An Inertial Extragradient Algorithm for Solving Variational Inequalities

Ximin Guo, Wenling Zhao*

Abstract—**In this paper, we introduce a new extragradient algorithm for solving variational inequality problems involving pseudomontone with Lipschitz continuous operators. The algorithm which combines the inertial technique and the extragradient algorithm. We show that the algorithm is globally convergence without any knowledge of the Lipschitz constant of the mapping. Besides, linear convergence is guaranteed under additional strong pseudo-monotonicity. Finally, compared with other algorithms, the numerical results indicate that our algorithm has a better behavior.**

Index Terms—**Variational inequality problem, Extragradient method, Inertial-type algorithm, Lipschitz continuity, Q-linear convergence.**

I. Introduction

IN this paper, let us consider the following classical variational problem, denoted by $VIP(F,C)$, is to find a point $x^* \in C$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \forall x \in C. \quad (1)$$

Where $F : R^m \rightarrow R^m$ is an operator, $C \in R^m$ is a nonempty closed convex set, $\langle \cdot, \cdot \rangle$ denotes the inner product in R^m and the solution set of $VIP(F,C)$ is denoted by C^*, respectively.

As an important part of nonlinear programming, variational inequalities have applications in many aspects and have been widely used in operational research problems, equilibrium problems in the economic field, and urban transportation network modeling [1]. It not only unifies the concepts in applied mathematics, but also strengthens the knowledge system of complementary problems, optimization problems and equilibrium problems [2]. In recent years, with more and more scholars studying the variational inequalities, it has achieved very outstanding success in the field of mathematics, and it has been widely applied in engineering, economics, operations research, game theory, and traffic assignment (see, e.g., [3], [4] and the references therein). There are many methods for solving the variational inequality problem and its variants [5]–[11]. In the above methods, the most notable and general popular methods are projection and regularized. In this paper, we mainly study projection algorithms, the oldest and simplest one is the following gradient projection method [12], [13],

$$x_{n+1} = P_C(x_n - \lambda F(x_n)),$$

where $\lambda \in (0, \frac{2\eta}{L^2})$, F is η-strongly monotone $L-Lipschitz$ continuous on C. Obviously, the assumption of this algorithm is very strong. In order to overcome the above problems, Korpelevich [14] proposed the following extragradient method(EGM):

$$\begin{cases} y_n = P_C(x_n - \lambda F(x_n)) \\ x_{n+1} = P_C(x_n - \lambda F(y_n)), \end{cases}$$

where $\lambda \in (0, \frac{1}{L})$, the operator F is monotone and $Lipschitz$ continuous. Obviously, the step size of the above two methods depends on the $Lipschitz$ constant. However, when the $Lipschitz$ constant is unknown or difficult to approximate, these methods are not applicable. Recently, the EGM has been interested and developed under suitable conditions [15]–[19]. Iusem [20] proposed a new algorithm that its convergence is guaranteed without the Lipschitz continuity, the algorithm's stepsize rule is as follow: $\lambda_n = \lambda^{l^{j_n}}$, where j_n is the smallest non-negative integer j satisfying $\lambda^{l^j} \|F(x_n) - F(y_n)\| \leq \mu \|x_n - y_n\|$; in particular, Trinh [21] introduced an self-adaptive step-size algorithm which without any information of $Lipschitz$ constant of operator(see Algorithm 1) for solving the pseudomonotone and Lipschitz continuous VIP as follows:

Algorithm 1

Initialization Choose arbitrarily $x_{-1}, x_0, y_0 \in C$; $\rho, \delta \in (0,1)$; $\lambda_{-1} \in (0, \infty)$. Set n:=0

Step 1 Given λ_{n-1}, y_n and x_n.
If $\lambda_{n-1}\|F(x_{n-1}) - F(y_n)\| \leq \rho\|x_{n-1} - y_n\|$ then set $\lambda_n = \lambda_{n-1}$, else set $\lambda_n = \lambda_{n-1}\delta$.
Compute

$$y_{n+1} = P_C(x_n - \lambda_n F(x_n)),$$

$$x_{n+1} = P_C(x_n - \lambda_n F(y_{n+1})).$$

Step 2 If $y_{n+1} = x_n$, then stop, else update $n := n+1$ and go to Step 1.

Note that, different from [20], this algorithm does not need to calculate values of the operator F many times at each iteration and not require its step-sizes tending to zero. Therefore, the algorithm reduces the amount of calculation and time consumed, it requires the mapping F being pseudomonotone and Lipschitz continuous only on the feasible set instead of on the whole space.

