Existence and Iteration of Monotone Positive Solutions for Fractional Boundary Value Problems with Riesz-Caputo Derivative

Meiling Li, Yunqing Wang*

Abstract—**As we all know, both the past and the future nonlocal memory effects can be characterized by the Riesz-Caputo derivative. A class of three-point boundary value problems for fractional differential equations with the Riesz-Caputo derivative is studied in this paper. We find the positive solutions for the fractional problem by applying the technique of monotone iterative. Moreover, an iterative scheme for approximating the solutions is given in the paper. Finally, an example is given.**

Index Terms—**Iteration; Monotone positive solution; Fractional differential equation; Riesz-Caputo derivative.**

I. Introduction

When we describe the hereditary properties and memory of various processes and materials, the fractional derivative plays a very important role. As a result, fractional differential equations is attracting more and more attention, see [1-5] and the references therein. Recently, there have been many discussions on the existence of solutions for fractional initial value problems and boundary value problems (see [6-15]). Most of the results where provided by use of the left Riemann-Liouville and Caputo derivative, these two fractional operators only reflected the past or future memory effect.

In the application of real world, there are many processes which started at the past states, also relying on its development in the future, for example, stock price option. Another example is the applications to anomalous diffusion problem, among which there is Riesz derivative. Nonlocality is implicit in the Riesz derivative, so it can be used to describe the diffusion concentration's dependence on path. All these practical problems prompt us to introduce Riesz fractional derivatives, which is a two-sided fractional operator including both left and right derivative. Past and future memory effects can be realized by Riesz fractional derivatives. Some natural systems including aquifers, rivers and heterogeneous soils involve space Riesz fractional diffusion equations. It is typically observed to be non-Fickian, also called anomalous (see [25]). A one component system is governed by the equation

$$\frac{\partial u}{\partial t} = K\frac{\partial^\alpha u}{\partial |x|^\alpha} + f(u,t),$$

where $\frac{\partial^\alpha u}{\partial |x|^\alpha}$ is the Riesz fractional derivative.

$$\begin{aligned}\frac{\partial^2 u(x,t)}{\partial t^2} \quad &+2\alpha\frac{\partial u(x,t)}{\partial t} + \beta^2 u(x,t)\\ &= \eta^R D_x^\gamma u(x,t) + f(x,t),\\ &a \le x \le b,\ 0 \le t \le T,\ 1 < \gamma \le 2,\end{aligned}$$

where $\alpha > \beta \ge 0,\ \eta > 0,\ {}^R D_x^\gamma$ is the Riesz fractional derivative. The above equation can be used to represent the fractional telegraph equation.

There is a lot of literature on numerical solutions of space fractional diffusion equations with Riesz derivative [16-19] and fractional variational problems with the Riesz Caputo derivative [20]. Nevertheless, there are less results about positive solutions for fractional boundary value problems with Riesz-Caputo derivative.

The authors of [21] discussed positive solutions of fractional differential equations with the Riesz space derivative

$${}_0^{RC}D_1^\alpha x(\xi) = h(\xi, x(\xi)), \quad \xi \in [0,1], \quad 0 < \alpha \le 1,$$

$$x(0) = x_0, \quad x(1) = x_1,$$

where ${}_0^{RC}D_1^\alpha$ is the Riesz Caputo derivative.

[24] obtained the existence results for anti-periodic boundary value problems with Riesz Caputo derivative

$${}_0^{RC}D_T^\gamma y(\tau) = g(\tau, y(\tau)), \quad \tau \in J, \quad J = [0,T], \quad 1 < \gamma \le 2,$$

$$y(0) + y(T) = 0, \quad y'(0) + y'(T) = 0,$$

where ${}_0^{RC}D_T^\gamma$ is the Riesz Caputo derivative.

