# A New Definition, a Generalisation and an Approximation for a Fractional Derivative with Applications to Stochastic Time Series Modeling

Jonathan Blackledge

*Abstract*—**A brief review of fractional differentiation, fractional integration and differo-integral operators is given based on the properties of the Fourier transform. This is undertaken to provide the reader with a quick-guide and a short background to the fractional calculus and includes a brief discussion on some of the principal characteristics of fractional differo-integral operators. The paper then presents a new definition for a fractional differo-integral based on the properties of the sign function and explores some related results. Using the properties of the Dirac delta function, a generalisation is developed in order to quantify the issue as to whether there is an upper bound to the number of definitions for a fractional differo-integral operator that can be developed. Finally, an approximation of a fractional differential is considered and used in the construction of a self-affine stochastic time series model based on the Kolmogorov-Feller equation for the memory function $t^{-\alpha}/\Gamma(1-\alpha),\ \alpha \in (0,1)$.**

*Index Terms*—**Fractional Differo-Integral Operators, Sign Function, Approximation of a Fractional Derivative, Stochastic Time Series Modelling.**

## I. Introduction

THE idea of a fractional derivative and/or a fractional integral (the fractional calculus) goes back to the time when (integer) calculus was being developed in the late seventeenth century. In a letter from Guillaume l'Hospital to Gottfried von Leibnitz written in 1695, l'Hospital asked the following question: 'Given that $d^n f/dx^n$ exists for all integers $n$, what if $n$ be $\frac{1}{2}$'. The reply from Leibnitz was all the more interesting: 'It will lead to a paradox ... From this paradox, one day useful consequences will be drawn'.

The foundations of fractional calculus were constructed later in the nineteenth century by mathematicians such as Joseph Liouville [1], and, in particular, the Norwegian mathematician, Niels Abel [2], who's work arguably led to the birth (or re-birth) of the subject [3]. During this period, many of the fundamental elements of the subject were conceived including the mutual inverse relationships between fractional-order integrals and differentials in terms of the same generalised operation - a fractional 'differo-integral operator', [4]. In terms of applications, one of the first was introduced by Oliver Heaviside, who considered the use of fractional differential operators in the analysis of electrical transmission lines in 1890.

Manuscript received on 10 June, 2020; revised version submitted on 11 September, 2020; final version submitted on 10 December, 2020.

Jonathan Blackledge is the Science Foundation Ireland Stokes Professor, Honorary Professor at the Technological University Dublin, Ireland, a Distinguished Professor at the Center for Advanced Studies, Warsaw University of Technology, Poland, Professor Extraordinaire and Honorary Professor at the University of Western Cape and the University of KwaZulu-Natal, South Africa, respectively, and, Visiting Professor, University of Wales, UK. https://www.linkedin.com/in/jonathan-blackledge-7643a5150/

Although both the theory and applications of fractional calculus continued to be developed over the twentieth century, relatively few papers, journals, conferences and books were devoted to the subject compared to other areas of mathematics. However, since the start of twenty-first century, there has been a considerable expansion of interests in the subject area coupled with a growing range of applications [5]. Some of these applications are based on modifications to the fundamental field equations of physics, namely, fractional partial differential equations that may be time and/or space fractional, and, in some cases, of variable order [6]. This includes, for example, the study of certain fractional calculus operators for specific functions [7], the evaluation of fractional integrals and derivatives of elementary functions [8], studies on the theory of discrete fractional calculus [9], time fractional systems of coupled equations [10], applications involving singularly perturbed nonlinear fractional differential equations [11], [12] and the use of time fractional differential equations in economics [13], [14].

A significant feature in the contributions that are prevalent in the development of fractional calculus, is the numerous new definitions for fractional derivatives and integrals that are based on different themes and/or generalisations of more standard results. In each case, the approach used to conceive and define a fractional integral/derivative is based on what is in effect, a generalisation of a corresponding result obtained using integer calculus.

If a definition for the multiple differentiation and/or integration of a function can be achieved (e.g. the differentiation of a function $n$ times where $n$ is a positive integer), then the equivalent definition for the fractional case is often taken to be given by letting $n$ be a non-integer value. This is undertaken on the strict basis that conformity and self-consistency is achievable when $n$ is re-assigned to be an integer, ideally within the context of a closed-form expression. This approach provides the basis for defining a fractional differo-integral operator.

A similar approach is taken in this paper to develop a new operator using the properties of the sign function. The result is then explored using some example sigmoid-type functions to obtain corresponding definitions of a fractional differential as a limiting case. In this context, and, given the growth in the number of definitions for a fractional differo-integral operator, a conjecture is presented whose argument leads to the concept that there may be an unlimited number of such operators.

The paper then considers an approximation to a fractional integral using a generalisation of a fractional derivative through application of the delta which is used to develop a model for a non-Ergodic stochastic times series field. This is based on using the Kolmogorov-Feller equation for the memory function $t^{-\alpha}/\Gamma(1-\alpha),\ \alpha \in (0,1)$ and applying an asymptotic analysis to derive a time-only dependent equation for the density function.

## II. Structure of the Paper

The structure of this paper is based on the following principal components:

(i) We begin by revisiting some of the standard definitions for a fractional integral, a fractional differential and generalised fractional differo-integral operators, all of which are shown to be related to the properties of the Fourier transform;
(ii) a short discussion on the properties of such operators is presented;
(iii) after providing a short list of some non-standard definitions for a fractional integral and differential, a new definition of a fractional differo-integral is presented which is based exclusively on the properties of the sign function;
(iv) given the rate of increase in which new definitions of such operators are being introduce (inclusive of those given in this paper), a conjecture is presented and justified (but without a formal proof) which implies that there may be an infinity of such definitions;
(v) an approximation to a factional differential is developed that provides a non-singular kernel;
(vi) asymptotic solutions are introduced that yield a model for a self-affine stochastic time series based on the Generalised Kolmogorov-Feller Equation coupled with a memory function that specifically determines the self-affine characteristics of the time series;
(vii) example results are presented which are based on the application of this approximation and time series model to simulate a non-Ergodic random walk field.

## III. Notation and Fundamental Results

We define the Fourier and inverse Fourier transforms in the 'non-unitary form' as

$$F(k)=\mathcal{F}[f(x)]\equiv\int_{-\infty}^{\infty} f(x)\exp(-ikx)dx$$

and

$$f(x)=\mathcal{F}^{-1}[F(k)]\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\exp(ikx)dk$$

respectively, where $\mathcal{F}$ is the forward Fourier transform operator, $\mathcal{F}^{-1}$ is the inverse Fourier transform operator, $x$ denotes the independent variable in a one-dimensional space and $k$ is the spatial frequency. These integral transforms define a Fourier transform pair which, in this paper, we express using the notation $F(k)\leftrightarrow f(x)$. Using this notation, we note the following Fourier transform pairs [15]:

$$\frac{1}{(ik)^\alpha}\leftrightarrow\frac{H(x)}{\Gamma(\alpha)}\frac{1}{x^{1-\alpha}},\ 0<\alpha<1 \tag{1}$$

where $H(x)$ is the Heaviside step function

$$H(x)=\begin{cases}1, & x\geq 0;\\ 0, & x<0.\end{cases}$$

and $\Gamma(\alpha)$ is the Gamma function defined as

$$\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}\exp(-x)dx,\ \ \mathrm{Re}[\alpha]>0.$$

For the Dirac delta function denoted by $\delta(k)$,

$$H(x)\leftrightarrow\frac{1}{ik}+\pi\delta(k).$$

For the sign function denoted by $\mathrm{sgn}(x)$,

$$\mathrm{sgn}(x)\leftrightarrow\frac{2}{ik} \tag{2}$$

where

$$\mathrm{sgn}(x)=\frac{x}{\mid x\mid}=\begin{cases}+1, & x>0;\\ -1, & x<0;\\ 0, & x=0.\end{cases}$$

Finally, for the function $\mid x\mid^\alpha,\ -1<\alpha<0$,

$$\mid x\mid^\alpha\leftrightarrow-\frac{2\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(1+\alpha)}{\mid k\mid^{1+\alpha}} \tag{3}$$

## IV. Conventional Definitions for a Fractional Integral and Differential

In order to provide a context to the remit of this paper, some conventional definitions for a fractional integral and (fractional) differential are introduced that are based exclusively on the inter-related properties of the Fourier transform and the convolution integral.

### A. Formulation based on the Fourier Transform

For a function $f(x)$ with Fourier transform $F(k)$, it is well known that multiple differentiation or integration (applied in the sense of integration being the anti-derivative and any constant of integration is ignored) yields the relationship

$$\frac{d^{\pm n}}{dx^{\pm n}}f(x)\leftrightarrow(ik)^{\pm n}F(k)$$

where $n$ is an integer. Because the order of the differential is $\pm n$, the differential is an integer differo-integral operator.

The key to deriving a Fourier transform based definition for a fractional differo-integral is to simply generalise this result to the case when $n$ is replaced by a non-integer $n+\alpha$ where $\alpha\in(0,1)$. Thus, suppose we wish to solve (i.e. to fractionally integrate) the fractional differential equation (for $n=0$)

$$\frac{d^\alpha}{dx^\alpha}g(x)=f(x)$$

to obtain a solution for $g(x)$ in terms of $f(x)$. Fourier transforming, we have

$$G(k)=\frac{F(k)}{(ik)^\alpha},\ F(k)\leftrightarrow f(x),\ G(k)\leftrightarrow g(x)$$

and using the convolution theorem, we can write

$$g(x)=h(x)\otimes f(x)\equiv\int_{-\infty}^{\infty} h(x-y)f(y)dy \tag{4}$$

where

$$h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(ikx)}{(ik)^{\alpha}}dk = \frac{H(x)}{\Gamma(\alpha)}\frac{1}{x^{1-\alpha}}$$

using Relationship (1).

