Growth of Solutions of Second Order Linear Complex Differential Equations with Completely Regular Growth Coefficient

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***Abstract*—In this paper we investigate the classical problem of finding conditions on the entire coefficients $A(z)$ and $B(z)$ to ensure that all nontrivial solutions of $f''+A(z)f'+B(z)f=0$ are of infinite order. We assume $A(z)$ is an entire function with completely regular growth and $B(z)$ satisfies three conditions respectively, (1) $B(z)$ is a transcendental entire function with lower order less than $1/2$; (2) $B(z)$ is a transcendental entire function with Fabry gaps; (3) $B(z)$ satisfies $T(r,B)\sim\alpha\log M(r,B)$ outside a set of finite logarithmic measure, we prove the solutions have infinite order in these three cases.**

***Index Terms*—entire function, infinite order, complex differential equation.**

I. Introduction

In this article, we assume the reader is familiar with the basic results of Nevanlinna theory in the complex plane **C** and standard notations, for example see [15], [31]. Nevanlinna theory plays an important role in the study of complex differential equations, and many results appear in recent years. In this paper, the order and lower order of an entire function f are defined respectively as

$$\rho(f) = \limsup_{r\to+\infty}\frac{\log^+ T(r,f)}{\log r} = \limsup_{r\to+\infty}\frac{\log^+\log^+ M(r,f)}{\log r}, \quad (1)$$

$$\mu(f) = \liminf_{r\to+\infty}\frac{\log^+ T(r,f)}{\log r} = \liminf_{r\to+\infty}\frac{\log^+\log^+ M(r,f)}{\log r}, \quad (2)$$

where $\log^+ x = \max\{\log x, 0\}$ and $M(r,f)$ denotes the maximum modulus of f on the circle $|z|=r$.

Our main purpose is to consider the second order linear differential equation

$$f''+A(z)f'+B(z)f=0, \quad (3)$$

where $A(z)$ and $B(z)$ are entire functions. It's well known that all solutions of (3) are entire functions. If $B(z)$ is transcendental and f_1, f_2 are two linearly independent solutions of this equation, then at least one of f_1, f_2 is of infinite order, see [13]. However, there exist some equations of form (3) that have a nontrivial solution of finite order. For example, $f(z)=e^z$ satisfies differential equation

$$f''+e^{-z}f'-(e^{-z}+1)f=0.$$

What conditions should $A(z)$ and $B(z)$ satisfy such that every solution $f(\not\equiv 0)$ of equation (3) is of infinite order? There have been many results in the literature with regard to this problem, see [13], [15]. For example, we collect some classical results and give the following theorem.

Theorem 1. *Let $A(z)$ and $B(z)$ be nonconstant entire functions, satisfying any one of the following additional hypotheses:*

1) *$\rho(A)<\rho(B)$, see [8];*
2) *$A(z)$ is a polynomial and $B(z)$ is transcendental, see [8];*
3) *$\rho(B)<\rho(A)\le\frac{1}{2}$, see [12],*

then every nontrivial solution f of equation (3) *has infinite order.*

This is a hot research object and a lot of works have been published, such as [3], [6], [12], [17], [19], [21], [22], [28], [29], [30], [32]. In the article we continue to study the above question. Since every nontrivial solution of (3) satisfies

$$\rho(f)\ge\max\{\rho(A),\rho(B)\},$$

so we consider the question under the condition

$$\max\{\rho(A),\rho(B)\}<\infty$$

in the following theorems.

At first, if $\rho(r)$ is positive, differentiable for large r and satisfies

$$\lim_{r\to\infty}\rho(r)=\rho\in(0,\infty),$$

$$\lim_{r\to\infty}\rho'(r)r\log r=0,$$

then $\rho(r)$ is called a proximate order, see [5, Section 2, Chapter 2]. In order to formulate our results, recall the indicator $h(\theta)$ of an entire function $A(z)$ of order ρ with respect to the proximate order $\rho(r)$ is defined by

$$h(\theta)=\limsup_{r\to\infty}\frac{\log|A(re^{i\theta})|}{r^{\rho(r)}}, \quad (4)$$

where $\rho(r)\to\rho$ as $r\to\infty$. The function $A(z)$ is said to be completely regular growth (in the sense of Levin and Pfluger) if there exist disks $D(a_k,s_k)$ satisfying

$$\sum_{|a_k|\le r}s_k=o(r) \quad (5)$$

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such that

$$\log |A(re^{i\theta})| = h(\theta) r^{\rho(r)} + o(r^{\rho(r)}), \tag{6}$$

$$re^{i\theta} \notin \bigcup_k D(a_k, s_k)$$

as $r \to \infty$, uniformly in θ. A union of disks satisfying (5) is called a C_0-set. We refer to Levin's book [18] for a thorough discussion of functions of completely regular growth. There have been some works about the coefficients of (3) involving completely regular growth, such as [11], [27]. In [11], the authors got the following result.