Motivated by the above mentioned results, in this paper, we discussed the following fractional problem

$${}_0^{RC}D_1^\gamma y(t) = f(t, y(t)), \quad t \in [0,1], \quad 0 < \gamma \le 1, \tag{1}$$

$$y(0) = a, \quad y(1) = by(\eta), \tag{2}$$

where ${}_0^{RC}D_1^\gamma$ is the Riesz Caputo derivative, $f \in C([0,1] \times [0,+\infty), [0,+\infty)),\ 0 < \eta < 1,\ a > 0,\ 0 < b < 2$. We found the positive solutions for the fractional problem (1), (2) by applying the technique of monotone iterative. Moreover, an iterative scheme for approximating the solutions was given in the paper. To our knowledge, this is the first paper to use the technique of monotone iterative to deal with fractional boundary value problem with Riesz Caputo derivative.

II. The preliminary lemmas

Let $\alpha > 0$ and $n-1 < \alpha \le n,\ n \in N$ and $n = [\nu]$, and $[\cdot]$ the ceiling of a number.

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Definition 2.1 [2] For a function $z(t),\ 0\le t\le T$, the Riesz-Caputo fractional derivative is

$$\begin{aligned} {}_{0}^{RC}D_T^{\alpha}z(t) &= \frac{1}{\Gamma(n-\alpha)}\int_0^T \frac{z^{(n)}(u)}{|t-u|^{\alpha+1-n}}du \\ &= \frac{1}{2}\big({}_{0}^{C}D_t^{\alpha}+(-1)^n {}_{t}^{C}D_T^{\alpha}\big)z(t), \end{aligned}$$

here ${}_{0}^{C}D_t^{\alpha}$ and ${}_{t}^{C}D_T^{\alpha}$ stand for the left and right Caputo derivative respectively.

$${}_{0}^{C}D_t^{\alpha}z(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{z^{(n)}(u)}{(t-u)^{\alpha+1-n}}du$$

and

$${}_{t}^{C}D_T^{\alpha}z(t)=\frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^T\frac{z^{(n)}(u)}{(u-t)^{\alpha+1-n}}du.$$

In particular, if $0<\alpha\le 1$ and $z(t)\in C(0,T)$, then

$${}_{0}^{RC}D_T^{\alpha}z(t)=\frac{1}{2}\big({}_{0}^{C}D_t^{\alpha}-{}_{t}^{C}D_T^{\alpha}\big)z(t).$$

Definition 2.2 [3]

$${}_{0}I_t^{\alpha}z(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-u)^{\alpha-1}z(u)du,$$

$${}_{t}I_T^{\alpha}z(t)=\frac{1}{\Gamma(\alpha)}\int_t^T(u-t)^{\alpha-1}z(u)du,$$

$${}_{0}I_T^{\alpha}z(t)=\frac{1}{\Gamma(\alpha)}\int_0^T|u-t|^{\alpha-1}z(u)du.$$

stand for the fractional left, right and Riemann-Liouville integrals of order α respectively.

Lemma 2.1 *[3]* If $z(t)\in C^n[0,T]$, then

$${}_{0}I_t^{\alpha}\,{}_{0}^{C}D_t^{\alpha}z(t)=z(t)-\sum_{l=0}^{n-1}\frac{z^{(l)}(0)}{l!}(t-0)^l$$

and

$${}_{t}I_T^{\alpha}\,{}_{t}^{C}D_T^{\alpha}z(t)=(-1)^n\Big[z(t)-\sum_{l=0}^{n-1}\frac{(-1)^l z^{(l)}(T)}{l!}(T-t)^l\Big].$$

$$\begin{aligned} &{}_{0}I_T^{\alpha}\,{}_{0}^{RC}D_T^{\alpha}z(t) \\ &=\frac{1}{2}\big({}_{0}I_t^{\alpha}\,{}_{0}^{C}D_t^{\alpha}+{}_{t}I_T^{\alpha}\,{}_{0}^{C}D_t^{\alpha}\big)z(t) \\ &+(-1)^n\frac{1}{2}\big({}_{0}I_t^{\alpha}\,{}_{t}^{C}D_T^{\alpha}+{}_{t}I_T^{\alpha}\,{}_{t}^{C}D_T^{\alpha}\big)z(t) \\ &=\frac{1}{2}\big({}_{0}I_t^{\alpha}\,{}_{0}^{C}D_t^{\alpha}+(-1)^n_t I_T^{\alpha}\,{}_{t}^{C}D_T^{\alpha}\big)z(t) \end{aligned}$$

can be got from the above definitions and lemmas.

