Application of a New Model for Complex Systems

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***Abstract*—In this article, we propose a three-parameter generalized inverted exponential inverted Weibull distribution (GIEIWD) and use the proposed distribution to model complex life data. The considered goodness-of-fit statistics and quantile-quantile (QQ) plots show the new model fits these data well. The fitting results show the usefulness, applicability and superiority of the proposed distribution.**

***Index Terms*—GIEIWD, MLEs, application.**

I. Introduction

In reliability engineering and survival analysis, data with upside-down bathtub shaped hazard rate are common. However, some well-known distributions such as the exponential, Rayleigh [1] and Weibull [2] distributions do not exhibit a upside-down bathtub shaped hazard rate function (HRF) and thus they can not be used to model the lifetimes of some complex systems in engineering. Hence, a number of extended distributions are introduced to overcome this shortage. For example, Sarhan and Kundu [3] proposed a modified Weibull distribution. Singla et al.[4] proposed a beta generalized Weibull distribution. Kundu and Raqab [5] proposed a generalized Rayleigh distribution. Sarhan and Kundu [6] proposed a generalized linear failure rate distribution. Nadarajah et al. [7] proposed a beta-modified Weibull distribution. It is well known that model with less parameters is convenient in practical application. In this paper, we propose a new distribution with less parameters and illustrate its usefulness, applicability and superiority using three real data sets from engineering.

II. The new model and estimation method

The cumulative distribution function (CDF), the probability density function (PDF) and HRF of the proposed distribution are given by

$$G(x;\alpha,\beta,\gamma) = 1-(1-e^{-\beta x^{-\alpha}})^{\gamma}, x>0; \alpha,\beta,\gamma>0, \quad (1)$$

$$g(x;\alpha,\beta,\gamma) = \alpha\beta\gamma e^{-\beta x^{-\alpha}}(1-e^{-\beta x^{-\alpha}})^{\gamma-1}x^{-\alpha-1}, \quad (2)$$

and

$$h(x;\alpha,\beta,\gamma) = \frac{\alpha\beta\gamma e^{-\beta x^{-\alpha}}x^{-\alpha-1}}{1-e^{-\beta x^{-\alpha}}}. \quad (3)$$

The new distribution includes some important special subclasses. For example, for $\alpha = 1$, model (1) can be deemed as a generalized inverted exponential distribution (GIED) with two parameters; for $\gamma = 1$, model (1) can be deemed as a inverse Weibull distribution (IWD) with two parameters. Therefore, model (1) can be deemed as generalization of the GIED and IWD, so we defined our model (1) as GIEIWD. The plot in Fig.1 indicates that the HRF of the new model can take upside-down bathtub forms.

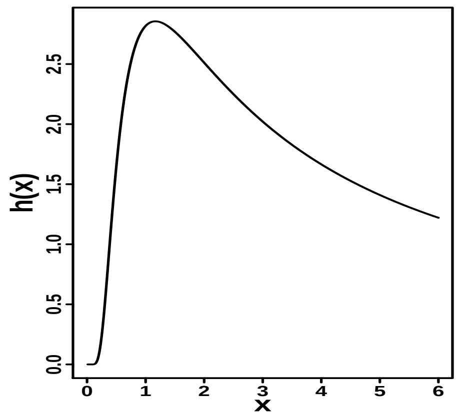

Fig. 1: Plot of the HRF for $\alpha = 0.9, \beta = 2, \gamma = 10$

We use maximum likelihood estimations (MLEs) to estimate the parameters (α,β,γ) of the proposed distribution. Let $(x_1, x_2, \cdots, x_n)$ be a random sample of size n from GIEIWD. The log-likelihood function $\log L$ is given by

$$\begin{aligned}\log L &= n\log\alpha + n\log\beta + n\log\gamma - \beta\sum_{i=1}^{n}x_i^{-\alpha} \\ &\quad +(\gamma-1)\sum_{i=1}^{n}\log(1-e^{-\beta x_i^{-\alpha}}) \\ &\quad -(\alpha+1)\sum_{i=1}^{n}\log x_i. \end{aligned} \quad (4)$$