### *B. The Riemann-Liouville Fractional Integral*

Expressing $g(x)$ in terms of the convolution integral $h(x)\otimes f(x)$ provides a primary definition for a fractional integral. The Riemann-Liouville fractional integral specifies limits on the convolution integral and is usually defined in terms of an anti-derivative operator denoted by $D^{-\alpha}$ as

$$_{a}D_{x}^{-\alpha}f(x) \equiv \frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{f(y)}{(x-y)^{1-\alpha}}dy \tag{5}$$

Equation (5) expresses the fractional integral in terms of an inverse differential operator $D^{-\alpha}$ where the limits of the integral are from $a$ to $x$. This can include the case when $a=-\infty$ subject to suitable restrictions on the function $f(x)$. The integral $_{-\infty}D_{x}^{-\alpha}$ was originally studied by Liouville who was one of the first mathematicians to re-engage with the concept of fractional calculus in 1832. Similarly, the Wyle fractional integral is defined as

$$_{x}D_{\infty}^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}\frac{f(y)}{(x-y)^{1-\alpha}}dy$$

### *C. Definition of a Fractional Derivative*

A fractional differential, denoted by the operator $D^{\alpha}\equiv d^{\alpha}/dx^{\alpha}$, can be defined in terms of a fractional integral given that

$$D^{\alpha}f(x) = D^{1}\ D^{\alpha-1}f(x) = D^{1}\ D^{-(1-\alpha)}f(x)$$

The precise definition of a fractional derivative is then specific to the limits associated with the fractional integral.

For the case when $x\in[a,b]$, using Leibniz integral rule,

$$\begin{aligned} _{a}D_{b}^{\alpha}f(x) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{a}^{b}\frac{f(y)}{(x-y)^{\alpha}}dy \\ &= \frac{-\alpha}{\Gamma(1-\alpha)}\int_{a}^{b}\frac{f(y)}{(x-y)^{1+\alpha}}dy \end{aligned} \tag{6}$$

Given that $\Gamma(1+\alpha)=\alpha\Gamma(\alpha)$, we can consider a definition for a differo-integral operator of the type

$$_{a}D_{b}^{\pm\alpha}f(x) = \frac{1}{\Gamma(\mp\alpha)}\int_{a}^{b}\frac{f(y)}{(x-y)^{1\pm\alpha}}dy \tag{7}$$

Note that because convolution is a commutative operation, we can also write this operator in the form

$$_{a}D_{b}^{\pm\alpha}f(x) = \frac{1}{\Gamma(\mp\alpha)}\int_{a}^{b}\frac{f(x-y)}{y^{1\pm\alpha}}dy$$

Also note that

$$_{-\infty}D_{\infty}^{\pm\alpha}f(x) = \frac{1}{\Gamma(\mp\alpha)}\int_{-\infty}^{\infty}\frac{f(y)}{(x-y)^{1\pm\alpha}}dy \tag{8}$$

### *D. The Caputo Fractional Derivative*

The Caputo fractional derivative is given by

$$_{a}D_{x}^{\nu}f(x) = \int_{a}^{x}K_{\nu}(x-y)f^{(n)}(y)dy$$

where

$$K_{\nu}(x-y) = \frac{(x-y)^{n-\nu-1}}{\Gamma(n-\nu)},\ n-1<\nu<n$$

This result is easily formulated via application of the inverse Fourier transform if we consider the result

$$\begin{aligned} _{-\infty}D_{\infty}^{-\nu}f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(ikx)}{(ik)^{\nu}}F(k)dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(ik)^{n}\exp(ikx)}{(ik)^{n+\nu}}F(k)dk \end{aligned}$$

so that we can write, using Relationship (1), as

$$\begin{aligned} _{-\infty}D_{\infty}^{-\nu}f(x) &= \frac{1}{2\pi}\frac{d^{n}}{dx^{n}}\int_{-\infty}^{\infty}\frac{\exp(ikx)}{(ik)^{n+\nu}}F(k)dk \\ &= \frac{d^{n}}{dx^{n}}\frac{H(x)}{\Gamma(n+\nu)}\frac{1}{x^{1-(n+\nu)}}\otimes f(x) \end{aligned}$$

where $n+\nu\in(0,1)$ and thus

$$\begin{aligned} _{-\infty}D_{\infty}^{\nu}f(x) &= \frac{H(x)}{\Gamma(n-\nu)}\frac{d^{n}}{dx^{n}}\frac{1}{x^{1-(n-\nu)}}\otimes f(x) \\ &= \frac{H(x)}{\Gamma(n-\nu)}\frac{1}{x^{1-(n-\nu)}}\otimes f^{(n)}(x) \end{aligned}$$

where $n-\nu\in(0,1)\ \Rightarrow\nu\in(n-1,n)$, the relationship between $\nu$ and $\alpha$ being given by $\nu=n-\alpha$.

### *E. Discussion*

From the examples given above for formulating a definition of the operators $_{a}D_{x}^{\alpha}$ and $_{a}D_{x}^{-\alpha}$, is it clear that they are based on a generalisation of the Fourier transform under differentiation and integration, respectively. This approach is analogous to the generalisation of the factorial of an integer number to that of a non-integer which yields the Gamma function, a function that arises as a natural consequence in the generalisations used to define many fractional differo-integral operators.

Traditional (integer) calculus goes hand-in-hand with a geometrical interpretation of the associated operations, starting with a differential defining the gradient of a function at a point, at least for a piecewise continuous function. This is not the case for generalised functions that are differentiable only in a distributional sense and classes of functions that are non-differentiable everywhere.

With the fractional calculus, the generalisations used do not easily lend themselves to a geometrical interpretation. Nevertheless, geometric and physical interpretations of a fractional derivative have been attempted, e.g. [16] and [17], including the connection between fractional calculus and fractal geometry [18].

A fundamental kernel of such interpretations is that the function $g(x)$ defined by Equation (4) is scale invariant, given that for some scale length $\lambda > 0$,

$$g_\lambda(x) = \frac{1}{\Gamma(\alpha)x^{1-\alpha}} \otimes f(\lambda x) = \frac{g(\lambda x)}{\lambda^\alpha}$$

This scaling property is a characteristic of self-affine functions and is consistent with a Power Spectral Density Function (PSDF) for $g(x)$ that is characterised by $\mid k \mid^{-2\alpha}$, assuming that the PSDF of $f(x)$ is a constant. This spectrum is a principal 'signature' for a 'fractal signal', the relationship between $\alpha$ and the fractal dimension $D \in (1,2)$ being given by $\alpha = 2.5 - D$ [19]. In this context, $g(x)$ can be taken to be a fractal signal and the solution to a fractional differential equation considered to yield a self-affine function.

The 'process' of generalising the differo-integral properties for the Fourier transform from integer to non-integer order is just one such generalisation that can be made. The same approach can be used for other integral transforms such as the Mellin transform. Thus, the operators defined by Equations (5) and (6), for example, are not unique and there are many definitions and generalisations of a fractional derivative which have been developed [20] and continue to be so [21]. In this context, a new example of a fractional differo-integral is provided in Section V and in Section VI, a generalised definition is considered.

Irrespective of the non-unique definition of a fractional derivative, there is one fundamental difference between a classical and a fractional derivative which is characterised by Equations (7) and (8), for example. A $n^{\text{th}}$ order derivative of a piecewise continuous function $f(x)$ can be defined at a single point on $x$ at $x_0$ say, and, is independent of any other values of $f(x)$ for $x < x_0$ or $x > x_0$. However, given that a fractional derivative involves the convolution of the function $f(x)$ with $1/x^{1+\alpha}$, for example, its value at a point $x_0$ depends on *a priori* values of $f(x),\ x < x_0$. In this context, the $n+\alpha$ derivative of a fractionally differentiable function at a point $x_0$ is only a local property of the function when $\alpha$ is 0 or 1. In other words, the fractional derivative at a point $x_0$ of a function $f(x)$ does not depend on values of $f(x)$ close to $x_0$ in the way that integer order derivatives do. The value of a fractional derivative of $f(x)$ depends on the 'history' of the function and therefore, unlike an integer derivative, a fractional derivative has 'memory'.

### *F. Further Example Definitions*

There have been and continue to be an increasing number of definitions of fractional integrals and derivatives. These results are typically based on generalisations of results which includes the multiple differentiation and/or integration of a function through a closed-form definition or transformation. Examples include the following:

**Fractional Integrals** $(0 < \alpha < 1)$

- Hadamard operator [22]:

$$_aD_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x \frac{1}{y} \left( \ln \frac{x}{y} \right)^{\alpha-1} f(y)dy,\ x > \alpha$$

  where $\ln \equiv \log_e$ denotes the natural logarithm.

- Erdélyi-Kober operator [23], [24]:

$$_0D_x^{-\alpha} f(x) = \frac{x^{1-\alpha-\nu}}{\Gamma(\alpha)} \int\limits_0^x \frac{y^{-\alpha-\nu}}{(y-x)^{1-\alpha}} dy$$

  which is a generalization of the Riemann-Liouville and Wyle fractional integrals.

- Atangana-Baleanu operator [25]:

$$_aD_x^{-\alpha} f(x) = \frac{1-\alpha}{N(\alpha)} + \frac{\alpha}{N(\alpha)} \frac{1}{\Gamma(\alpha)} \int\limits_a^x \frac{f(y)dy}{(x-y)^{1-\alpha}}$$

  for normalization condition $N(0) = N(1) = 1$.