In particular, if $0<\alpha\le 1$ and $z(t)\in C(0,T)$, then

$${}_{0}I_T^{\alpha}\,{}_{0}^{RC}D_T^{\alpha}z(t)=z(t)-\frac{1}{2}(z(0)+z(T)). \tag{3}$$

Lemma 2.2 Assume that $f\in C([0,1],R)$. A function $y\in C[0,1]$ given by

$$\begin{aligned} y(t) &= \frac{a}{2}+\frac{ab}{4-2b}+\frac{b}{(2-b)\Gamma(\gamma)}\int_0^{\eta}(\eta-u)^{\gamma-1}f(u)du \\ &+\frac{b}{(2-b)\Gamma(\gamma)}\int_{\eta}^{1}(u-\eta)^{\gamma-1}f(u)du \\ &+\frac{1}{\Gamma(\gamma)}\int_0^t(t-u)^{\gamma-1}f(u)du \\ &+\frac{1}{\Gamma(\gamma)}\int_t^1(u-t)^{\gamma-1}f(u)du \end{aligned} \tag{4}$$

is a unique solution of the following fractional boundary value problem

$${}_{0}^{RC}D_1^{\gamma}y(t)=f(t),\quad t\in[0,1],\quad 0<\gamma\le 1, \tag{5}$$

$$y(0)=a,\quad y(1)=by(\eta). \tag{6}$$

Proof:

$$\begin{aligned} y(t) &= \frac{1}{2}y(0)+\frac{1}{2}y(1)+\frac{1}{\Gamma(\gamma)}\int_0^1|t-u|^{\gamma-1}f(u)du \\ &= \frac{1}{2}y(0)+\frac{1}{2}y(1)+\frac{1}{\Gamma(\gamma)}\int_0^t(t-u)^{\gamma-1}f(u)du \\ &+\frac{1}{\Gamma(\gamma)}\int_t^1(u-t)^{\gamma-1}f(u)du \end{aligned} \tag{7}$$

can be got by (3) and the equality (5). Then

$$\begin{aligned} y(\eta) &= \frac{1}{2}y(0)+\frac{1}{2}y(1)+\frac{1}{\Gamma(\gamma)}\int_0^{\eta}(\eta-u)^{\gamma-1}f(u)du \\ &+\frac{1}{\Gamma(\gamma)}\int_{\eta}^1(u-\eta)^{\gamma-1}f(u)du. \end{aligned}$$

By the boundary conditions $y(0)=a,\quad y(1)=by(\eta)$, we get

$$\begin{aligned} y(1) &= \frac{1}{2}ab+\frac{1}{2}by(1)+\frac{b}{\Gamma(\gamma)}\int_0^{\eta}(\eta-u)^{\gamma-1}f(u)du \\ &+\frac{b}{\Gamma(\gamma)}\int_{\eta}^1(u-\eta)^{\gamma-1}f(u)du, \end{aligned}$$

thus,

$$\begin{aligned} y(1) &= \frac{ab}{2(1-\frac{b}{2})}+\frac{b}{(1-\frac{b}{2})\Gamma(\gamma)}\int_0^{\eta}(\eta-u)^{\gamma-1}f(u)du \\ &+\frac{b}{(1-\frac{b}{2})\Gamma(\gamma)}\int_{\eta}^1(u-\eta)^{\gamma-1}f(u)du. \end{aligned}$$