Theorem 2. *Let $A(z)$ be an entire function of completely regular growth, and let $B(z)$ be any entire function such that $\rho(B) < \rho(A)$. Define*

$$E = \{\theta \in [-\pi, \pi) : h(\theta) \leq 0\}.$$

Then every nonzero solution of (3) *satisfies*

$$\rho(f) \geq \max\{\rho(A), \left(21\sqrt{m(E)}\right)^{-1} - 1\},$$

where $\rho(f) = \infty$ if $m(E) = 0$, here $m(E)$ is the Lebesgue measure of E.

From the above theorem, it's easy to see that if $h(\theta) > 0$ for almost every $\theta \in [0, 2\pi)$, then the nonzero solutions of (3) have infinite order. In this paper, we release the restriction on $h(\theta)$, that is, assume that $h(\theta)$ can take negative values for θ in some intervals which are contained in $[0, 2\pi)$. Moreover, we should give some conditions for $B(z)$, then the order of solutions of (3) are of infinite order.

The first main result relates to the well known $\cos(\pi\rho)$ theorem, which is due to Barry [1].

Theorem 3. *Let $A(z)$ be a completely regular growth entire function and the set*

$$E = \{\theta \in [0, 2\pi) : h(\theta) = 0\}$$

is of Lebesgue measure zero, and let $B(z)$ be a transcendental entire function with lower order $\mu(B) < 1/2$ and $\rho(A) \neq \rho(B)$. Then every nontrivial solution of (3) *is of infinite order.*

For an entire function

$$B(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n},$$

if $B(z)$ satisfies the gaps condition

$$\frac{\lambda_n}{n} \to \infty$$

as $n \to \infty$, we call $B(z)$ is an entire function with Fabry gaps. It has positive order which was shown in [10, p.651]. We apply this property to $B(z)$ in equation (3) and establish the following result.

Theorem 4. *Let $A(z)$ be a completely regular growth entire function and the set*

$$E = \{\theta \in [0, 2\pi) : h(\theta) = 0\}$$

is of Lebesgue measure zero, and let $B(z)$ be a transcendental entire function with Fabry gaps and $\rho(A) \neq \rho(B)$. Then every nontrivial solution of (3) *is of infinite order.*

In the last result, we assume $B(z)$ is a transcendental entire function satisfying

$$T(r, B) \sim \alpha \log M(r, B) \tag{7}$$

as $r \to \infty$ outside a set of finite logarithmic measure, where $0 < \alpha \leq 1$. This method was ever used in [20] and [26, Lemma 2.7]. The function $B(z)$ in Theorem 5 really exists. For example, entire function having Fejér gaps. Here,

$$f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$$

is said to have Fejér gaps if

$$\sum_{n=1}^{\infty} \lambda_n^{-1} < \infty,$$

see [23]. A result involved Fejér gaps concerning infinite growth of the solution of equation (3) was given in [16].

Theorem 5. *Let $A(z)$ be a completely regular growth entire function and the set*

$$E = \{\theta \in [0, 2\pi) : h(\theta) = 0\}$$

is of Lebesgue measure zero, and let $B(z)$ be a transcendental entire function satisfying $T(r, B) \sim \log \alpha M(r, B)$ as $r \to \infty$, where $0 < \alpha \leq 1$, outside a set of finite logarithmic measure such that $\rho(A) \neq \rho(B)$. Then every nontrivial solution of (3) *is of infinite order.*

We note that there exist many functions satisfy the condition (7). A simple example is the exponent function $B(z) = e^z$, for which the condition (7) holds for $\alpha = \frac{1}{\pi}$ without an exceptional set.