It is also known that the inertial-type algorithm is one of the effective methods for speeding up the convergence properties of fundamental algorithms, see [22]. The main feature of inertial algorithms is that the next iterate is constructed from the two previous iterates. Motivated and inspired by inertial-type method [23] and above algorithms [24], we will introduce a kind of inertial extragradient algorithm.

Our paper structure is as follows. In Section 2, we first introduce some concepts and preliminary results used in this paper. Section 3 propose an inertial extragradient algorithm and analyze the convergence of it. Some experiment results are presented in Section 4.

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II. Preliminaries

In this section, we introduce some concepts and lemmas that will be used. Throughout the paper, let $C \in R^m$ is a nonempty closed convex set and the operator $F : C \to R^m$ is pseudomonotone and $Lipschitz$ continuous.

In this paper, the orthogonal projection of x onto C is denoted by $P_C(x)$ such that

$$P_C(x) = argmin_{y \in C}\|y - x\|.$$

and the natural residual of $VIP(F, C)$ be denoted by

$$r(x) = x - P_C(x - \lambda F(x)), \quad where \quad \lambda > 0.$$

For all $\lambda > 0$, x^* is a solution of (1) if and only if

$$r(x^*) = 0.$$

Definition 2.1: Let $F : C \to R^m$ be a mapping, then
(1) F is monotone on C if for all $x, y \in C$, we have

$$\langle F(x) - F(y), x - y\rangle \geq 0.$$

(2) F is pseudomonotone on C, if for all $x, y \in C$, we have

$$\langle F(x), y - x\rangle \geq 0 \Longrightarrow \langle F(y), y - x\rangle \geq 0.$$

(3) F is γ-strongly pseudomonotone on C, if there exists a constant $\gamma > 0$, for all $x, y \in C$, we have

$$\langle F(x), y - x\rangle \geq 0 \Longrightarrow \langle F(y), y - x\rangle \geq \gamma\|x - y\|^2.$$

(4) F is called $L - Lipschitz$ continuous, if there exists a constant $L > 0$, for all $x, y \in C$, we have

$$\|F(x) - F(y)\| \leq L\|x - y\|.$$

Obviously, (3) is included in (2), (1) is included in (2) , but the converse is not true. This paper will use the following lemmas:

Lemma 2.1: [24] Let C be a nonempty, closed and convex subset of R^m, for $\forall x \in R^m$. Then
(1) $\langle P_C(x) - x, y - P_C(x)\rangle \geq 0, \forall y \in C$;
(2) $\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle, \forall y \in R^m$;
(3) If $y \in C$, then $\|P_C(x) - y\|^2 \leq \|x - y\|^2 - \|x - P_C(x)\|^2$.

Lemma 2.2: [23] Let $\{a_n\}, \{b_n\} and \{c_n\}$ be sequences in $[0, \infty)$ such that

$$a_{n+1} \leq a_n + b_n(a_n - a_{n-1}) + c_n \forall n \geq 1, \Sigma_{n=1}^{\infty} c_n < \infty,$$

and there exists a real number b with $0 \leq b_n \leq b < 1$ for all $n \in N$. Then the following results hold:
(1) $\Sigma_{n=1}^{\infty}\Sigma_{n=1}^{\infty}[a_n - a_{n-1}]_+ < \infty$, where $[t]_+ := \max\{t, 0\}$;
(2) there exists $a^* \in [0, \infty)$ such that $\lim_{n\to\infty} a_n = a^*$.

III. Main results

In this section, under mild assumptions, we give a modified algorithm which is called a self-adaptive inertial extragradient algorithm for solving variational inequalities, the information of L is not necessary to be known. Based on Algorithm 1, our algorithm adds an inertial term. Now we assume the following conditions that will be used in the proof process.

Condition 1: C is a nonempty, closed, and convex set. We always assume that the solution set of the variational inequality is nonempty i.e., $\exists x^* \in C^*$.

Condition 2: The operator F is $L - Lipschitz$ continuous and pseudomonotone on C.

Condition 3: The operator F is $L - Lipschitz$ continuous and γ-strongly pseudomonotone on C.