Substituting $y(1)$ into (7), we have

$$\begin{aligned} y(t) &= \frac{a}{2}+\frac{ab}{4-2b}+\frac{b}{(2-b)\Gamma(\gamma)}\int_0^{\eta}(\eta-u)^{\gamma-1}f(u)du \\ &+\frac{b}{(2-b)\Gamma(\gamma)}\int_{\eta}^{1}(u-\eta)^{\gamma-1}f(u)du \\ &+\frac{1}{\Gamma(\gamma)}\int_0^t(t-u)^{\gamma-1}f(u)du \\ &+\frac{1}{\Gamma(\gamma)}\int_t^1(u-t)^{\gamma-1}f(u)du. \end{aligned}$$

■

III. Main results

Let the space $X=C[0,1]$ be endowed with the maximum norm $\|y\|=\max\limits_{0\le t\le 1}|y(t)|$. It is well known that X is a Banach space. Define the cone $K\subset X$ by

$$K=\{y\in X:\ y(t)\ge 0,\quad 0\le t\le 1\}. \tag{8}$$

Let $T:\ K\to K$ be the operator defined by

$$\begin{aligned} Ty(t) &= \frac{a}{2}+\frac{ab}{4-2b} \\ &+\frac{b}{(2-b)\Gamma(\gamma)}\int_0^{\eta}(\eta-u)^{\gamma-1}f(u,y(u))du \\ &+\frac{b}{(2-b)\Gamma(\gamma)}\int_{\eta}^{1}(u-\eta)^{\gamma-1}f(u,y(u))du \\ &+\frac{1}{\Gamma(\gamma)}\int_0^t(t-u)^{\gamma-1}f(u,y(u))du \\ &+\frac{1}{\Gamma(\gamma)}\int_t^1(u-t)^{\gamma-1}f(u,y(u))du. \end{aligned} \tag{9}$$

Lemma 3.1 The operator $T:\ K\rightarrow K$ is completely continuous.

Proof: Firstly, we claim that $T:\ K\rightarrow K$ is continuous. Since $f\in C([0,1]\times[0,+\infty),[0,+\infty))$, let $y_1,y_2\in[0,+\infty)$ and for $\forall\ \varepsilon>0,\ \exists\delta>0$, when $|y_1(t)-y_2(t)|<\delta$, we have

$$|f(t,y_1(t))-f(t,y_2(t))|<\frac{(2-b)\Gamma(\gamma+1)\varepsilon}{b[\eta^\gamma+(1-\eta)^\gamma]+2(2-b)}.$$

Thus, we have

$$\begin{aligned}
&|Ty_1(t)-Ty_2(t)|\\
&\le\frac{b}{(2-b)\Gamma(\gamma)}\int_0^\eta(\eta-u)^{\gamma-1}|f(u,y_1(u))-f(u,y_2(u))|du\\
&+\frac{b}{(2-b)\Gamma(\gamma)}\int_\eta^1(u-\eta)^{\gamma-1}|f(u,y_1(u))-f(u,y_2(u))|du\\
&+\frac{1}{\Gamma(\gamma)}\int_0^t(t-u)^{\gamma-1}|f(u,y_1(u))-f(u,y_2(u))|du\\
&+\frac{1}{\Gamma(\gamma)}\int_t^1(u-t)^{\gamma-1}|f(u,y_1(u))-f(u,y_2(u))|du\\
&<\frac{(2-b)\Gamma(\gamma+1)\varepsilon}{b[\eta^\gamma+(1-\eta)^\gamma]+2(2-b)}\\
&\bigg[\frac{b}{(2-b)\Gamma(\gamma)}\int_0^\eta(\eta-u)^{\gamma-1}du\\
&+\frac{b}{(2-b)\Gamma(\gamma)}\int_\eta^1(u-\eta)^{\gamma-1}du\\
&+\frac{1}{\Gamma(\gamma)}\int_0^t(t-u)^{\gamma-1}du+\frac{1}{\Gamma(\gamma)}\int_t^1(u-t)^{\gamma-1}du\bigg]\\
&<\frac{(2-b)\Gamma(\gamma+1)\varepsilon}{b[\eta^\gamma+(1-\eta)^\gamma]+2(2-b)}\\
&\left[\frac{b[\eta^\gamma+(1-\eta)^\gamma]}{(2-b)\Gamma(\gamma)\gamma}+\frac{t^\gamma+(1-t)^\gamma}{\Gamma(\gamma)\gamma}\right]\\
&<\frac{(2-b)\Gamma(\gamma+1)\varepsilon}{b[\eta^\gamma+(1-\eta)^\gamma]+2(2-b)}\\
&\frac{b[\eta^\gamma+(1-\eta)^\gamma]+2(2-b)}{(2-b)\Gamma(\gamma+1)}=\varepsilon
\end{aligned}$$