In general, fractional integral operators of the Riemann-Liouville type and the Wyle type have the general forms (through induction)

$$_{-\infty}D_x^{-\alpha} f(x) = x^{\alpha-1} \int\limits_{-\infty}^x \Phi\left(\frac{y}{x}\right) y^{-\alpha} f(y)dy$$

and

$$_xD_\infty^{-\alpha} f(x) = x^{-\alpha} \int\limits_x^\infty \Phi\left(\frac{x}{y}\right) y^{\alpha-1} f(y)dy$$

respectively, where the kernel $\Phi$ is an arbitrary continuous function so that the integrals above make sense in sufficiently large functional spaces.

**Fractional Derivatives** $(q = n + \alpha,\ 0 < \alpha < 1)$

- Wyle derivative:

$$D_x^q f(x) = \sum_{n=-\infty}^{\infty} a_n (in)^q \exp(inx)$$

  where $a_0 = 0$.

- Reisz derivative:

$$D_x^q f(x) = -\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \mid k \mid^q F(k) \exp(ikx) dk$$

  where $F(k)$ is the Fourier transform of $f(x)$.

- Laplace transform based derivative (using the Bromwich integral):

$$D_x^q f(x) = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} p^q \exp(px) F(p) dp$$

  where $F(p)$ is the Laplce transform of $f(x)$ given by

$$F(p) = \int\limits_0^\infty f(x) \exp(-px) dx$$

- Mellin transform based derivative [26]:

$$D_x^q f(x) = \frac{(-1)^q}{2\pi i x^q} \int\limits_{c-i\infty}^{c+i\infty} \frac{\Gamma(q+s)}{\Gamma(s)x^s} F(s) ds$$

where $F(s)$ is the Mellin transform of $f(x)$, i.e.

$$F(s) = \int_0^\infty x^{s-1} f(x)dx, \ f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} F(s) ds$$

and it is noted that

$$\frac{d^q}{dx^q} x^{-s} = (-1)^q \frac{\Gamma(q+s)}{\Gamma(s)} x^{-(q+s)}, \ q \geq 0$$

- Caputo-Fabrizio derivative [27]:

$$_aD_x^\alpha f(x) = \frac{1}{1-\alpha} \int_a^x \left[\frac{d}{dx} f(x)\right] \exp\left(-\alpha \frac{x-y}{1-\alpha}\right) dy, \tag{9}$$

a definition whose kernel is non-singular.

- Grünwald-Letnikov derivative [28]:

$$D^q f(x) =$$

$$\lim_{\Delta \to 0} \frac{1}{\Delta^q} \times \sum_{0 \leq m < \infty} (-1)^m \frac{\Gamma(1+q) f[x + (q-m)\Delta]}{\Gamma(1+m)\Gamma(1+q-m)}$$

## V. New Definition of a Fractional Differential based on the Sign Function

Given the increasing number of definitions for a fractional differo-integral, we now consider another example of such a case which is based on the following theorem.

### Theorem

The fractional differo-integral $f^{(\gamma)}(x)$ of a function $f(x)$ can be written, without loss of generality, as

$$f^{(\gamma)}(x) = \frac{1}{2} f(x) \otimes \text{sgn}^{(1+\gamma)}(x), \ \gamma \in (-\infty, \infty)$$

### Proof

For any positive or negative integer $n$ (including $n = 0$),

$$f^{(n)}(x) \leftrightarrow (ik)^n F(k)$$

and since we can write

$$(ik)^n F(k) = \frac{1}{2} \frac{2}{ik} (ik)^{1+n} F(k),$$

then, with the application of Relationship (2) and the convolution theorem, we have

$$\frac{1}{2} \frac{2}{ik} (ik)^{1+n} F(k) \leftrightarrow \frac{1}{2} \text{sgn}(x) \otimes \delta^{(1+n)}(x) \otimes f(x)$$

$$= \frac{1}{2} \text{sgn}^{(1+n)}(x) \otimes f(x)$$

Hence, generalizing this result for any integer or non-integer value $\gamma$, we can write

$$f^{(\gamma)}(x) = \frac{1}{2} \text{sgn}^{(1+\gamma)}(x) \otimes f(x) \tag{10}$$

### Corollary 1

Using the result

$$\frac{1}{2} \text{sgn}^{(1+\gamma)}(x) \otimes f(x) = \frac{1}{2} \text{sgn}^{(1+\gamma)}(x) \otimes f(x) \otimes \delta(x)$$

$$= \frac{1}{2} \text{sgn}(x) \otimes f^{(1+\gamma)}(x) \otimes \delta(x) = \frac{1}{2} \text{sgn}(x) \otimes f(x) \otimes \delta^{(1+\gamma)}(x)$$

we can consider the following specific cases to illustrate the compatibility of the definition given by Equation (10) with conventional Fourier based definitions of an integer derivative and (integer) integral.

For $\gamma = 1$:

$$\frac{1}{2} \text{sgn}^{(2)}(x) \otimes f(x) = \frac{1}{2} \text{sgn}(x) \otimes f(x) \otimes \delta^{(2)}(x)$$
$$\leftrightarrow \frac{1}{2} \frac{2}{ik} F(k)(ik)^2 = ikF(k)$$

For $\gamma = -1$:

$$\frac{1}{2} \text{sgn}^{(0)}(x) \otimes f(x) = \frac{1}{2} \text{sgn}(x) \otimes f(x) \otimes \delta^{(0)}(x)$$
$$\leftrightarrow \frac{1}{2} \frac{2}{ik} F(k) = \frac{F(k)}{ik}$$

Also, for $\gamma = 0$:

$$\frac{1}{2} \text{sgn}^{(1)}(x) \otimes f(x) = \frac{1}{2} \text{sgn}(x) \otimes f(x) \otimes \delta^{(1)}(x)$$
$$\leftrightarrow \frac{1}{2} \frac{2}{ik} ikF(k) = F(k)$$

### Corollary 2

If we write

$$\frac{F(k)}{(ik)^{n+\alpha}} = F(k) \frac{1}{(ik)^\alpha} \frac{1}{2^n} \prod_{m=1}^n \frac{2}{(ik)}$$

where

$$\prod_{m=1}^n \frac{2}{ik} \equiv \frac{2}{ik} \frac{2}{ik} \ldots = \frac{2^n}{(ik)^n}$$

then, using the convolution theorem and noting Relationship (1), we obtain

$$f^{-(n+\alpha)}(x) = \frac{1}{2^n} f(x) \otimes \prod_{m=1}^{n}{}^{\otimes} \ \text{sgn}(x) \otimes \frac{1}{\Gamma(\alpha)} \frac{H(x)}{x^{1-\alpha}}$$

where, for $n$ convolutions,

$$\prod_{m=1}^{n}{}^{\otimes} \ \text{sgn}(x) \equiv \text{sgn}(x) \otimes \text{sgn}(x) \otimes \ \ldots \ \otimes \text{sgn}(x)$$

This provides an expression for the fractional integral of $n+\alpha$ in terms of $n$ convolutions of the sign function.

### COROLLARY 3

To define a fractional differential of order $n-\alpha,\ n>\alpha$ using the same principle, we note that

$$F(k)(ik)^{n-\alpha}=F(k)(ik)^{n}\frac{1}{(ik)^{\alpha}}=F(k)(ik)^{2n}\frac{1}{(ik)^{n}}\frac{1}{(ik)^{\alpha}}$$
$$=F(k)(ik)^{2n}\frac{1}{(ik)^{\alpha}}\frac{1}{2^{n}}\prod_{m=1}^{n}\frac{2}{(ik)}$$

and thus, using the convolution theorem,

$$f^{(n-\alpha)}(x)=\frac{1}{2^{n}}f^{(2n)}(x)\otimes\prod_{m=1}^{n}\!\!\otimes\ \mathrm{sgn}(x)\otimes\frac{1}{\Gamma(\alpha)}\frac{H(x)}{x^{1-\alpha}}$$
$$=\frac{1}{2^{n}}f(x)\otimes\prod_{m=1}^{n}\!\!\otimes\ \mathrm{sgn}^{(2)}(x)\otimes\frac{1}{\Gamma(\alpha)}\frac{H(x)}{x^{1-\alpha}}$$

which expresses the fractional derivative in terms of $n$ convolutions of the second differential of the sign function. In this context, noting that $\mathrm{sgn}(x)\simeq\tanh(ax),\ a>>1$, we can use the approximation

$$\mathrm{sgn}^{(2)}(x)\simeq-2a^{2}\mathrm{sech}^{2}(ax)\tanh(ax)$$

for application in the numerical computation of a $(n-\alpha)$-order fractional derivative based on the application of multiple convolution sums.