When the Condition1 and Condition2 hold, our self-adaptive inertial extragradient algorithm is as follows:

Algorithm 2

Initialization Choose arbitrarily $x_0, x_1 \in C$, $\rho, \delta \in (0, 1)$ and λ_1, $a \in (0, \infty)$.
Set n:=1

Step 1 Computer $\omega_n = x_n + a_n(x_n - x_{n-1})$. Where

$$a_n = \begin{cases} \min\{\frac{1}{n^2\|x_n - x_{n-1}\|^2}, a\}, & if \quad x_n \neq x_{n-1}, \\ a, & otherwise. \end{cases}$$

Step 2 Compute

$$y_n = P_C(\omega_n - \lambda_n F(\omega_n)).$$

$$x_{n+1} = P_C(\omega_n - \lambda_n F(y_n)).$$

step 3 If $y_n = \omega_n$, then stop, y_n is a solution of (1). Otherwise update

$$\lambda_{n+1} = \begin{cases} \lambda_n, & if \quad \lambda_n\|F(\omega_n) - F(y_n)\| \leq \rho\|\omega_n - y_n\|, \\ \lambda_n\delta, & otherwise. \end{cases} \tag{2}$$

Set $n := n + 1$ and go to **step 1** .

Remark 1: Since $\delta \in (0, 1)$, then the sequence $\{\lambda_n\}$ is nonincreasing. Meanwhile, if $\lambda_n \to 0$, then which contradicts the operator F is $Lipschitz$ continuous, so there has a constant $q > 0$ such that $\lim_{n\to\infty} \lambda_n = q$.

Lemma 3.1: Assume that Condition1 and Condition2 hold, let $\{x_n\}$ be the sequence generated by Algorithm 2, we have

$$\|x_{n+1} - x^*\|^2 \leq \|\omega_n - x^*\|^2 - (1 - \rho)\|x_{n+1} - y_n\|^2 - (1 - \rho)\|y_n - \omega_n\|^2. \tag{3}$$

Proof: As the sequence $\{\lambda_n\}$ is lower bounded, there exists $n_0 > 0$ such that $\lambda_{n+1} = \lambda_n$, and

$$\lambda_n\|F(\omega_n) - F(y_n)\| \leq \rho\|\omega_n - y_n\|, \quad \forall n \geq n_0. \tag{4}$$

Since $y_n = P_C(\omega_n - \lambda_n F(\omega_n))$, by Lemma 2.1 we have

$$\langle z - y_n, \omega_n - \lambda_n F(\omega_n) - y_n\rangle \leq 0, \quad for \quad \forall z \in C,$$

equivalently,

$$\langle y_n - \omega_n, y_n - z\rangle \leq \lambda_n\langle F(\omega_n), z - y_n\rangle, \quad for \quad \forall z \in C. \tag{5}$$

Besides, from projection properties and the definition of x_{n+1}, we get

$$\langle z - x_{n+1}, \omega_n - \lambda_n F(y_n) - x_{n+1}\rangle \leq 0, \quad for \quad \forall z \in C,$$

similarly, we get

$$\langle x_{n+1} - \omega_n, x_{n+1} - z\rangle \leq \lambda_n\langle F(y_n), z - x_{n+1}\rangle, \quad for \quad \forall z \in C. \tag{6}$$

Then

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
&=\|x_{n+1}-y_n-z+y_n\|^2\\
&=\|x_{n+1}-y_n\|^2+\|y_n-z\|^2+2\langle x_{n+1}-y_n,y_n-z\rangle\\
&=-\|x_{n+1}-y_n\|^2+\|y_n-\omega_n+\omega_n-z\|^2\\
&+2\langle x_{n+1}-y_n,x_{n+1}-z\rangle\\
&=-\|x_{n+1}-y_n\|^2+\|y_n-\omega_n\|^2+\|\omega_n-z\|^2\\
&+2\langle y_n-\omega_n,\omega_n-z\rangle+2\langle x_{n+1}-y_n,x_{n+1}-z\rangle\\
&=-\|x_{n+1}-y_n\|^2+\|\omega_n-z\|^2-\|y_n-\omega_n\|^2\\
&+2\langle y_n-\omega_n,y_n-z\rangle+2\langle x_{n+1}-y_n,x_{n+1}-z\rangle\\
&=-\|x_{n+1}-y_n\|^2+\|\omega_n-z\|^2-\|y_n-\omega_n\|^2\\
&+2\langle y_n-\omega_n,y_n-x_{n+1}\rangle+2\langle x_{n+1}-\omega_n+\omega_n,x_{n+1}-z\rangle.
\end{aligned} \tag{7}$$