In the following, we give an example to illustrate the condition for $A(z)$ in the above results really exist. Firstly we introduce the definition so called $SCRG$. An example of completely regular growth function is the exponential sum

$$A(z) = \sum_{k=0}^{n-1} a_k \exp(b_k z),$$

provided

$$\arg b_k < \arg b_{k+1} < \arg b_k + \pi$$

for

$$0 \leq k \leq n-2$$

and

$$\arg b_0 < \arg b_{n-1} - \pi,$$

see details in [25]. In fact, exponential polynomial form an important subclass of functions of completely regular growth. It's well known [24] that the zeros of exponential sums are close to certain rays. Motivated by this we consider the functions satisfying the following condition, which are more general than the exponential polynomials.

Definition 6. If $A(z)$ is an entire function satisfying the following items, then we say $A(z)$ has the $SCRG$ (special completely regular growth) property.

1) Let the rays $\arg z = \theta_j$ be the accumulated lines of zeros of $A(z)$, where $j = 1, 2, \cdots, m$ and

$$\theta_1 < \theta_2 < \cdots < \theta_m < \theta_{m+1} = \theta_1 + 2\pi;$$

2) Let $h(\theta)$ be the indicator of $A(z)$ in the angle
$$S(\theta_j, \theta_{j+1}) = \{z : \theta_j < \arg z < \theta_{j+1})\},$$
$j = 1, 2, \cdots, m$ and $\rho(r)(\to \rho)$ be a proximate order of $A(z)$;
3)
$$\varepsilon(r) = 1/\log^N(r)$$
for some $N \in \mathbb{N}$, where $\log^N$ denotes the N-th iterate of the logarithm;
4)
$$\log|A(re^{i\theta})| = h(\theta)r^\rho + O(r^{\rho(r)}\varepsilon(r))$$
for $|\theta - \theta_j| > \varepsilon(r)$, $j = 1, 2, \cdots, m$.

The $SCRG$ property was firstly used in [2], in which some functions satisfying this property were given and the complex dynamical properties of entire function satisfying $SCRG$ were investigated. By Lemma 5 in the second section, it's easy to see that the function having $SCRG$ property satisfies the condition for $A(z)$ in Theorem 3 and 4. Inspired by this, we assume the coefficient $A(z)$ involving the $SCRG$ property and get the following result.

Corollary 7. *Let $A(z)$ be an entire function satisfying the $SCRG$ property, and let $B(z)$ be a transcendental entire function satisfying the condition for $B(z)$ in Theorem 3. Then every nontrivial solution of* (3) *is of infinite order.*

Corollary 8. *Let $A(z)$ be an entire function satisfying the $SCRG$ property, and let $B(z)$ be a transcendental entire function satisfying the condition for $B(z)$ in Theorem 4. Then every nontrivial solution of* (3) *is of infinite order.*

Corollary 9. *Let $A(z)$ be an entire function satisfying the $SCRG$ property, and let $B(z)$ be a transcendental entire function satisfying the condition for $B(z)$ in Theorem 5. Then every nontrivial solution of* (3) *is of infinite order.*

II. Preliminary lemmas and auxiliary results

The Lebesgue linear measure of a set $E \subset [0, \infty)$ is
$$meas(E) = \int_E dt,$$
and the logarithmic measure of a set $F \subset [1, \infty)$ is
$$m_l(F) = \int_F \frac{dt}{t}.$$
The upper and lower logarithmic densities of $F \subset [1, \infty)$ are given by
$$\overline{\log dens} F = \limsup_{r\to\infty} \frac{m_l(F \cap [1, r])}{\log r}$$
and
$$\underline{\log dens} F = \liminf_{r\to\infty} \frac{m_l(F \cap [1, r])}{\log r}$$
respectively. We say F has logarithmic density if
$$\overline{\log dens}(F) = \underline{\log dens}(F).$$
By the definitions of the logarithmic measure and the logarithmic density, we see that if the logarithmic density is positive, then the logarithmic measure is infinite.

In order to present the following two lemmas, we set
$$M(r, B) = \sup\{|B(z)| : |z| = r\},$$
$$m(r, B) = \inf\{|B(z)| : |z| = r\}$$
for an entire function $B(z)$. The first one is Barry's $\cos(\pi\rho)$ theorem.