### COROLLARY 4

Since $\mathrm{sgn}(x)=2H(x)-1$, for $\gamma\geq0$, the convolution kernel in Equation (10) can be written as

$$\frac{1}{2}\mathrm{sgn}^{(1+\gamma)}(x)=H^{(1+\gamma)}(x)$$

Moreover, because

$$H(x)\leftrightarrow\frac{1}{ik}+\pi\delta(k)$$

then

$$\frac{1}{2}\mathrm{sgn}^{(1+\gamma)}(x)=\frac{1}{2}\mathrm{sgn}(x)\otimes\delta^{(1+\gamma)}(x)$$
$$=H(x)\otimes\delta^{(1+\gamma)}(x)\leftrightarrow\left[\frac{1}{ik}+\pi\delta(k)\right](ik)^{1+\gamma}=(ik)^{\gamma}$$

given that

$$\pi\delta(k)(ik)^{1+\gamma}=\begin{cases}0, & \mid k\mid=0;\\ 0, & \mid k\mid>0.\end{cases}$$

and hence, we can write Equation (10) as

$$f^{(\gamma)}(x)=H^{(1+\gamma)}(x)\otimes f(x),\ \gamma\geq0$$

### REMARK 1

Relationship (2), as used in the derivation of Equation (10), can be obtained by noting that

$$\mathrm{rect}(ax)=\frac{1}{2}[\mathrm{sgn}(x+a)-\mathrm{sgn}(x-a)]$$

where

$$\mathrm{rect}(ax)=\begin{cases}1, & \mid x\mid\leq a;\\ 0, & \mid x\mid>a.\end{cases}$$

Then, since

$$\mathrm{rect}(ax)\leftrightarrow2a\mathrm{sinc}(ka)=2a\frac{\sin(ka)}{ka}$$
$$=\frac{1}{ik}[\exp(ika)-\exp(-ika)]$$

we can write

$$\mathrm{sgn}(x+a)-\mathrm{sgn}(x-a)\leftrightarrow\exp(ika)\frac{2}{ik}-\exp(-ika)\frac{2}{ik}$$

Thus, using the shift theorem for the Fourier transform, i.e.

$$f(x\pm a)\leftrightarrow\exp(\pm ika)F(k)$$

it is clear that

$$\mathrm{sgn}(x+a)\leftrightarrow\exp(\pm ika)\frac{2}{ik}\ \Rightarrow\ \mathrm{sgn}(x)\leftrightarrow\frac{2}{ik}$$

### REMARK 2

Compared to other definitions of a fractional derivative, Equation (10) places emphasis on computing a fractional differential through the convolution of a function with the fractional derivative of the sign function. In addition to Corollary 2 and Corollary 3, this result is sufficient to deserve further investigation in order to establish its value and/or role (if any) in the fractional calculus. A first approach to initiating such an investigation, is to derive the $n^{\text{th}}$ derivative for different Sigmoid-type functions (the Hyperbolic tangent, Gudermannian function and the Error function, for example), and, where appropriate (in regard to generated a self-consistent and closed-form formula) generalise the result for the non-integer case. In this context, some example definitions based on Sigmoid-type functions are derived below.

### DEFINITION USING THE INVERSE TANGENT FUNCTION

Suppose we work with the inverse tangent Sigmoid-type function $\tan^{-1}(x)$ given that

$$\mathrm{sgn}(x)\simeq\frac{2}{\pi}\tan^{-1}(a\pi x)$$

In this case, a closed form formula for the $n^{\text{th}}$ derivative of $\tan^{-1}(x)$ is [29]

$$\frac{d^{n}}{dx^{n}}\tan^{-1}(x)=\frac{(-1)^{n-1}(n-1)!}{(1+x^{2})^{n/2}}$$
$$\times\sin\left[n\sin^{-1}\left(\frac{1}{\sqrt{1+x^{2}}}\right)\right],\ n=1,2,3,...$$

Thus, given Equation (10), we can write the convolution kernel as (for $\gamma\geq0$)

$$\frac{1}{2}\mathrm{sgn}^{(1+\gamma)}(x)\simeq\frac{(-\pi)^{\gamma}a^{1+\gamma}\Gamma(1+\gamma)}{[1+(a\pi x)^{2}]^{(1+\gamma)/2}}$$
$$\sin\left[(1+\gamma)\sin^{-1}\left(\frac{1}{\sqrt{1+(a\pi x)^{2}}}\right)\right]\qquad(11)$$

thereby providing yet another new and original (judging from the open literature) definition of a fractional differential. To fractionally integrate using this definition, if we let $\gamma=n-\alpha,\ \alpha\in(0,1)$, then the convolution kernel for a fractional integral can be written in the form

$$\frac{1}{2}\mathrm{sgn}^{(1-\gamma)}(x)=\frac{1}{2}\mathrm{sgn}^{(1-n)}(x)\otimes\frac{1}{2}\mathrm{sgn}^{(1+\alpha)}(x)\qquad(12)$$

### Definition based on the Logistic Function

By way of another example, the sign function can be approximated by the logistic function, i.e.

$$\operatorname{sgn}(x) \simeq f(x) = \frac{1}{1+\exp(-x)},$$

This function has the property

$$\frac{d}{dx} f(x) = f(x)[1 - f(x)]$$

or alternatively,

$$\frac{d}{dx} \ln[f(x)] = 1 - f(x)$$

Thus,

$$\frac{d}{dx} f(x) = -\frac{d^2}{dx^2} \ln[f(x)]$$

$$\Rightarrow \frac{d^n}{dx^n} f(x) = -\frac{d^{2+n}}{dx^{2+n}} \ln[f(x)], \ n = 1, 2, 3, ...$$

so that upon generalising this results, we can write

$$\frac{d^\gamma}{dx^\gamma} f(x) = -\frac{d^{2+\gamma}}{dx^{2+\gamma}} \ln[f(x)]$$

leading to the result

$$\frac{1}{2}\operatorname{sgn}^{(1+\gamma)}(x) \simeq -\frac{1}{2}\frac{d^{3+\gamma}}{dx^{3+\gamma}} \ln[f(x)], \ \gamma \geq 0$$

If we now let $\gamma = n - \alpha$, then

$$\frac{1}{2}\operatorname{sgn}^{(1+\gamma)}(x) \simeq -\frac{1}{2}\frac{d^{3+n}}{dx^{3+n}}\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{\infty}\frac{\ln[f(y)]}{(x-y)^{1-\alpha}}dy$$

$$= \frac{1}{2}\frac{d^{3+n}}{dx^{3+n}}\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{\infty}\frac{\ln[(1+\exp(-y)]}{(x-y)^{1-\alpha}}dy \qquad (13)$$

The associated fractional integral is then obtained through Equation (12).

### Definition using the Hyperbolic Tangent Function

The logistic function $f(x)$ defined above, is related to the hyperbolic tangent function $\tanh$ by

$$f(x) = \frac{1}{2}[1 + \tanh(x/2)]$$

and thus, from Equation (13), we can write

$$\frac{1}{2}\operatorname{sgn}^{(1+\gamma)}(x) \simeq$$

$$\frac{1}{2}\frac{d^{3+n}}{dx^{3+n}}\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{\infty}\frac{0.6931 - \ln\left[1+\tanh(y/2)\right]}{(x-y)^{1-\alpha}}dy \qquad (14)$$

## VI. Formulation and Generalisation of a Fractional Derivative based on the Delta Function

The continued number and and variable types of definitions for a fractional differo-integral operator, raises the question as to whether there is a finite or infinite class of such operators. In this section, we discuss this question by considering another related approach to defining a fractional differential through application of the delta function.

Given that

$$f(x) = \delta(x) \otimes f(x)$$

we have

$$f^{(n)}(x) = \delta(x) \otimes f^{(n)}(x) = \delta^{(n)}(x) \otimes f(x)$$

where

$$f^{(n)}(x) \equiv \frac{d^n}{dx^n} f(x)$$

Generalizing this result to the non-integer case, we write

$$f^{(\alpha)}(x) = \delta^{(\alpha)}(x) \otimes f(x)$$

where, given that

$$\delta^{(n)}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(ik)^n \exp(ikx)dk$$

we consider the case when

$$\delta^{(\alpha)}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(ik)^\alpha \exp(ikx)dk$$

We can then write, using Relationship (1),

$$\delta^{(\alpha)}(x) = \frac{d}{dx}\delta^{(\alpha-1)}(x) = \frac{d}{dx}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(ik)^{1-\alpha}}\exp(ikx)dk$$

$$= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\frac{H(x)}{x^\alpha} = \frac{1}{\Gamma(1-\alpha)}\frac{\delta(x)}{x^\alpha} - \frac{\alpha}{\Gamma(1-\alpha)}\frac{H(x)}{x^{1+\alpha}}$$

### Conjecture

There is no upper bound to the number of self-consistent definitions that can be derived to define a fractional differential.

### Justification

The basis for this conjecture is that

$$f^{(\alpha)}(x) = \int_{-\infty}^{\infty}\delta^{(\alpha)}(x-y)f(y)dy$$

where we let

$$\delta^{(\alpha)}(x-y) = D^1 \, {}_aD_x^{-(1-\alpha)}\delta(x-y)$$

Consider a class of linear differential operators $\mathcal{L}$, for which there exits a corresponding Green's function $g(x,y)$ such that

$$\mathcal{L}g(x,y) = \delta(x-y) \qquad (15)$$

where the operator is taken to operate on the independent variable $x$ alone. If the operator has constant coefficients with respect to $x$ and is thereby translation invariant, then this Green's function can be taken to be a convolution kernel, i.e.

$g(x, y) = g(x - y)$. Irrespective of this property, the linear operation on the Green's function defines the delta function. Thus, in principle, we can consider the definition

$$f^{(\alpha)}(x) = f(x) \otimes [D^1 \ {}_aD_x^{-(1-\alpha)} \mathcal{L}g(x, y)]$$
$$= f(x) \otimes \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{\mathcal{L}g(x, y)}{(x-y)^\alpha} dy \quad (16)$$

so that if we assume that there is an unlimited number of linear differential operators $\mathcal{L}$ for which the Green's function exists, then the conjecture can be supported. However, this presupposes a limitless number of linear operators and corresponding Green's function for which Equation (15) holds and this is not proven.

### A. Interpretation based on the Green's Function

A better way of interpreting this result is to consider the Green's function in terms of providing a solution to the equation

$$\mathcal{L}u(x) = f(x), \ x \in [a, b] \quad (17)$$

when

$$u(x) = \int_a^b g(x, y) f(y) dy \quad (18)$$

The critical issue here, is that the Green's function can always be obtained if the equivalent eigenvalue problem can be solved for the same operator $\mathcal{L}$, i.e.

$$\mathcal{L}u(x) = \lambda u(x) \quad (19)$$

as shall now be demonstrated.