Which together with (3.4) and (3.5) shows that

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
&\le-\|x_{n+1}-y_n\|^2+\|\omega_n-z\|^2-\|y_n-\omega_n\|^2\\
&+2\lambda_n\langle F(\omega_n),x_{n+1}-y_n\rangle+2\langle F(y_n),z-x_{n+1}\rangle\\
&=-\|x_{n+1}-y_n\|^2+\|\omega_n-z\|^2-\|y_n-\omega_n\|^2\\
&+2\lambda_n[\langle F(\omega_n)-F(y_n),x_{n+1}-y_n\rangle+\langle F(y_n),x_{n+1}-y_n\rangle]\\
&+2\lambda_n[\langle F(y_n),z-y_n\rangle+\langle F(y_n),y_n-x_{n+1}\rangle]\\
&=\|\omega_n-z\|^2-\|x_{n+1}-y_n\|^2-\|y_n-\omega_n\|^2\\
&+2\lambda_n\langle F(\omega_n)-F(y_n),x_{n+1}-y_n\rangle\\
&+2\lambda_n\langle F(y_n),z-y_n\rangle.
\end{aligned} \tag{8}$$

Using (4) and (8), when $z=x^*\in C$, for $\forall n\ge n_0$, we obtain

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
&\le\|\omega_n-x^*\|^2-\|x_{n+1}-y_n\|^2-\|y_n-\omega_n\|^2\\
&+2\lambda_n\langle F(y_n),x^*-y_n\rangle+2\rho\|\omega_n-y_n\|\|x_{n+1}-y_n\|\\
&\le\|\omega_n-x^*\|^2-(1-\rho)\|x_{n+1}-y_n\|^2\\
&-(1-\rho)\|y_n-\omega_n\|^2+2\lambda_n\langle F(y_n),x^*-y_n\rangle.
\end{aligned} \tag{9}$$

And because the operator F is pseudo-monotonic, we have

$$\langle F(y_n),x^*-y_n\rangle\le 0,$$

therefore, (3) hold. ■

Lemma 3.2: Assume that Condition1 and Condition2 hold, then the $\{x_n\}$ generated by Algorithm 2 is bounded.

Proof: According to the definition of ω_n, we obtain

$$\begin{aligned}
&\|\omega_n-x^*\|^2\\
&=\|x_n+a_n(x_n-x_{n-1})-x^*\|^2\\
&=(1+a_n^2+2a_n)\|x_n-x^*\|^2+a_n^2\|x_{n-1}-x^*\|^2\\
&-2a_n\langle x_n-x^*,x_{n-1}-x^*\rangle-2a_n^2\langle x_n-x^*,x_{n-1}-x^*\rangle\\
&=(1+a_n)\|x_n-x^*\|^2+a_n^2\langle x_n-x^*,x_n-x^*\rangle\\
&+a_n^2\langle x_{n-1}-x^*,x_{n-1}-x^*\rangle\\
&-a_n^2\langle x_n-x^*,x_{n-1}-x^*\rangle-a_n^2\langle x_n-x^*,x_{n-1}-x^*\rangle\\
&+a_n\langle x_n-x^*,x_n-x^*\rangle-a_n\langle x_n-x^*,x_{n-1}-x^*\rangle\\
&-a_n\langle x_n-x^*,x_{n-1}-x^*\rangle\\
&=(1+a_n)\|x_n-x^*\|^2+a_n^2\langle x_n-x^*,x_n-x_{n-1}\rangle\\
&-a_n^2\langle x_{n-1}-x^*,x_n-x_{n-1}\rangle\\
&+a_n\langle x_n-x^*,x_n-x^*\rangle-a_n\langle x_n-x^*,x_{n-1}-x^*\rangle\\
&-a_n\langle x_n-x^*,x_{n-1}-x^*\rangle\\
&=(1+a_n)\|x_n-x^*\|^2+a_n^2\langle x_n-x_{n-1},x_n-x_{n-1}\rangle\\
&+a_n\langle x_n-x^*,x_n-x_{n-1}\rangle-a_n\langle x_n-x^*,x_{n-1}-x^*\rangle\\
&=(1+a_n)\|x_n-x^*\|^2-a_n\|x_{n-1}-x^*\|^2\\
&+(1+a_n)a_n\|x_n-x_{n-1}\|^2
\end{aligned} \tag{10}$$

which together with Lemma 3.1 suggests that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&\le(1+a_n)\|x_n-x^*\|^2-a_n\|x_{n-1}-x^*\|^2\\
&+(1+a_n)a_n\|x_n-x_{n-1}\|^2\\
&-(1-\rho)\|x_{n+1}-y_n\|^2-(1-\rho)\|y_n-\omega_n\|^2\\
&\le\|x_n-x^*\|^2+a_n(\|x_n-x^*\|^2-\|x_{n-1}-x^*\|^2)\\
&+(1+a)a_n\|x_n-x_{n-1}\|^2.
\end{aligned} \tag{11}$$