Lemma 1. *([1]) Let $B(z)$ be an entire function with the lower order $\mu(B) < 1/2$, and*
$$\mu = \mu(B) < \rho(B) = \rho.$$
If
$$\mu \le \delta < \min\{\rho, 1/2\}$$
and
$$\delta < \alpha < 1/2,$$
then the set
$$G = \{r : \log m(r, B) > \cos\pi\alpha \log M(r, B) > r^\delta\} \tag{8}$$
satisfies
$$\underline{\log dens}(G) \ge C(\rho, \delta, \alpha),$$
where $C(\rho, \delta, \alpha)$ is a positive constant only dependent on ρ, δ, α.

We state the second lemma in regard to entire function with Fabry gaps. It can be found in [4, Theorem 1] and [9, Lemma 2.2].

Lemma 2. *([4, Theorem 1]) Let*
$$B(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$$
be an entire function of finite order with Fabry gaps. Then, for any given $\varepsilon > 0$,
$$\log m(r, B) > (1 - \varepsilon) \log M(r, B) \tag{9}$$
holds outside a set of logarithmic density 0.

The proofs of our results highly rely on the estimation of logarithmic derivatives, which is due to Gundersen [7].

Lemma 3. *[7] Let f be a transcendental meromorphic function of finite order $\rho(f)$. Let $\varepsilon > 0$ be a given real constant, and let k and j be two integers such that $k > j \ge 0$. Then there exists a set $E \subset (1, \infty)$ with $m_l(E) < \infty$, such that for all z satisfying $|z| \notin (E \cup [0, 1])$, we have*
$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le |z|^{(k-j)(\rho(f)-1+\varepsilon)}. \tag{10}$$

Lemma 4. *(Phragmén-Lindelöf principle) [14, Theorem 7.3] Let $f(z)$ be an analytic function of $z = re^{i\theta}$, regular in a region D between rays making an angle π/α at the origin and on the straight lines themselves. Suppose that $|f(z)| \le M$ on the lines and as $r \to \infty$,*
$$f(z) = O(e^{r^\beta}),$$
where $\beta < \alpha$ uniformly. Then $|f(z)| \le M$ throughout D.

Lemma 5. *[18, p.115, Corollary] If the zeros of entire function $A(z)$ of proximate order $\rho(r)$ are regular distribution for the index $\rho(r)$, and if the density of the set of zeros within some angle $S(\alpha, \beta)$ is equal to zero, then the indicator function within this angle is a trigonometric function, i. e.*
$$h(\theta) = a\cos\rho\theta + b\sin\rho\theta, \tag{11}$$

where $\alpha \leq \theta \leq \beta$, a and b are constants. If, however, inside this angle there are no zeros of the function, then for $\alpha < \theta < \beta$ there exists the limit

$$h(\theta) = \lim_{r\to\infty} \frac{\log|A(re^{\theta})|}{r^{\rho(r)}}, \tag{12}$$

where the variable tends to the limit uniformly when $\alpha+\varepsilon \leq \theta \leq \beta-\varepsilon$ for any given $\varepsilon > 0$.

The following result due to Gundersen [8, Theorem 3] shows the asymptotic properties of finite order solutions of equation (3).

Lemma 6. *Let $A(z)$ and $B(z)(\not\equiv 0)$ be two entire functions such that for real constants $\alpha, \beta, \theta_1, \theta_2$, where $\alpha > 0, \beta > 0$ and $\theta_1 < \theta_2$,*

$$|A(z)| \geq \exp\{(1+o(1))\alpha|z|^{\beta}\}$$

and

$$|B(z)| \leq \exp\{o(1)|z|^{\beta}\}$$

as $z \to \infty$ in

$$\overline{S}(\theta_1, \theta_2) = \{z : \theta_1 \leq \arg z \leq \theta_2\}.$$

Let $\varepsilon > 0$ be a given small constant, and let

$$\overline{S}(\theta_1+\varepsilon, \theta_2-\varepsilon) = \{z : \theta_1+\varepsilon \leq \arg z \leq \theta_2-\varepsilon\}.$$

If f is a nontrivial solution of (3) with $\rho(f) < \infty$, then the following conclusions hold.

1) *There exists a constant $b(\neq 0)$ such that $f(z) \to b$ as $z \to \infty$ in $\overline{S}(\theta_1+\varepsilon, \theta_2-\varepsilon)$. Furthermore,*
$$|f(z)-b| \leq \exp\{-(1+o(1))\alpha|z|^{\beta}\}$$
as $z \to \infty$ in $\overline{S}(\theta_1+\varepsilon, \theta_2-\varepsilon)$.