### B. Complete Eigenfunctions Analysis

If we let

$$u(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) \quad (20)$$

and

$$f(x) = \sum_{n=0}^{\infty} f_n \phi_n(x), \ f_n = \int_a^b f(y) \phi_n^*(y) dy$$

then, given Equations (17), (18) and (19),

$$\mathcal{L}u(x) = \mathcal{L} \sum_{n=0}^{\infty} c_n \phi_n(x) = \sum_{n=0}^{\infty} c_n \mathcal{L} \phi_n(x)$$
$$= \sum_{n=0}^{\infty} c_n \lambda_n \phi_n(x) = \sum_{n=0}^{\infty} f_n \phi_n(x)$$

so that after equating the coefficients of the independent eigenfunctions we can write

$$c_n = \frac{f_n}{\lambda_n} = \frac{1}{\lambda_n} \int_a^b f(y) \phi_n^*(y) dy$$

From Equation (20),

$$u(x) = \sum_{n=0}^{\infty} \frac{1}{\lambda_n} \left[ \int_a^b f(y) \phi_n^*(y) dy \right] \phi_n(x)$$
$$= \int_a^b \left[ \sum_{n=0}^{\infty} \frac{1}{\lambda_n} \phi_n^*(y) \phi_n(x) \right] f(y) dy$$

and thus, by comparing this result with Equation (18),

$$g(x, y) = \sum_{n=0}^{\infty} \frac{1}{\lambda_n} \phi_n^*(y) \phi_n(x)$$

which gives the Green's function in terms of the eigenvalues and eigenfunctions of the operator $\mathcal{L}$. Further, from this bilinear expansion of the Green's function,

$$\mathcal{L}g(x, y) = \sum_{n=0}^{\infty} \frac{1}{\lambda_n} \phi_n^*(y) \mathcal{L} \phi_n(x) = \sum_{n=0}^{\infty} \phi_n^*(y) \phi_n(x)$$

Hence we can write the delta function in the form

$$\delta(x - y) = \sum_{n=0}^{\infty} \phi_n(x) \phi_n^*(y) \quad (21)$$

In the context of Equation (21), the conjecture relies on an assumed unlimited number of linear operators whose eigenfunctions are complete when, for any such function $\phi(x)$,

$$f^{(\alpha)}(x) = f(x) \otimes \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \sum_{n=0}^{\infty} \phi_n(x) \int_a^x \frac{\phi_n^*(y)}{(x-y)^\alpha} dy \quad (22)$$

### C. A Quantum Mechanical Argument

In (non-relativistic) quantum mechanics, the linear operator $\mathcal{L}$ is the Hamiltonian energy operator $H$, the eigenvalues are the (discrete) energy levels $E_n$ and the eigenfunctions $\psi_n$ are the wave functions of the electrons. The linear operator and hence, the eigenfunctions, are determined by the potential energy $V$ through the time-independent Schrödinger equation.

For the three-dimension case, when $\mathbf{r} \in \mathcal{R}^3$, this equation is given by

$$H\psi_n(\mathbf{r}) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) \right] \psi_n(\mathbf{r}) = E_n \psi_n(\mathbf{r})$$

where $\hbar$ is the Dirac constant and $m$ is the mass of an electron. The eigenfunctions describe the standing wave patterns of the electrons. There are many one-dimensional models used in quantum mechanics which allows the problem to be reduced to the simpler form (for $\hbar = m = 1$)

$$\left[ -\frac{1}{2} \frac{d^2}{dx^2} + V(x) \right] \psi_n(x) = E_n \psi_n(x)$$

Numerous potentials [i.e. different model functions for the potential $V(x)$] are then considered and, where possible, the eigenfunctions (and corresponding eigenvalues) computed analytically, if possible, and numerically, as necessary. Such models are a very limited representation of reality in terms of the dimensionality that is used and the potentials considered.

In quantum chemistry, the (three dimensional) Hamiltonian operator depends upon the total potential energy associated with the arrangement of nuclei in the atoms of a molecule. The eigenfunctions describe the shapes and orientations of orbitals that characterise electrons in the molecule. Analytical solutions for the eigenfunction are only possible for nuclei with one electron (i.e. 'hydrogenic systems') and for all other cases, approximation methods and/or numerical methods of computation are required. Irrespective of this, for any stable molecular structure, the electrons form a complete set of eigenfunctions, each eigenfunction corresponding to different energy levels for which there are an infinite number beyond which an electron become a free ion. The point here, is that there are, in principle, a limitless number of stable molecular structures that can be formed, each structure relating to a different Hamiltonian for which there exists a complete set of eigenfunctions. Thus, on the basis of non-relativistic quantum chemistry alone, there are as many potentials as there are stable molecular structures. In each case, we can construct a Hamiltonian for which there is an infinite number of eigenfunctions to satisfy the equation

$$\delta^3(\mathbf{r}-\mathbf{s})=\sum_{n=0}^{\infty}\phi_n(\mathbf{r})\phi_n^*(\mathbf{s})$$

even though the eigenfunctions can not be derived analytically (for non-hydrogenic cases). In this context, the infinity of definitions for a fractional differential based on Equations (16) and (21) may be conjectured albeit for the one-dimensional case.

## VII. Approximation to a Fractional Differential

An approximation is considered that is based on the result

$$\mathcal{F}_1^{-1}[\exp(-\mid k\mid^{\alpha})]=\mathcal{F}_1^{-1}[1]+\sum_{n=1}^{\infty}(-1)^n\frac{1}{n!}\mathcal{F}_1^{-1}[\mid k\mid^{n\alpha}]$$
$$=\delta(x)-\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{1}{n!}\left[\frac{\sin\left(\frac{\pi n\alpha}{2}\right)\Gamma(1+n\alpha)}{\mid x\mid^{1+n\alpha}}\right]$$
$$\sim\frac{1}{\mid x\mid^{1+\alpha}},\ \mid x\mid\to\infty$$

using Relationship (3) when we can write

$$\exp(-\mid x\mid^{\alpha-1})\leftrightarrow\frac{1}{\mid k\mid^{\alpha}},\ \mid k\mid\to\infty$$

as a first order approximation.

### A. Approximation Method

Using the approximation

$$\exp(-\mid x\mid^{\alpha-1})\sim 1-\mid x\mid^{\alpha-1},\ \mid x\mid<<1$$

we can write

$$\frac{H(x)}{\Gamma(\alpha)}\exp(-\mid x\mid^{\alpha-1})\sim\frac{H(x)}{\Gamma(\alpha)}-\frac{H(x)}{\Gamma(\alpha)}\frac{1}{\mid x\mid^{1-\alpha}}$$

or

$$\frac{H(x)}{\Gamma(\alpha)}\frac{1}{\mid x\mid^{1-\alpha}}\sim\frac{H(x)}{\Gamma(\alpha)}[1-\exp(-\mid x\mid^{\alpha-1})]$$

and hence,

$$f^{(-\alpha)}(x)=f(x)\otimes\frac{H(x)}{\Gamma(\alpha)}\frac{1}{\mid x\mid^{1-\alpha}}$$
$$\sim f(x)\otimes\frac{H(x)}{\Gamma(\alpha)}[1-\exp(-\mid x\mid^{\alpha-1})]$$

Thus, for $n>\alpha$ we obtain the approximation

$$f^{(n-\alpha)}(x)\sim f^{(n)}(x)\otimes\frac{H(x)}{\Gamma(\alpha)}[1-\exp(-\mid x\mid^{\alpha-1})]\quad(23)$$

This approximation eliminates the singularity of the convolution kernel associated with other definitions of fractional differentials. In this context, the approximation has a synergy with Equation (9).

### B. Asymptotic Analysis

For $n=0$, Equation (23) can be written in the form

$$f^{(-\alpha)}(x)\sim\int_{-\infty}^{\infty}f(x-y)g(y)dy=\int_{-\infty}^{\infty}f(y)g(x-y)dy$$

where

$$g(x)=\frac{H(x)}{\Gamma(\alpha)}[1-\exp(-\mid x\mid^{\alpha-1})]$$

Consider the case when $y<<1$, so that upon applying a Taylor series to $g(x-y)$, we can write

$$g(x-y)\simeq g(x)-y\frac{d}{dx}g(x),\ \mid y\mid<<1$$

Noting that

$$\frac{d}{dx}g(x)=\frac{\delta(x)}{\Gamma(\alpha)}[1-\exp(-x^{\alpha-1})]$$
$$+H(x)[(\alpha-1)x^{\alpha-2}\exp(-x^{\alpha-1})]$$
$$=(\alpha-1)x^{\alpha-2}\exp(-x^{\alpha-1}),\ x>0$$

we obtain an approximation for $f^{(-\alpha)}(x)$ given by

$$f^{(-\alpha)}(x)\sim\frac{1}{\Gamma(\alpha)}[1-\exp(-\mid x\mid^{\alpha-1})]\int_a^x f(y)dy$$
$$+\frac{1}{\Gamma(\alpha)}(\alpha-1)x^{\alpha-2}\exp(-\mid x\mid^{\alpha-1})\int_a^x yf(y)dy,\ a>0\quad(24)$$

It is then clear that as $\alpha\to 1$,

$$f^{(-\alpha)}(x)\sim[1-\exp(-1)]\int_a^x f(y)dy=0.6231\int_a^x f(y)dy$$

given that $\Gamma(\alpha)\to 1$ as $\alpha\to 1$, and

$$f^{(-\alpha)}(x)\sim 0,\ \alpha\to 0$$

given that $\Gamma(\alpha)\to\infty$ as $\alpha\to 0$. Further, $\forall\ \alpha\in(0,1)$

$$f^{(-\alpha)}(x)\sim\frac{\alpha-1}{\Gamma(\alpha)x^{2-\alpha}}\int_a^x yf(y)dy,\ x\to 0$$

and

$$f^{(-\alpha)}(x)\sim\frac{1}{\Gamma(\alpha)}\int_a^x f(y)dy,\ x\to\infty$$

where, in the latter case, when $\alpha=1$

$$f^{(-1)}(x)\equiv\int_a^x f(y)dy$$

thereby re-establishing a conventional integral as an anti-derivative.