Where we get the last inequality follows from the definition of $0\le a_n\le a$. Moreover, since $a_n\|x_n-x_{n-1}\|^2\le\frac{1}{n^2}$(for $\forall n\in N$), so we have $\Sigma_{n=1}^{\infty}a_n\|x_n-x_{n-1}\|^2<\infty$, this implies that, $\lim_{n\to\infty}\|x_n-x_{n-1}\|=0$. Now, we let $a_n=\|x_n-x^*\|^2$ and $c_n=(1+a)a_n\|x_n-x_{n-1}\|^2$, then, by Lemma 2.2 and (11), there has a constant α such that

$$\lim_{n\to\infty}\|x_n-x^*\|^2=\alpha,$$

which means the sequence $\{x_n\}$ is bounded. ■

Next, we start to analyze the above algorithm convergence by proving the following theorem.

Theorem 3.1: If Condition1 and Condition2 hold, then the sequence $\{x_n\}$ generated by Algorithm 2 converges to a solution x^* of (1).

Proof: By Lemma 2.2, we obtain

$$\sum_{n=1}^{\infty}[\|x_n-x^*\|^2-\|x_{n-1}-x^*\|^2]_+<\infty,$$

where $t_+=\max\{t,0\}$. Consequently,

$$\lim_{n\to\infty}[\|x_n-x^*\|^2-\|x_{n-1}-x^*\|^2]_+=0. \tag{12}$$

Using (11) and the definition of a_n, we also obtain

$$\begin{aligned}
&(1-\rho)\|y_n-\omega_n\|^2+(1-\rho)\|x_{n+1}-y_n\|^2\\
\le&\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+a_n[\|x_n-x^*\|^2\\
&-\|x_{n-1}-x^*\|^2]_++(1+a)a_n\|x_n-x_{n-1}\|^2.
\end{aligned} \tag{13}$$

From (12) and (13), we obtain

$$\lim_{n\to\infty} \|y_n - \omega_n\| = 0, \tag{14}$$

also from the definition of ω_n, we have

$$\|\omega_n - x_n\|^2 = a_n^2\|x_n - x_{n-1}\|^2 \to 0, \quad n \to \infty.$$

Hence

$$\lim_{n\to\infty} \|\omega_n - x_n\| = 0, \tag{15}$$

Therefore, by (9) and (15), we know the sequence $\{\|x_n - x^*\|\}$ is nonincreasing. Since the sequence $\{x_n\}$ is bounded, $\{\|x_n - x^*\|\}$ converges, and there exists subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\} \to \bar{x} \in C$. Therefore ,we only need to prove $\bar{x} \in C^*$: by (4) and (5), for $\forall z \in C, n_i \geq n_0$, we have

$$\begin{aligned}&\langle y_{n_i} - \omega_{n_i}, y_{n_i} - z\rangle + \lambda_{n_i}\langle F(\omega_{n_i}), y_{n_i} - \omega_{n_i}\rangle \\ &\leq \lambda_{n_i}\langle F(\omega_{n_i}), y_{n_i} - \omega_{n_i}\rangle.\end{aligned} \tag{16}$$

Since the F is continuous, combining (14) and (16), letting $i \to \infty$, we obtain

$$\langle F(\bar{x}), z - \bar{x}\rangle \geq 0,$$

from the definition of $VIP(F, C)$, this implies that $\bar{x} \in C^*$. Therefore, we have

$$\lim_{n\to\infty} \|x_n - \bar{x}\| = \lim_{i\to\infty} \|x_{n_i} - \bar{x}\| = 0.$$

■

Now, let's analyze the convergence rate of the Algorithm 2. Before that, we give the following Lemma.

Lemma 3.3: If Condition1 and Condition2 hold, then (1) has a unique solution.