2) *For each integer $k \geq 1$,*
$$|f^{(k)}(z)| \leq \exp\{-(1+o(1))\alpha|z|^{\beta}\}$$
as $z \to \infty$ in $\overline{S}(\theta_1+\varepsilon, \theta_2-\varepsilon)$.

III. Proof of Theorems

A. Proof of Theorem 3

The case $\rho(A) < \rho(B)$ has been proved by Gundersen [8, Theorem 2], thus we assume $\rho(A) > \rho(B)$. Suppose that there is a nontrivial solution f of (3) with finite order. Set

$$E^* = \{\theta \in [0, 2\pi) : h(\theta) \leq 0\}.$$

We divide into two cases on basis of $meas(E^*) = 0$ or $meas(E^*) > 0$.

Case 1. Assume that $meas(E^*) = 0$, then the indicator of $A(z)$ satisfies $h(\theta) > 0$ for every $\theta \in [0, 2\pi) \setminus E^*$. We give the details following the idea from [29] for the convenience of reading. By (6), we have

$$\log|A(re^{i\theta})| = h(\theta)r^{\rho(r)} + o(r^{\rho(r)})$$

for $z = re^{i\theta}$ satisfying $\theta \in [0, 2\pi) \setminus E^*$ and except a C_0 set, where $\rho(r) \to \rho(A)$ as $r \to \infty$. Then for any given $\delta \in (0, \frac{\pi}{4\rho(A)})$ and $\eta \in (0, \frac{\rho(A)-\rho(B)}{4})$, we have

$$|A(z)| \geq \exp\{(1+o(1))\alpha|z|^{\rho(A)-\eta}\}, \tag{13}$$

$$\begin{aligned} |B(z)| &\leq \exp\{|z|^{\rho(B)+\eta}\} \\ &\leq \exp\{|z|^{\rho(A)-2\eta}\} \\ &\leq \exp\{o(1)|z|^{\rho(A)-\eta}\} \end{aligned} \tag{14}$$

as $z = re^{i\theta}(\to \infty)$ satisfying $\theta \in [0, 2\pi) \setminus E^*$ and except a C_0 set, where α is a positive constant depending on δ. Then by Lemma 6, there exist corresponding constants $b_j \neq 0$ such that

$$|f(z) - b_j| \leq \exp\{-(1+o(1))\alpha|z|^{\rho(A)-\eta}\} \tag{15}$$

as $z = re^{i\theta}(\to \infty)$ satisfying $\theta \in [0, 2\pi) \setminus E^*$ and except a C_0 set. Then $f(z)$ is bounded in the whole complex plane by the Phragmén-Lindelöf principle. Therefore, f is a constant in the complex plane by Liouville's theorem. Obviously, this is a contradiction.

Case 2. Assume $meas(E^*) > 0$, then there exist some angles in which the indicator of $A(z)$ satisfying $h(\theta) < 0$. We can choose a ray $\arg z = \theta^*$ in these angles such that $h(\theta^*) < 0$.

By Lemma 3, there exists a set $E \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E \cup [0, 1]$,

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq |z|^{2\rho(f)}, k = 1, 2. \tag{16}$$

Then, by Lemma 1, there exists a set $G \subset (1, \infty)$ with infinite logarithmic measure such that (8) holds for $r \in G$. The for $|z| = r \in G \setminus E$ as r sufficiently large, we obtain

$$\begin{aligned} \exp\{r^{\delta}\} &< m(r, B) \\ &\leq |B(z)| \\ &\leq \left|\frac{f''(z)}{f(z)}\right| + |A(z)|\left|\frac{f'(z)}{f(z)}\right| \\ &\leq (1+o(1))r^{2\rho(f)}, \end{aligned} \tag{17}$$

where $\delta > 0$. Thus we obtain a contradiction. Thus, we complete the proof.

Remark 1. The situation of case 1 can really happen, see Theorem 1.5 and the following content in paper [2].

B. Proof of Theorem 4

As the similar argument in subsection III-A, we only need to consider the situation $\rho(A) > \rho(B)$. Suppose that there is a nontrivial solution f of (3) with finite order. We treat two cases on basis of $meas(E^*) = 0$ or $meas(E^*) > 0$.

Case 1. Assume that $meas(E^*) = 0$, then the indicator of $A(z)$ satisfies $h(\theta) > 0$ for every $\theta \in [0, 2\pi) \setminus E^*$. The arguments are similar as Case 1 in subsection III-A.