## VIII. Application: Stochastic Times Series

The purpose of this section is to show how Equation (24) can be used to model a stochastic time series. This is accomplished in the context of the evolution equation, and, specifically the evolution equation for a memory function that provides a self-affine time series to which Equation (24) can then be applied.

### A. The Evolution Equation

For Probability Density Function (PDF) $p(x)$, the density field $u(x,t)$ describing the density (particles per unit length) of a canonical ensemble of particles undergoing independent random motion due to elastic interactions is given by

$$u(x,t+\tau)=p(x)\otimes u(x,t)+s(x,t) \tag{25}$$

where $s(x,t)$ is a stochastic source function. For any instant in time $t$, Equation (25) shows that the spatial behaviour of the density field at some future time $\tau$ is given by the convolution of the density of particles at a previous time with the PDF of the system that governs its 'statistical evolution'. In this sense, $p(x)$ is analogous to the Impulse Response Function of a linear stationary system when, for an initial condition $u_0(x)\equiv u(x,t=0)$, say,

$$u(x,t)=g(\mid x\mid,t)\otimes u_0(x)$$

where $g(\mid x\mid,t)$ is the characteristic free-space Green's function of the system [30]. However, in this case $u(x,t)$ denotes a deterministic function associated with the behaviour of a deterministic system, whereas in Equation (25), $u(x,t)$ is the density function associated with the evolution of a distribution for a stochastic system. This 'system' is taken to be stationary in a statistical sense because it is assumed that $p(x)$ does not vary in time. The time evolution model given by Equation (25) therefore assumes that the system is 'Ergodic'.

Equation (25) is an evolution equation first derived by Albert Einstein in 1905 [31] from which he derived the diffusion equation. This was achieved by applying a Taylor series in time to the function $u(x,t+\tau)$ to first order and applying a Taylor series in space to the function $u(x,t)$ in the convolution integral to second order as follows:

$$u(x,t)+\tau\frac{\partial}{\partial t}u(x,t)$$

$$\simeq\int\limits_{-\infty}^{\infty}\left[p(y)\left(u(x,t)-y\frac{\partial}{\partial x}u(x,t)+\frac{y^2}{2}\frac{\partial^2}{\partial x^2}u(x,t)\right)\right]dy$$

$$+s(x,t)=u(x,t)+\frac{\partial^2}{\partial x^2}u(x,t)\int\limits_{-\infty}^{\infty}\frac{y^2}{2}p(y)dy+s(x,t)$$

given that $p(x)$ is symmetric so that

$$\int\limits_{-\infty}^{\infty}p(y)dy=1 \text{ and } \int\limits_{-\infty}^{\infty}yp(y)dy=0$$

Hence, we derive the diffusion equation for diffusivity $D$ given by [with $s(t):=s(t)/\tau$]

$$\frac{\partial}{\partial t}u(x,t)=D\frac{\partial^2}{\partial x^2}u(x,t)+s(x,t),\ D=\frac{1}{\tau}\int\limits_{-\infty}^{\infty}\frac{y^2}{2}p(y)dy$$

This is a derivation of the diffusion equation (a parabolic partial differential equation) without reference to physical laws such as the heat flux being proportional to temperature gradient (Fourier's Law) coupled with the continuity equation which yields the heat equation and describes how heat diffuses through a given region. In this context, Equation (25) is a fundamental field equation of statistical physics which can be used derive a range of models for the density function $u(x,t)$ other than the diffusion equation. In this respect, the Kolmogorov-Feller equation provides a valuable framework.

### B. The Kolmogorov-Feller Equation

Consider the following Taylor series for the function $u(x,t+\tau)$ in Equation (25):

$$u(x,t+\tau)=u(x,t)+\tau\frac{\partial}{\partial t}u(x,t)+\frac{\tau^2}{2!}\frac{\partial^2}{\partial t^2}u(x,t)+...$$

For $\tau<<1$

$$u(x,t+\tau)\simeq u(x,t)+\tau\frac{\partial}{\partial t}u(x,t)$$

and from Equation (25), we obtain the Classical Kolmogorov-Feller Equation (CKFE) [32], [33]

$$\tau\frac{\partial}{\partial t}u(x,t)=-u(x,t)+u(x,t)\otimes p(x)+s(x,t) \tag{26}$$

which is a representation of Equation (25) when $\tau<<1$.

Equation (26) is based on a critical assumption which is that the time evolution of the density field $u(x,t)$ is influenced only by short time events and that longer term events have no influence on the behaviour of the field at any later time. This is to say that the 'system' described by Equation (26) has no 'memory'. This statement is the physical basis upon which the condition $\tau<<1$ can be imposed thereby facilitating the Taylor series expansion of the function $u(\mathbf{r},t+\tau)$ to first order alone. It means that if a time series is taken to be described by $u(x_0,t)$ (for some fixed position in space $x_0$) then the behaviour at a time $t$ is not influenced by the behaviour at some earlier time $<t$. This is the basis for understanding classical diffusion, for example.

Given that Equation (26) is memory invariant, the question arises as to how longer temporal influences can be modelled, other than by taking an increasingly larger number of terms in the Taylor expansion of $u(\mathbf{r},t+\tau)$ which is not analytically consequential, i.e. writing Equation (25) in the form

$$\sum_{n=1}^{\infty}\frac{\tau^n}{n!}\frac{\partial^n}{\partial t^n}u(x,t)=-u(x,t)+u(x,t)\otimes p(x)+s(x,t)$$

The key to solving this problem is to consider the idea of expressing the infinite series on the left hand side of the equation above in terms of a 'memory function' $m(t)$ and write

$$\tau m(t)\otimes\frac{\partial}{\partial t}u(x,t)=-u(x,t)+u(x,t)\otimes p(x)+s(x,t) \tag{27}$$

This is the Generalised Kolmogorov-Feller Equation (GKFE). In addition to specifying the source function $s(x,t)$ and the PDF $p(x)$ in order to develop a solution for $u(x,t)$, this equation also requires a memory function to be specified. In this case, if a time series is taken to be described by $u(x_0,t)$ then the behaviour at a time $t$ is influenced by the behaviour at some time $<t$ according to the characteristics of the memory function. This is an example of a stochastic process in which the past influences the future. The time scale over which this effect is possible then depends on the 'width' in time of the memory function where it is noted that the GKFE reduces to the CKFE when $m(t)=\delta(t)$ which is equivalent to applying the condition $\tau<<1$.

### C. Orthonormal Memory Functions

For any inverse function or class of inverse functions of the type $n(t)$, say, such that

$$n(t) \otimes m(t) = \delta(t),$$

the GKFE can be written in the form

$$\tau \frac{\partial}{\partial t} u(x,t)$$
$$= -n(t) \otimes u(x,t) + n(t) \otimes u(x,t) \otimes p(x) + n(t) \otimes s(x,t) \quad (28)$$

where the CKFE is again recovered when $n(t) = \delta(t)$. The function $n(t)$ is a orthonormal function of $m(t)$. Writing the GKFE in this form facilitates the development of solutions for $u(x,t)$ when $n(t)$ can be derived from $m(t)$. This is possible, given that in Fourier space, the orthonormality relationship between $m(t)$ and $n(t)$ is (using the convolution theorem)

$$N(\omega) = \frac{1}{M(\omega)} \quad (29)$$

where $N(\omega)$ and $M(\omega)$ are the Fourier transforms of $n(t)$ and $m(t)$, respectively.

### D. Time Series Models

Equation (28) is a description for a density field that is dependent on both space and time. If we are interested in using this equation for simulating and analysing data that are time series alone, it is necessary to develop a time-only series model. A method of doing this is to note that when $p(x) = \delta(x)$, we can write

$$\tau \frac{d}{dt} u(t) = n(t) \otimes s(t) \quad (30)$$

where

$$u(t) \equiv \int u(x,t)dx \text{ and } s(t) \equiv \int s(x,t)dx$$

A more general method is to consider an asymptotic result for the spatial component of Equation (28). To do this, we note that using a Taylor expansion for the convolution integral, we can write

$$u(x,t) \otimes p(x) = \int p(x-y)u(y,t)dy$$
$$= \int \left[ p(x) - y\frac{d}{dx}p(x) + ... \right] u(y,t)dy$$
$$= p(x) \int u(y,t)dy + \frac{d}{dx}p(x) \left[ \int yu(y,t)dy \right] + ...$$

If the PDF is such that $p^{(n)}(x) \sim 0$ as $x \to \infty, \ \forall \ n = 0, 1, 2, ...$ then,

$$u(x,t) \otimes p(x) \sim 0 \text{ as } x \to \infty$$

The contribution of the term $n(t) \otimes u(x,t) \otimes p(x)$ in Equation (28) then becomes insignificant, and we can consider the time-only dependent asymptotic equation

$$\tau \frac{d}{dt} u(t) = -n(t) \otimes u(t) + n(t) \otimes s(t) \quad (31)$$

where $u(t) \equiv u(x,t), \ x \to \infty$ and $s(t) \equiv s(x,t), \ x \to \infty$.