Proof: Suppose $x_1^*, x_2^* \in C^*$ and $x_1^* \neq x_2^*$, for any $y, z \in C$, because F is strongly pseudomonotonic, we get

$$\langle F(x_1^*), y - x_1^*\rangle \geq 0 \Rightarrow \langle F(y), y - x_1^*\rangle \geq \gamma\|y - x_1^*\|^2,$$

$$\langle F(x_2^*), z - x_2^*\rangle \geq 0 \Rightarrow \langle F(z), z - x_2^*\rangle \geq \gamma\|z - x_2^*\|^2,$$

when $y = x_2^*$, we have the following inequality

$$\langle F(x_1^*), x_2^* - x_1^*\rangle \geq 0 \Rightarrow \langle F(x_2^*), x_2^* - x_1^*\rangle \geq \gamma\|x_2^* - x_1^*\|^2. \tag{17}$$

Similarly, when $z = x_2^*$, we obtain

$$\langle F(x_2^*), x_1^* - x_2^*\rangle \geq 0 \Rightarrow \langle F(x_1^*), x_1^* - x_2^*\rangle \geq \gamma\|x_1^* - x_2^*\|^2. \tag{18}$$

From (17) and (18), we have

$$\gamma\|x_1^* - x_2^*\|^2 \leq 0,$$

It implies that $x_1^* = x_2^*$, this contradicts the condition. Consequently, (1) has a unique solution. ■

Theorem 3.2: If $\lambda_1 < \frac{1}{L}$, Condition1 and Condition3 hold, then sequence $\{x_n\}$ generated by Algorithm2 Q-linear converges to the unique solution of (1), denoted by x^*.

Proof: From Lemma 2.1, we have

$$\begin{aligned}&\langle x_{n+1} - y_n, \omega_n - y_n - \lambda_n F(y_n)\rangle \\ &= \langle x_{n+1} - y_n, \omega_n - y_n - \lambda_n F(\omega_n)\rangle \\ &+ \lambda_n\langle x_{n+1} - y_n, F(\omega_n) - F(y_n)\rangle \\ &\leq \lambda_n\langle x_{n+1} - y_n, F(\omega_n) - F(y_n)\rangle.\end{aligned} \tag{19}$$

Since $x^* \in C^*$, from the pseudomonotonicity of F, this shows that

$$\langle F(x^*), y_n - x^*\rangle \geq 0 \Rightarrow \langle F(y_n), y_n - x^*\rangle \geq 0,$$

or equivalently,

$$\langle F(y_n), x_{n+1} - x^*\rangle \geq \langle F(y_n), x_{n+1} - y_n\rangle. \tag{20}$$

Let $z_n = \omega_n - \lambda_n F(y_n)$, then

$$\begin{aligned}&2\langle z_n - P_C(z_n), x^* - P_C(z_n)\rangle \\ &= 2\|z_n - P_C(z_n)\|^2 + 2\langle P_C(z_n) - z_n, z_n - x^*\rangle \leq 0,\end{aligned}$$

this implies that

$$\begin{aligned}&\|z_n - P_C(z_n)\|^2 + 2\langle P_C(z_n) - z_n, z_n - x^*\rangle \\ &\leq -\|z_n - P_C(z_n)\|^2.\end{aligned} \tag{21}$$

By (19), (20) and (21) we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &= \|P_C(z_n) - x^*\|^2 \\ &= \|z_n - x^*\|^2 + \|z_n - P_C(z_n)\|^2 + 2\langle P_C(z_n) - z_n, z_n - x^*\rangle \\ &\leq \|z_n - x^*\|^2 - \|z_n - P_C(z_n)\|^2 \\ &= \langle \omega_n - \lambda_n F(y_n) - x^*\rangle\langle \omega_n - \lambda_n F(y_n) - x^*\rangle \\ &- \langle \omega_n - \lambda_n F(y_n) - x_{n+1}\rangle\langle \omega_n - \lambda_n F(y_n) - x_{n+1}\rangle \\ &= \|\omega_n - x^*\|^2 - 2\langle \lambda_n F(y_n), \omega_n - x^*\rangle \\ &- \|\omega_n - x_{n+1}\|^2 + 2\langle \lambda_n F(y_n), \omega_n - x_{n+1}\rangle \\ &= \|\omega_n - x^*\|^2 - \|\omega_n - x_{n+1}\|^2 + 2\lambda_n\langle F(y_n), x^* - x_{n+1}\rangle \\ &\leq \|\omega_n - x^*\|^2 - \|\omega_n - x_{n+1}\|^2 + 2\lambda_n\langle y_n - x_{n+1}, F(y_n)\rangle \\ &= \|\omega_n - x^*\|^2 - \langle \omega_n - y_n + y_n - x_{n+1}, \omega_n - y_n + y_n - x_{n+1}\rangle \\ &+ 2\lambda_n\langle y_n - x_{n+1}, F(y_n)\rangle \\ &= \|\omega_n - x^*\|^2 - \|\omega_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 \\ &+ 2\langle x_{n+1} - y_n, \omega_n - y_n - \lambda_n F(y_n)\rangle \\ &\leq \|\omega_n - x^*\|^2 - \|\omega_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 \\ &+ 2\lambda_n\|x_{n+1} - y_n\|\|F(\omega_n) - F(y_n)\| \\ &\leq \|\omega_n - x^*\|^2 - \|\omega_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 \\ &+ 2\lambda_n L\|x_{n+1} - y_n\|\|\omega_n - y_n\|.\end{aligned} \tag{22}$$