which implies that T is continuous. Similar to [21], we can prove T is compact. ■

Theorem 3.2 Let $0<\gamma\le1,\ f\in C([0,1]\times[0,+\infty),[0,+\infty)),\ 0<\eta<1,\ a>0,\ 0<b<2$. Suppose that there exist two positive constants r_1,r_2 ($r_2>a+\dfrac{ab}{2-b}>r_1$) satisfying

(H_1) $f(t,v)\le\dfrac{(2-b)\Gamma(\gamma+1)r_2}{2[b(\eta^\gamma+(1-\eta)^\gamma)+2(2-b)]}$ for $(t,v)\in[0,1]\times[0,r_2]$;

(H_2) $f(t,v)\ge\dfrac{(2-b)\Gamma(\gamma+1)r_1}{2[b(\eta^\gamma+(1-\eta)^\gamma)+(2-b)]}$ for $(t,v)\in[0,1]\times[0,r_1]$.

Then there is at least one positive solution of fractional boundary value problem (1)(2).

Proof: Let $\Omega_1=\{y\in K:\ \|y\|<r_1\}$. For $y\in K\cap\partial\Omega_1$, one has $0\le y(t)\le r_1,\ 0\le t\le1$. From (H_2), we get

$$\begin{aligned}
Ty(1)\ &=\frac{a}{2}+\frac{ab}{4-2b}\\
&+\frac{b}{(2-b)\Gamma(\gamma)}\int_0^\eta(\eta-u)^{\gamma-1}f(u,y(u))du\\
&+\frac{b}{(2-b)\Gamma(\gamma)}\int_\eta^1(u-\eta)^{\gamma-1}f(u,y(u))du\\
&+\frac{1}{\Gamma(\gamma)}\int_0^1(1-u)^{\gamma-1}f(u,y(u))du\\
&>\frac{r_1}{2}+\frac{(2-b)\Gamma(\gamma+1)r_1}{2[b(\eta^\gamma+(1-\eta)^\gamma)+(2-b)]}\\
&\bigg[\frac{b}{(2-b)\Gamma(\gamma)}\int_0^\eta(\eta-u)^{\gamma-1}du\\
&+\frac{b}{(2-b)\Gamma(\gamma)}\int_\eta^1(u-\eta)^{\gamma-1}du\\
&+\frac{1}{\Gamma(\gamma)}\int_0^1(1-u)^{\gamma-1}du\bigg]\\
&=\frac{r_1}{2}+\frac{(2-b)\Gamma(\gamma+1)r_1}{2[b(\eta^\gamma+(1-\eta)^\gamma)+(2-b)]}\\
&\left[\frac{b(\eta^\gamma+(1-\eta)^\gamma)}{(2-b)\Gamma(\gamma)\gamma}+\frac{1}{\Gamma(\gamma)\gamma}\right]\\
&=r_1.
\end{aligned}$$

So, $\|Ty\|\ge r_1=\|y\|$ for $y\in K\cap\partial\Omega_1$.

