 1, q = 1$	$A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4$
$p = 2, q = 1$	$A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4$
$p = 5, q = 1$	$A_3 \succ A_5 \succ A_2 \succ A_4 \succ A_1$
$p = 10, q = 1$	$A_3 \succ A_5 \succ A_1 \succ A_4 \succ A_2$
$p = 0, q = 0.01$	$A_3 \succ A_2 \succ A_5 \succ A_1 \succ A_4$
$p = 0, q = 0.1$	$A_3 \succ A_2 \succ A_5 \succ A_1 \succ A_4$
$p = 0, q = 1$	$A_3 \succ A_2 \succ A_5 \succ A_1 \succ A_4$
$p = 0, q = 2$	$A_3 \succ A_5 \succ A_2 \succ A_4 \succ A_1$
$p = 0, q = 5$	$A_3 \succ A_5 \succ A_4 \succ A_1 \succ A_2$
$p = 0, q = 10$	$A_3 \succ A_5 \succ A_4 \succ A_1 \succ A_2$
$p = q = 2$	$A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4$
$p = q = 5$	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$
$p = q = 10$	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$

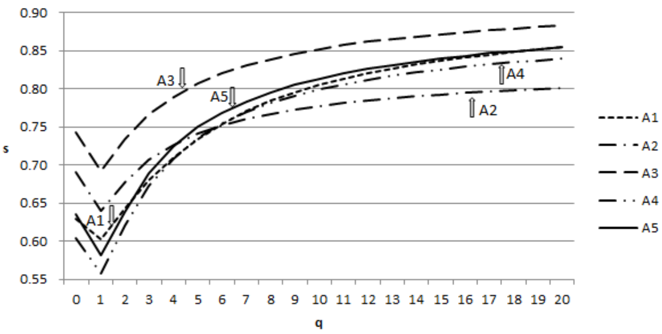

Fig.1. Score values of all the alternatives when $p = 1, q \in [0, 20]$ utilizing PHFNWBM operator.

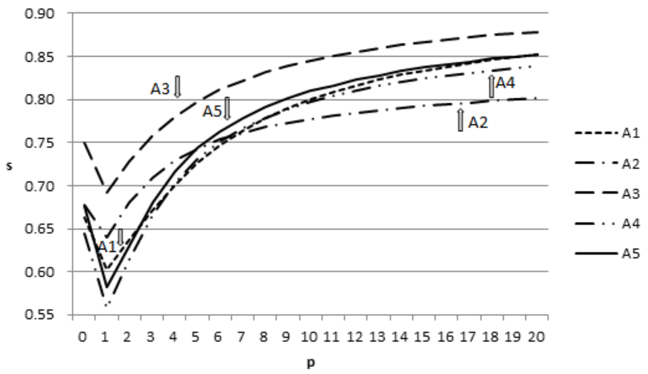

Fig.2. Score values of all the alternatives when $q = 1, p \in [0, 20]$ utilizing PHFNWBM operator.

As we can see from Figure 1, when $p = 1, q \in [0, 20]$, the optimal alternative is always A_3. With the increasing of parameter q, the worst alternative will change from A_4 to A_2. When $q = 1, p \in [0, 20]$, the trend of Figure 2 is similar to that of Figure 1.

C. Comparative Analysis

To further explore the superiorities of the proposed MADM method, we provide a comparative analysis using our method with the existing methods [9] [29]. The comparative analysis contains two aspects, one is performed under picture hesitant fuzzy environment [9], and the other is performed under picture fuzzy environment [29].The comparative results using different methods with PHFS are shown in TABLE II.

TABLE II
THE COMPARATIVE RESULTS WITH PHFS

p,q	Ranking
Wang[9]	$A_3 \succ A_2 \succ A_5 \succ A_1 \succ A_4$ $A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4$
Ates [29]	$A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4$
Our method	$A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4$

From TABLE II, we can clearly observe that the final ranking is $A_3 \succ A_2 \succ A_5 \succ A_1 \succ A_4$ or $A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4$ utilizing picture hesitant fuzzy weighted average (PHFWA) operator or picture hesitant fuzzy weighted geometric (PHFWG) operator proposed by Wang [9]. The ranking orders are slightly different between our method and Wang's method. However, the optimal vendor is always A_3, and the worst vendor is always A_4.

The PHFWA and PHFWG operators in [9] consider the importance of attributes, but ignore the interrelationship between input arguments. However, the PHFNWBM operator developed in this paper can reflect both the importance of attributes and the correlations of input values. Furthermore, the operational rules employed in the method of Wang [9] are based on Algebraic operations, while the proposed operator in this paper is based on Einstein operations. According to decision-makers' preference, we can assign different values to parameters p and q, then different ranking can be gained. Therefore, the proposed method in this paper is more practical.

If our method is applied to handle with the same example with picture fuzzy information in Ates [29], the final ranking is $A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4$, where the result is the same as that utilizing Ates's method, the optimal vendor is always A_3, while the worst vendor is always A_4. In PHFS, if there is only one value for positive, neutral, and negative degree, then the PHFS is reduced to the PFS. Thus, PFS is a special case of PHFS. The PFNWBM operator based on Algebraic operations was proposed by Ates [29] and extended to fuse picture fuzzy information, which is a special case of the proposed method in this paper. Therefore, the proposed method in this paper is more general.

V. CONCLUSION

Although FS theory is an effective tool to deal with MCDM problems in real life, it cannot represent all kinds of uncertainty information in different situation. Therefore, PHFS as an extended form of PFS and HFS is proposed, which is more general and practical for handling complicated decision making information.

The main contributions of the work are as follows.

1) According to the existing researches regarding PHFS, the comparative method and operational rules are all based on Algebraic operations, and ignore Einstein operations for PHFEs. Thus, comparative method and operational rules on the basis of Einstein operations were defined in this manuscript.

2) Since NWBM operator can reflect both the importance of attributes and the interrelationships between input arguments, it has become a hot issue in recent years. Thus, it is meaningful to solve MADM problems with NWBM operator under PHFEs environment. Therefore, the typical NWBM operator was extended to PHF environment, namely, PHFNWBM operator was proposed, and its desirable properties were also discussed in this paper.

3) To demonstrate the application and powerfulness of the proposed method, an example of MADM problem based on the developed operator was performed, together with sensitivity analysis and comparative analysis. The analysis results demonstrated our method is more practical and general.

In future, we will further explore different aggregated operators for dealing with PHFES and applied them to handle with different MCDM problem. Meanwhile, we will investigate the measures of PHFS, such as similarity, entropy, and so on.

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