The essential difference between Equation (31) and Equation (30) is compounded in the inclusion or otherwise of the term $-n(t) \otimes u(t)$, respectively. In the former case, the spectral response of $u(t)$ to $s(t)$ is determined by the Transfer Function

$$T(\omega) = \frac{N(\omega)}{N(\omega) + i\omega\tau}$$

In the latter case the Transfer Function is

$$T(\omega) = \frac{N(\omega)}{i\omega\tau}$$

In both cases, the stochastic behaviour of the density field $u(t)$ depends on the source function $s(t)$ and the memory function $m(t)$.

## IX. Self-Affine Random Walk Fields

The solution to Equation (30) requires the distribution of the source term to be quantified and the memory function to be specified. In the latter case, let the memory function be given by

$$m(t) = \frac{1}{\Gamma(1-\alpha)t^{\alpha}}, \ t > 0, \ \alpha \in (0,1) \quad (32)$$

The reason for adopting this particular function is that the solution to Equation (30) then yields a self-affine time-series. To show this, we note, from Relationship (1) (for independent variable $t$ rather than $x$), that

$$m(t) \leftrightarrow \frac{1}{(i\omega)^{1-\alpha}} \ \Rightarrow \ N(\omega) = (i\omega)^{1-\alpha}$$

given Equation (29).

### A. Solution for Equation (30)

In Fourier space, Equation (30) is given by (using the convolution theorem)

$$i\omega\tau U(\omega) = N(\omega)S(\omega)$$

where $U(\omega)$ and $S(\omega)$ are the Fourier transforms of $u(t)$ and $s(t)$, respectively. Thus, we can write

$$U(\omega) = \frac{1}{i\omega\tau}(i\omega)^{1-\beta}S(\omega), \ \beta \in (0,1)$$

or, using the convolution theorem again,

$$u(t) = \frac{1}{\tau\Gamma(\beta)t^{1-\beta}} \otimes s(t) \quad (33)$$

where we change the notation for the parameter $\alpha$ to $\beta$ in regard to the analysis given in the following section which will require differentiation between two such parameters. The solution for $u(t)$ is then expressed in terms of the Riemann-Liouville integral subject to a scaling constant $\frac{1}{\tau}$. Further, given the function $u(t)$, it is possible to compute $\beta$ through application of a regression method since, for some real constant $C$,

$$\ln \mid U(\omega) \mid^2 = C - 2\beta \ln(\omega), \ \omega > 0 \quad (34)$$

where $\mid U(\omega) \mid^2$ is the power spectrum of $u(t)$ and it is assumed that the PSDF of $s(t)$ is the constant $C$.

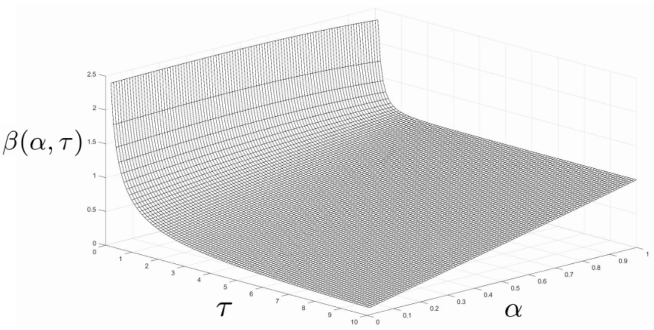


Fig. 1. Surface plot of $\beta(\alpha,\tau)$, $\alpha \in (0,1)$, $\tau \in (0,10]$ based on Equation (37) illustrating that for $\tau >> 1$, there is a linear relationship between $\alpha$ and $\beta$.

### B. Solution for Equation (31)

Given Equation (32), and, following the analysis provided in Section IX.A, the solution to Equation (31) becomes

$$u(t) = -\frac{1}{\tau\Gamma(\alpha)t^{1-\alpha}} \otimes u(t) + \frac{1}{\tau\Gamma(\alpha)t^{1-\alpha}} \otimes s(t) \quad (35)$$

which has the transfer function

$$T(\omega) = \frac{1}{1+\tau(i\omega)^{\alpha}} \quad (36)$$

Since the transfer function for Equation (33) is $1/\tau(i\omega)^{\beta}$, if we can relate $\alpha$ in Equation (35) to $\beta$, then it becomes possible to compare the solution given by Equation (33) and the solution we now require to Equation (35) for $u(t)$. To do this, we equate the power spectra of the two transfer functions. This yields a relationship between $\alpha$ and $\beta$ for $\tau$ given by

$$\tau^2 \mid \omega \mid^{2\beta} = 1 + \tau^2 \mid \omega \mid^{2\alpha} + 2\tau \mid \omega \mid^{\alpha} \cos(\alpha\pi/2)$$

which is a transcendental equation for $\omega$. Since this equation is applicable for any value of $\omega$, we let $\omega = e$ giving the equation

$$\beta = \frac{1}{2}\ln[1+\tau^2\exp(2\alpha) + 2\tau\exp(\alpha)\cos(\alpha\pi/2)] - \ln\tau \quad (37)$$

Figure 1 shows a plot of $\beta$ for $\alpha \in (0,1)$ and $\tau \in (0,10]$ based on Equation (37) illustrating that as $\tau > 1$ increases, there develops a linear relationship between $\alpha$ and $\beta$, e.g. for $\tau = 10$, a linear fit between the two parameters yields $\beta = 0.9016\alpha + 0.09393$. Since Equation (33) and Equation (35) are both scaled by $1/\tau$, this result implies that for $\tau >> 1$ the two equations become effectively equivalent in the sense that the solution for $u(t)$ given by Equation (33) is equivalent to a solution for $u(t)$ given by Equation (35) with $\alpha \simeq 1.1\beta - 0.1$. Further, any estimate for $\beta$ obtained from knowledge of $u(t)$ based on Equation (33) and application of a regression method using Equation (34) becomes equivalent to estimating $\alpha$ from $u(t)$ defined by Equation (35).

### C. Example Numerical Results

We consider some example numerical simulations based on computing $u(t)$ for Equation (35) using the approximation defined by Equation (24). On the basis of the analysis given in the previous section, this is equivalent to computing the function $u(t)$ based on the approximation

$$u(t) \sim \frac{1}{\tau\Gamma(\beta)}[1-\exp(-t^{\beta-1})]\int_a^t s(\xi)d\xi$$

$$+\frac{1}{\tau\Gamma(\beta)}(\beta-1)t^{\beta-2}\exp(-t^{\beta-1})\int_a^t \xi s(\xi)d\xi, \ t>0 \quad (38)$$

where $\beta \simeq 0.9\alpha + 0.1, \ \alpha \in (0,1)$. Figure 2 shows an example of a discrete stochastic field consisting of 100 signals $u_n, \ n = 1,2,...,N$ for $N = 100$ where: (i) $\tau = 10$; (ii) $s_n$ is zero mean Gaussian distributed with a standard deviation of $\sim 0.2$; (iii) values of $\alpha \in (0,1)$ conform to a Gaussian distribution with a mean of $\sim 1$ and standard deviation $\sim 0.2$; (iv) application of the MATLAB function *randn* for computing random Gaussian number streams; (v) application of Euler's method for computing both integrals for a step size of 1 with an initial condition $a = 1$. From Figure 2, it is clear that the field is concentrated around zero when $\beta \sim 0.1$ and $\Gamma(\beta) \sim 9.5$. This is an example of a non-Ergodic field which is due to the random variations in the values of $\alpha$ considered.

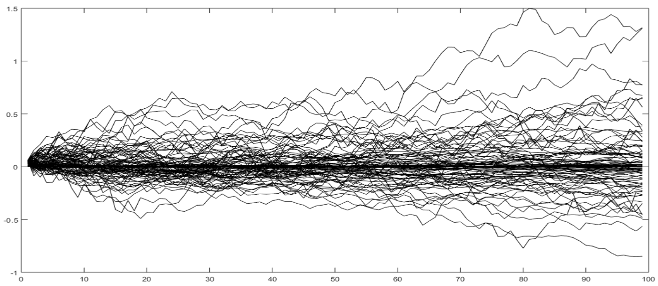

Fig. 2. Example of a stochastic field consisting of 100 signals $u_n, \ n = 1,2,...,N$ obtained through application of Equation (38) using Euler's method for computing the integrals with $a = 1$ for $N = 100$ and a step length of 1.

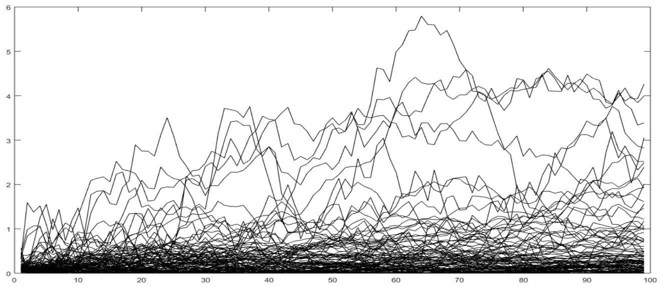

Fig. 3. A stochastic field consisting on 100 signals given by $\mid u_n \mid$ obtained using Equation (38) with variable values of $\tau \in [1,10]$ conforming to a uniform distribution.

Figure 3 shows an example of a stochastic field generated using exactly the same conditions and methods as those itemised above but where $\tau \in [1,10]$ varies randomly for each signal and conforms to a uniform distribution. In this case, the plot of each signal is $\mid u_n \mid$ and the vast majority of the field is clustered close to zero relative to a few signals whose scale is significantly greater due to those cases when both $\tau$ and $\beta$ are close to or equal to 1.