Let $\Omega_2=\{y\in K:\ \|y\|<r_2\}$. For $y\in K\cap\partial\Omega_2$, one has $0\le y(t)\le r_2,\ 0\le t\le1$. From (H_1), we have

$$\begin{aligned}
Ty(t)\ &=\frac{a}{2}+\frac{ab}{4-2b}\\
&+\frac{b}{(2-b)\Gamma(\gamma)}\int_0^\eta(\eta-u)^{\gamma-1}f(u,y(u))du\\
&+\frac{b}{(2-b)\Gamma(\gamma)}\int_\eta^1(u-\eta)^{\gamma-1}f(u,y(u))du\\
&+\frac{1}{\Gamma(\gamma)}\int_0^t(t-u)^{\gamma-1}f(u,y(u))du\\
&+\frac{1}{\Gamma(\gamma)}\int_t^1(u-t)^{\gamma-1}f(u,y(u))du\\
&<\frac{r_2}{2}+\frac{(2-b)\Gamma(\gamma+1)r_2}{2[b(\eta^\gamma+(1-\eta)^\gamma)+2(2-b)]}\\
&\bigg[\frac{b}{(2-b)\Gamma(\gamma)}\int_0^\eta(\eta-u)^{\gamma-1}du\\
&+\frac{b}{(2-b)\Gamma(\gamma)}\int_\eta^1(u-\eta)^{\gamma-1}du\\
&+\frac{1}{\Gamma(\gamma)}\int_0^t(t-u)^{\gamma-1}du+\frac{1}{\Gamma(\gamma)}\int_t^1(u-t)^{\gamma-1}du\bigg]\\
&\le\frac{r_2}{2}+\frac{(2-b)\Gamma(\gamma+1)r_2}{2[b(\eta^\gamma+(1-\eta)^\gamma)+2(2-b)]}\\
&\left[\frac{b(\eta^\gamma+(1-\eta)^\gamma)}{(2-b)\Gamma(\gamma)\gamma}+\frac{2}{\Gamma(\gamma)\gamma}\right]\\
&=r_2.
\end{aligned}$$

So, $\|Ty\|\le\|y\|$ for $y\in K\cap\partial\Omega_2$. Therefore, an application of Krasnosel'skii fixed point theorem implies the fractional boundary value problem (1)(2) has at least one positive solution. ■

Theorem 3.3 Let $0<\gamma\le1,\ f\in C([0,1]\times[0,+\infty),[0,+\infty)),\ 0<\eta<1,\ a>0,\ 0<b<2$. If there exists $c>\dfrac{a}{2}+\dfrac{ab}{4-2b}$ such that

(H_3) $f(t,x_1)\le f(t,x_2)$ for any $0\le t\le1,\ 0\le x_1\le x_2\le c$;

(H_4) $\max\limits_{0\le t\le 1} f(t,c) \le \dfrac{\left[c-\left(\frac{a}{2}+\dfrac{ab}{4-2b}\right)\right](2-b)\Gamma(\gamma+1)}{b(\eta^\gamma+(1-\eta)^\gamma)+2(2-b)}$.

Then the fractional boundary value problem (1)(2) has at least one positive solution ω^* with $0<\omega^*\le c$ and $\lim\limits_{n\to\infty}\omega_n=\lim\limits_{n\to\infty}T^n\omega_0=\omega^*$,where $\omega_0(t)=c,\ 0\le t\le 1$.

Proof: Let

$$K_c=\{y\in K|\ \ \|y\|<c\}$$

and

$$\overline{K_c}=\{y\in K|\ \ \|y\|\le c\}.$$

Then, in what follows, we first prove $T:\ \overline{K_c}\to\overline{K_c}$. Let $y\in\overline{K_c}$, then $\|y\|\le c$, which implies

$$0\le y(t)\le\max_{0\le t\le 1}|y(t)|\le\|y\|\le c.$$

By $(H_3),(H_4)$, we get

$$\begin{aligned}0\ &\le f(t,y(t))\le f(t,c)\le\max_{0\le t\le 1}f(t,c)\\ &\le\frac{\left[c-\left(\frac{a}{2}+\dfrac{ab}{4-2b}\right)\right](2-b)\Gamma(\gamma+1)}{b(\eta^\gamma+(1-\eta)^\gamma)+2(2-b)},\ 0\le t\le 1.\end{aligned}$$