On the other hand, since

$$\begin{aligned}&(\lambda_n L\|\omega_n - y_n\| - \|x_{n+1} - y_n\|)^2 \\ &= (\lambda_n L)^2\|\omega_n - y_n\|^2 + \|x_{n+1} - y_n\|^2 \\ &- 2\lambda_n L\|\omega_n - y_n\|\|x_{n+1} - y_n\| \geq 0,\end{aligned}$$

which implies that

$$\begin{aligned}&2\lambda_n L\|\omega_n - y_n\|\|x_{n+1} - y_n\| \\ &\leq (\lambda_n L)^2\|\omega_n - y_n\|^2 + \|x_{n+1} - y_n\|^2.\end{aligned} \tag{23}$$

Combining (22) and (23), we obtain

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\leq \|\omega_n - x^*\|^2 - \|\omega_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 \\ &+ (\lambda_n L)^2\|\omega_n - y_n\|^2 + \|y_n - x_{n+1}\|^2 \\ &= \|\omega_n - x^*\|^2 - (1 - \lambda_n^2 L^2)\|\omega_n - y_n\|^2.\end{aligned}$$

From the γ-strongly pseudo-monotone of F on C it follows

$$\langle F(y_n), y_n - x^*\rangle \geq \gamma\|y_n - x^*\|^2,$$

then

$$\begin{aligned}&\langle F(\omega_n), x^* - y_n\rangle \\ &= \langle F(\omega_n) - F(y_n), x^* - y_n\rangle - \langle F(y_n), y_n - x^*\rangle \\ &\leq \|F(\omega_n) - F(y_n)\|\|y_n - x^*\| - \gamma\|y_n - x^*\|^2 \\ &\leq L\|\omega_n - y_n\|\|y_n - x^*\| - \gamma\|y_n - x^*\|^2. \qquad (24)\end{aligned}$$

Also from Lemma 2.1, we have

$$\langle x^* - y_n, y_n - \omega_n + \lambda_n F(\omega_n)\rangle \geq 0,$$

which yields

$$\begin{aligned}&\langle x^* - y_n, \omega_n - y_n\rangle \\ &\leq \lambda_n\langle F(\omega_n), x^* - y_n\rangle \\ &\leq \lambda_n L\|\omega_n - y_n\|\|y_n - x^*\| - \lambda_n\gamma\|y_n - x^*\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\lambda_n\gamma\|y_n - x^*\|^2 \\ &\leq \lambda_n L\|\omega_n - y_n\|\|y_n - x^*\| - \langle x^* - y_n, \omega_n - y_n\rangle \\ &\leq \lambda_n L\|\omega_n - y_n\|\|y_n - x^*\| + \|x^* - y_n\|\|\omega_n - y_n\| \\ &= (1 + \lambda_n L)\|\omega_n - y_n\|\|y_n - x^*\|, \qquad (25)\end{aligned}$$

then it holds.

$$\begin{aligned}\|\omega_n - x^*\| &= \|\omega_n - y_n + y_n - x^*\| \\ &\leq \|\omega_n - y_n\| + \|y_n - x^*\| \\ &\leq \|\omega_n - y_n\| + \frac{1 + \lambda_n L}{\lambda_n\gamma}\|\omega_n - y_n\| \\ &= \frac{1 + \lambda_n L + \lambda_n\gamma}{\lambda_n\gamma}\|\omega_n - y_n\|. \qquad (26)\end{aligned}$$

Replacing (26) into (24), we get

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\leq \|\omega_n - x^*\|^2 - (1 - \lambda_n^2 L^2)(\frac{\lambda_n\gamma}{1 + \lambda_n L + \lambda_n\gamma})^2\|\omega_n - x^*\|^2.\end{aligned}$$