## X. Summary and Conclusions

The principal contributions that have been made in this paper are given in the proof of Equation (10), the approximation compounded in Equation (23) and the conjecture introduced in Section VI. Equation (10) puts the 'focus' for evaluating a fractional differential of a function on the fractional differentiation of a sign function. The implications of this focus requires further consideration. In this context, Equation (11) provides an example of a Sigmoid-type function through which an explicit definition can been obtained for the fractional differentiation of the sign function as a limiting case, a result that is complemented through the definitions given by Equations (13) and (14).

The approximation given by Equation (23) coupled with the asymptotic analysis presented in Section VII.B yields a new result compounded in Equation (24). This equation replaces the need to compute a convolution integral to evaluate a fractional differential with the computation of integrals over the function itself, i.e. integration of the functions $f(x)$ and $xf(x)$. On the basis of the evolution equation discussed in Section VIII and the asymptotic analysis used to derive a time-only dependent model, Equation (24) has been used to develop a stochastic time series model. In the context of Equation (38), if $\mid u(t) \mid$ is taken to be a model for any non-Ergodic field over a uniform period of time, then the amplitude of this function is reduced as $\tau \to \infty$ and $\alpha \to 0$. In the latter case, this implies that the memory function given by Equation (32) reduces to a constant.

## Acknowledgments

The author is grateful to Professor Roy Hoskins for his help and advice in regard to the remit of this paper.

## References

[1] S. Dugowson, *Metaphysical Differences*, History and Philosophy of the Generalisation of the Order of Derivation, Thesis, Université Paris Nord, 1984. http://s.dugowson.free.fr/recherche/dones

[2] N. H. Abel, *Solving a Few Tasks using Specific Integrals*, Magazin for Naturvidenskaberne. Kristiania (Oslo), pp 55–68, 1823. https://www.abelprize.no/nedlastning/verker/oeuvres_1881_del1/oeuvres_completes_de_abel_nouv_ed_1_kap02_opt.pdf

[3] I. Podlubny, R. L. Magin and I. Trymorush, *Niels Henrik Abel and the Birth of Fractional Calculus*. Fractional Calculus and Applied Analysis, vol 20, no. 5, pp 1068–1075, 2017. https://www.degruyter.com/view/journals/fca/20/5/article-p1068.xml

[4] B. Ross (1977). *The Development of Fractional Calculus 1695-1900*, Historia Mathematica. vol. 4, pp 75–89, 1977.

[5] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, 2014.

[6] R. Hermann, *Fractional Calculus: An Introduction for Physicists*, World Scientific, 2014.

[7] S. M. Khairnar and M. More, *A Subclass of Uniformly Convex Functions Associated with Certain Fractional Calculus Operator*, IAENG International Journal of Applied Mathematics, vol. 39, no. 3, pp 184-191, 2009. http://www.iaeng.org/IJAM/issues_v39/issue_3/IJAM_39_3_07.pdf

[8] R. Garrappa, E. Kaslik and M Popolizio, *Evaluation of Fractional Integrals and Derivatives of Elementary Functions: Overview and Tutorial*, MDPI Mathematics Mathematics, vol. 7, no. 407, pp 1-21, 2019. doi:10.3390/math7050407

[9] Q. Feng, *Some New Gronwall-Bellmann Type Discrete Fractional Inequalities Arising in the Theory of Discrete Fractional Calculus*, IAENG International Journal of Applied Mathematics, vol. 46, no. 4, pp 527-534, 2016. http://www.iaeng.org/IJAM/issues_v48/issue_3/IJAM_48_3_07.pdf

[10] H. M. Jaradat, S. Al-Shar'a, Q. J.A. Khan, M. Alquran and K. Al-Khaled, *Analytical Solution of Time-Fractional Drinfeld-Sokolov-Wilson System Using Residual Power Series Method*, IAENG International Journal of Applied Mathematics, vol. 46, no. 1, 64-70, 2016. http://www.iaeng.org/IJAM/issues_v46/issue_1/IJAM_46_1_09.pdf

[11] A. M. Bijura, *Systems of Singularly Perturbed Fractional Integral Equations*, Journal of Integral Equations and Applications, vol. 24, no. 2, pp 195-211, 2012

[12] A. M. Bijura, *Systems of Singularly Perturbed Fractional Integral Equations II*, IAENG International Journal of Applied Mathematics, vol. 42, no. 4, pp 198-203, 2012. http://www.iaeng.org/IJAM/issues_v42/issue_4/IJAM_42_4_01.pdf

[13] S. O. Edeki, O. O. Ugebor and E. A. Owoloko, *Analytical Solution of the Time-fractional Order Black-Scholes Model for Stock Option Valuation on No Dividend Yield Basis*, IAENG International Journal of Applied Mathematics, vol. 47, no. 4, pp 407-416, 2017. http://www.iaeng.org/IJAM/issues_v47/issue_4/IJAM_47_4_06.pdf

[14] J. M. Blackledge, D. Kearney, M. Lamphiere, R. Rani and P. Walsh, *Econophysics and Fractional Calculus: Einstein's Evolution Equation, the Fractal Market Hypothesis, Trend Analysis and Future Price Prediction*, MDPI Special Issue on Mathematical Economics: Application of Fractional Calculus, pp 1-57, 2019. https://www.mdpi.com/journal/mathematics/special_issues/Mathematical_Economics

[15] *The Fourier Transform*, Tables of Important Fourier Transforms, Wikipedia, 2019. https://en.wikipedia.org/wiki/Fourier_transform

[16] I. Podlubny, *Geometric and Physical Interpretation of Fractional Integration and Fractional Differentiation*, Fractional Calculus and Applied Analysis, vol. 5, no. 4, pp 367-386, 2002.

[17] M. H. Tavassoli, A. Tavassoli and M. R. Ostad Rahimi, *The Geometric and Physical Interpretation of Fractional Order Derivatives of Polynomial Functions*, Differential Geometry - Dynamical Systems, vol. 15, pp 93-104, 2003.

[18] S. Butera and M. Di Paola, *A Physically Based Connection Between Fractional Calculus and Fractal Geometry*, Annals of Physics, vol. 350, pp 146-158, 2014.

[19] Turner, M. J., Blackledge, J. M. and Andrews, P. R.: Fractal Geometry in Digital Imaging, Academic Press, 1998. ISBN-10: 0127039708

[20] V. Kiryakova, *Generalised Fractional Calculus and Applications*, Longman, 1994.

[21] Z. Zheng, W. Zhao and H. Dai, *A New Definition of Fractional Derivative*, International Journal of Non-Linear Mechanics, vol. 108, pp 1-6, 2019.

[22] J. Hadamard, *Essay on the Study of Functions Given by Their Taylor Expansion*, Journal de Mathématiques Pures et Appliquées. vol. 4, no. 8, pp 101–186, 1892. http://sites.mathdoc.fr/JMPA/PDF/JMPA_1892_4_8_A4_0.pdf

[23] A. Erdélyi, *On some Functional transformations*, Reports of the Mathematical Seminar of the University and Polytechnic of Turin, vol. 10, pp 217–234, 1950.

[24] H. Kober, *On Fractional Integrals and Derivatives*, The Quarterly Journal of Mathematics, vol. 11, no. 1, 193–211, 1940. https://academic.oup.com/qjmath/article-abstract/os-11/1/193/1574937?redirectedFrom=fulltext

[25] A. Atangana and D. Baleanu, (2016), *New Fractional Derivatives with Non-local and Non-Singular Kernel: Theory and Application to Heat Transfer Model*, Thermal Science, vol. 20, no. 2, pp 763-769, 2016. http://www.doiserbia.nb.rs/img/doi/0354-9836/2016/0354-98361600018A.pdf

[26] D. Lokenath and B. Dambaru, *Integral Transforms and Their Applications*, CRC Press, 2016, ISBN 978-1-4200-1091-6.

[27] M. Caputo and M. Fabrizio, *A new Definition of Fractional Derivative without Singular Kernel*. Progress in Fractional Differentiation and Applications, vol. 1, no. 2, pp 73–85, 2015. http://www.naturalspublishing.com/files/published/0gb83k287mo759.pdf

[28] M. D. Ortigueira and F. Coito, From Differences to Derivatives, Fractional Calculus and Applied Analysis, vol. 7, no. 4, pp 459-471, 2004. http://www.diogenes.bg/fcaa/volume7/fcaa74/74_Ortigueira_Coito.pdf

[29] K. Adegoke and O. Layeni, *The Higher Derivatives of the Inverse Tangent Function and Rapidly Convergent BBP-type Formulas*, Applied Mathematics E-Notes, vol. 10(2010), pp 70-75, 2010. http://www.kurims.kyoto-u.ac.jp/EMIS/journals/AMEN/2010/090408-2.pdf

[30] G. Evans, J. M. Blackledge and P. Yardley, *Analytic Solutions to Partial Differential Equations*, Springer Undergraduate Mathematics Series, Springer-Verlag, London, 1999.

[31] A. Einstein, *On the Motion of Small Particles Suspended in Liquids at Rest Required by the Molecular-Kinetic Theory of Heat*, Annalen der Physik, vol. 17, pp 549-560, 2016.

[32] A. N. Kolmogorov, *On Analytic Methods in Probability Theory*, Selected Works of A. N. Kolmogorov, Volume II: Probability Theory and Mathematical Statistics (Ed. Shiryaev, A. N.), Kluwer, Dordrecht, pp 61-108, 1992.

[33] W. Feller, *On Boundaries and Lateral Conditions for the Kolmogorov Differential Equations*, The Annals of Mathematics, Second Series, vol. 65, no. 3, pp 527-570, 1957.