Then by (9), we have

$$\begin{aligned}&Ty(t)=\frac{a}{2}+\frac{ab}{4-2b}\\ &+\frac{b}{(2-b)\Gamma(\gamma)}\int_0^\eta(\eta-u)^{\gamma-1}f(u,y(u))du\\ &+\frac{b}{(2-b)\Gamma(\gamma)}\int_\eta^1(u-\eta)^{\gamma-1}f(u,y(u))du\\ &+\frac{1}{\Gamma(\gamma)}\int_0^t(t-u)^{\gamma-1}f(u,y(u))du\\ &+\frac{1}{\Gamma(\gamma)}\int_t^1(u-t)^{\gamma-1}f(u,y(u))du\\ &<\frac{a}{2}+\frac{ab}{4-2b}+\frac{\left[c-\left(\frac{a}{2}+\dfrac{ab}{4-2b}\right)\right](2-b)\Gamma(\gamma+1)}{b(\eta^\gamma+(1-\eta)^\gamma)+2(2-b)}\\ &\left[\frac{b}{(2-b)\Gamma(\gamma)}\int_0^\eta(\eta-u)^{\gamma-1}du\right.\\ &+\frac{b}{(2-b)\Gamma(\gamma)}\int_\eta^1(u-\eta)^{\gamma-1}du\\ &\left.+\frac{1}{\Gamma(\gamma)}\int_0^t(t-u)^{\gamma-1}du+\frac{1}{\Gamma(\gamma)}\int_t^1(u-t)^{\gamma-1}du\right]\\ &\le\frac{a}{2}+\frac{ab}{4-2b}\\ &+\frac{\left[c-\left(\frac{a}{2}+\dfrac{ab}{4-2b}\right)\right](2-b)\Gamma(\gamma+1)}{b(\eta^\gamma+(1-\eta)^\gamma)+2(2-b)}\\ &\left[\frac{b(\eta^\gamma+(1-\eta)^\gamma)}{(2-b)\Gamma(\gamma+1)}+\frac{2}{\Gamma(\gamma+1)}\right]\\ &=\frac{a}{2}+\frac{ab}{4-2b}+\left[c-\left(\frac{a}{2}+\frac{ab}{4-2b}\right)\right]\\ &=c.\end{aligned}$$

Thus, we have

$$\|Ty\|\le c.$$

So, we obtain $T:\ \overline{K_c}\to\overline{K_c}$. Denote

$$\omega_0(t)=c,\ 0\le t\le 1.$$

Let $\omega_1=T\omega_0,\ \omega_2=T\omega_1=T^2\omega_0$, then we write

$$\omega_{n+1}=T\omega_n=T^n\omega_0,\ n=1,2,3,\cdots,\tag{10}$$

Since $T:\ \overline{K_c}\to\overline{K_c}$, we have

$$\omega_n\in T\overline{K_c}\subseteq\overline{K_c},\ n=1,2,3,\cdots.$$

By Lemma 3.1, T is compact. We claim that $\{\omega_n\}_{n=1}^\infty$ has a convergent subsequence $\{\omega_{nk}\}_{k=1}^\infty$ and there exists $\omega^*\in\overline{K_c}$, such that $\omega_{nk}\to\omega^*$. Now since