By (15), for $n \to \infty$, we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - (1 - \lambda_n^2 L^2)(\frac{\lambda_n\gamma}{1 + \lambda_n L + \lambda_n\gamma})^2\|x_n - x^*\|^2 \\ &= [1 - (1 - \lambda_n^2 L^2)(\frac{\lambda_n\gamma}{1 + \lambda_n L + \lambda_n\gamma})^2]\|x_n - x^*\|^2. \qquad (27)\end{aligned}$$

Hence, according to $\{\lambda_n\}$ is nonincreasing and $\lambda_1 < \frac{1}{L}$, we obtain

$$\sigma = [1 - (1 - \lambda_n^2 L^2)(\frac{\lambda_n\gamma}{1 + \lambda_n L + \lambda_n\gamma})^2]^{\frac{1}{2}} \in (0, 1),$$

which suggests that sequence $\{x_n\}$ converges to x^* with a Q-linear rate. ■

IV. Numerical experiments

In this section, we give two examples and experimental data to illustrate the proposed algorithm. The python codes are on a PC(with CPU Intel (R)) under Python Version 3.7.3.

Example 1 Consider the following problem: the operator F:$R^m \to R^m$ by

$$F(x) = Mx + q,$$

where $M \in R^{m\times m}$ is a positive semi-definite matrix, q is a vector in R^m and the feasible set C is given by

$$C = \{x \in R^m | -1 \leq x_i \leq 1, \forall i = 1, ..., m\}.$$

Example 2 In this example, take $F : R^2 \to R^2$ be defined by

$$F(x + y + \cos(x), -x + y + \cos(y)), \quad \forall x, y \in C.$$

Where C is same as the above example.

Obviously, the operator F in the above examples are continuous and $Lipschitz$ continuous.

Throughout the numerical experiment, we choose the same starting points in this example: $x_0 = x_1 = (1, ..., 1)$. We use the error 'ε' as the algorithm stopping rule, $'Iter'$ represents the number of iterations, $'CPU'$ represents the total time of the algorithm termination. We compare Algorithm 2($SIEGM$) with two algorithms, these algorithms were introduced by T.N.Hai [21]($IEGM$), D.V.thong [25]($MTEGM$), respectively. The following tables summarize the numerical results of this experiment under different dimensional m.

In Table I and Table II, we used the parameters as follows:
$SIEGM$ and $IEGM$: $\rho = 0.6$, $\delta = 0.8$, $a = 1$, choose the same initial step size is 0.8;
$MTEGM$: $\lambda = 0.8$, $a = 1$.
In Table III, we used the parameters as follows:
$SIEGM$,$IEGM$: $\rho = 0.8$, $\delta = 0.3$, $a = 1$, choose the same initial step size is 1;
$MTEGM$: $\lambda = 0.5$, $a = 0.3$.

It is clearly seen that compared with Algorithm1, our algorithm has good results on running time and number of iterations from the above tables; compared with another algorithm, we have fewer iterations. Therefore, we proposed algorithm have the competitive advantages.

V. Conclusions

In this paper, we first give an inertial extragradient algorithm for solving variational inequality problem. Under the assumptions of pseudo-monotonicity and $Lipschitz$ continuity of F, we verify the global convergence of the algorithm and analyze its convergence rate. Compared with several related algorithms, the effectiveness of the algorithm is verified.

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TABLE I
$\varepsilon = 10^{-3}$

Algorithm	$m = 3$		$m = 10$		$m = 50$	
	Iter	CPU(s)	Iter	CPU(s)	Iter	CPU(s)
SIEGM	24	0.116272	27	0.530530	31	11.438988
IEGM	27	0.147087	29	0.652585	32	13.379145
MTEGM	47	0.129601	50	0.517724	55	10.869507

TABLE II
$\varepsilon = 10^{-5}$

Algorithm	$m = 3$		$m = 10$		$m = 50$	
	Iter	CPU(s)	Iter	CPU(s)	Iter	CPU(s)
SIEGM	42	0.203993	43	0.881706	43	16.358555
IEGM	43	0.353989	45	0.979863	52	30.089164
MTEGM	73	0.194398	75	0.843850	79	17.430304

TABLE III
$m = 2$

Algorithm	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-5}$		$\varepsilon = 10^{-8}$	
	Iter	CPU(s)	Iter	CPU(s)	Iter	CPU(s)
SIEGM	22	0.077274	38	0.137620	62	0.236612
IEGM	25	0.107664	41	0.173531	65	0.322066
MTEGM	35	0.074360	59	0.125782	97	0.260141

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