Case 2. Assume $meas(E^*) > 0$, then there exist some angles in which the indicator of $A(z)$ satisfying $h(\theta) < 0$. Hence, there must exist an interval $I_A \in [0, 2\pi)$ such that $h(\theta) < 0$ for all $\theta \in I_A$. By Lemma 3, there exists a set $E_1 \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E_1 \cup [0, 1]$, (16) holds. In view of Lemma 2, there exists a set $G \subset (1, \infty)$ with infinite logarithmic measure such that (9) holds. Thus, combining (6), (9) with (16) we have, for any given $\varepsilon > 0$,

$$\begin{aligned} M(r,B)^{1-\varepsilon} &< m(r,B) \le |B(z)| \\ &\le \left|\frac{f''(z)}{f(z)}\right| + |A(z)| \left|\frac{f'(z)}{f(z)}\right| \\ &\le (1+o(1)) r^{2\rho(f)} \end{aligned} \tag{18}$$

for $z = re^{i\theta}$, $r \in G \setminus E_1, \theta \in I_A$ and outside a C_0 set. Thus,

$$\begin{aligned} (1-\varepsilon)T(r,B) &\le (1-\varepsilon)\log M(r,B) \\ &\le 2\rho(f)\log r + o(1) \end{aligned} \tag{19}$$

as $r \in G \backslash E_1$ sufficiently large. Since $B(z)$ is transcendental, we have

$$\lim_{r\to\infty} \frac{T(r,B)}{\log r} = \infty.$$

Thus, we get a contradiction from (19). Then we complete the proof.

C. Proof of Theorem 5

As the similar argument in subsection III-A, we only need to consider the situation $\rho(A) > \rho(B)$. Suppose that there is a nontrivial solution f of (3) with finite order. We treat two cases.

Case 1. Assume that the indicator of $A(z)$ satisfying $h(\theta) > 0$ for every $|\theta - \theta_j| > \varepsilon(r), j = 1, 2, \cdots, m$. The arguments is similar as Case 1 in subsection III-A.

Case 2. There exists θ^* satisfying $|\theta^* - \theta_j| > \varepsilon(r)$ such that $h(\theta^*) < 0$. Since the indicator of $A(z)$ is trigonometric function by Lemma 5, there must exist an interval $I_A \in [0, 2\pi)$ which contains θ^* such that $h(\theta) < 0$ for all $\theta \in I_A$. By Lemma 3, there exists a set $E_1 \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E_1 \cup [0, 1]$, (16) holds. For given $0 < c < 1$, set

$$I_B(r) = \{\theta \in [0, 2\pi) : \log |B(re^{i\theta})| \le c \log M(r, B)\} \tag{20}$$

and denote its Lebesgue measure by $meas(I_B(r))$. It follows from the definition of proximate function that

$$\begin{aligned} T(r,B) &= m(r,B) \\ &= \frac{1}{2\pi}\int_{I_B(r)} \log^+ |B(re^{i\theta})| d\theta \\ &+ \frac{1}{2\pi}\int_{[0,2\pi)\backslash I_B(r)} \log^+ |B(re^{i\theta})| d\theta \\ &\le c\frac{meas(I_B(r))}{2\pi} \log M(r,B) \\ &+ \left(\frac{2\pi - meas(I_B(r))}{2\pi}\right) \log M(r,B). \end{aligned} \tag{21}$$

Therefore, $T(r, B) \sim \alpha \log M(r, B)$ outsides a set E_2 of finite linear measure implies that $meas(I_B(r)) \le \frac{2\pi(1-\alpha)}{1-c}$ as $r(\notin E_2) \to \infty$. Hence, for $\alpha \in (0, 1)$ close enough to 1 we have $meas(I_B(r)) < meas(I_A)$. Combining (6), (16) with (20), it leads to

$$\begin{aligned} M(r,B)^c &\le |B(re^{i\theta})| \\ &\le \left|\frac{f''(re^{i\theta})}{f(re^{i\theta})}\right| + |A(re^{i\theta})| \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| \\ &\le (1+o(1)) r^{2\rho(f)} \end{aligned} \tag{22}$$

for $r(\notin E_1 \cup E_2 \cup [0, 1])$ sufficiently large and $\theta \in I_A \setminus I_B(r)$. Since $B(z)$ is transcendental, we obtain a contradiction.

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