$$\begin{aligned}\omega_1(t)\ &=T\omega_0(t)=\frac{a}{2}+\frac{ab}{4-2b}\\ &+\frac{b}{(2-b)\Gamma(\gamma)}\int_0^\eta(\eta-u)^{\gamma-1}f(u,\omega_0(u))du\\ &+\frac{b}{(2-b)\Gamma(\gamma)}\int_\eta^1(u-\eta)^{\gamma-1}f(u,\omega_0(u))du\\ &+\frac{1}{\Gamma(\gamma)}\int_0^t(t-u)^{\gamma-1}f(u,\omega_0(u))du\\ &+\frac{1}{\Gamma(\gamma)}\int_t^1(u-t)^{\gamma-1}f(u,\omega_0(u))du\\ &\le\frac{a}{2}+\frac{ab}{4-2b}\\ &+\frac{\left[c-\left(\frac{a}{2}+\dfrac{ab}{4-2b}\right)\right](2-b)\Gamma(\gamma+1)}{b(\eta^\gamma+(1-\eta)^\gamma)+2(2-b)}\\ &\left[\frac{b(\eta^\gamma+(1-\eta)^\gamma)}{(2-b)\Gamma(\gamma+1)}+\frac{2}{\Gamma(\gamma+1)}\right]\\ &=\frac{a}{2}+\frac{ab}{4-2b}+\left[c-\left(\frac{a}{2}+\frac{ab}{4-2b}\right)\right]\\ &=c=\omega_0(t),\qquad 0\le t\le 1.\end{aligned}$$

Then we have $\omega_1(t)\le\omega_0(t),\quad 0\le t\le 1$. So,

$$\omega_2(t)=T\omega_1(t)\le T\omega_0(t)=\omega_1(t),\quad 0\le t\le 1.$$

Hence, by induction, we have $\omega_{n+1}(t)\le\omega_n(t),\quad 0\le t\le 1,\quad (n=0,1,2\cdots)$. Thus, we assert that $\omega_n\to\omega^*$. Let $n\to\infty$ in (10), we have $T\omega^*=\omega^*$ since T is continuous. Since $\dfrac{a}{2}+\dfrac{ab}{4-2b}>0$, then the zero function is not the solution of (1)(2). Thus $\max\limits_{0\le t\le 1}|\omega^*|>0$, we conclude that $\omega^*(t)>0,\quad t\in(0,1)$. Therefore, ω^* is a positive solution of problem (1)(2). ■

IV. EXAMPLE

Example 4.1 Consider the fractional boundary value problem

$${}_0^{RC}D_1^{\frac{1}{2}}y(t)=\frac{1}{10}t+e^{-y},\quad t\in[0,1],\tag{11}$$

$$y(0)=1,\quad y(1)=y(\frac{1}{2}),\tag{12}$$

we notice that $\gamma=\frac{1}{2},\ a=1,\ b=1,\ \eta=\frac{1}{2},\ f(t,y)=\frac{1}{10}t+e^{-y}$, it follows from a direct calculation that $\Gamma(\gamma+1)=\Gamma(\frac{3}{2})\approx 0.886,\quad \eta^\gamma=(\frac{1}{2})^{\frac{1}{2}}\approx 0.707,\quad (1-\eta)^\gamma=(\frac{1}{2})^{\frac{1}{2}}\approx 0.707$. Choose $r_1=1,\quad r_2=10$, we get $a+\dfrac{ab}{2-b}=2$, so, $r_2>a+\dfrac{ab}{2-b}>r_1$ holds. Furthermore,

$$\begin{aligned}f(t,y)\ &=\tfrac{1}{10}t+e^{-y}\le 1.1<1.296\\ &\le\frac{(2-b)\Gamma(\gamma+1)r_2}{2[b(\eta^\gamma+(1-\eta)^\gamma)+2(2-b)]}\\ &\ \text{for }(t,y)\in[0,1]\times[0,10];\end{aligned}$$

$$
\begin{aligned}
f(t,y) &= \frac{1}{10}t + e^{-y} \geq e^{-y} > 0.368 \geq 0.184 \\
&\geq \frac{(2-b)\Gamma(\gamma+1)r_1}{2[b(\eta^{\gamma}+(1-\eta)^{\gamma})+(2-b)]} \\
&\text{for } (t,y) \in [0,1] \times [0,1].
\end{aligned}
$$

Then, all conditions of Theorem 3.2 hold. Thus, with Theorem 3.2, fractional boundary value problem (11)(12) has at least one positive solution.

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