Then

$$\begin{aligned}\frac{\partial \log L}{\partial \alpha} &= \frac{n}{\alpha} + \beta\sum_{i=1}^{n}x_i^{-\alpha}\log x_i + \beta(\gamma-1) \\ &\quad \times\sum_{i=1}^{n}\frac{-e^{-\beta x_i^{-\alpha}}}{1-e^{-\beta x_i^{-\alpha}}}x_i^{-\alpha}\log x_i - \sum_{i=1}^{n}\log x_i,\end{aligned}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n}x_i^{-\alpha} + (\gamma-1)\sum_{i=1}^{n}x_i^{-\alpha}\frac{-e^{-\beta x_i^{-\alpha}}}{1-e^{-\beta x_i^{-\alpha}}}$$

and

$$\frac{\partial \log L}{\partial \gamma} = \frac{n}{\gamma} + \sum_{i=1}^{n}\log(1-e^{-\beta x_i^{-\alpha}}).$$

The MLEs of α, β and γ can be obtained by solving the

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log-likelihood equations

$$\frac{\partial \log L}{\partial \alpha} = 0, \frac{\partial \log L}{\partial \beta} = 0 \text{ and } \frac{\partial \log L}{\partial \gamma} = 0.$$

In order to solve the above equations, one can apply suitable iterative procedure such as Newton-Raphson method by using nlm($\cdot$) function in R-software.

III. Application

In this section, three real data sets are considered to illustrate how the new model works in practice. All used models and the MLEs of parameters are listed in Appendix A.

A. *Example 1*

The first data set represents the failure times of the air conditioning system of an air plane (in hours), and has been analyzed by many authors such as Guptu and Kundu [8] and Mokhtari et al. [9]. It is given in Table I.

TABLE I: Failure times of the air conditioning system

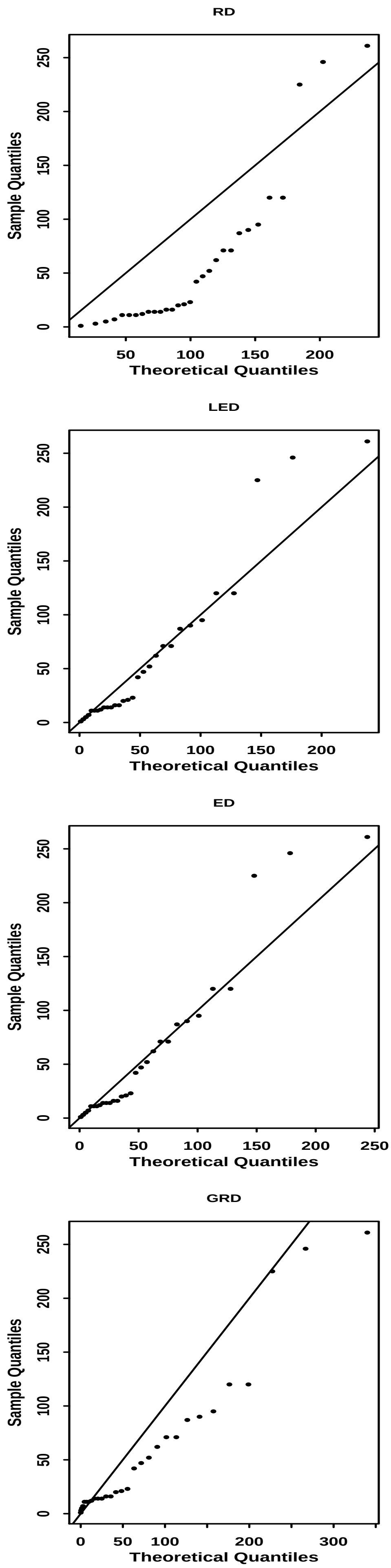
1	3	5	7	11	11	11	12	14	14
14	16	16	20	21	23	42	47	52	62
71	71	87	90	95	120	120	225	246	261

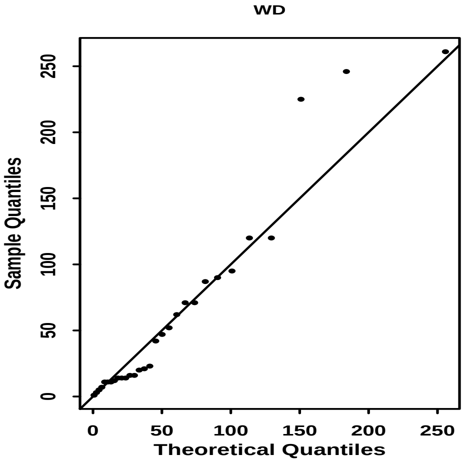
For comparison purposes, we will consider eight alternative models that may be good competitive distributions to the GIEIWD. To check the adequacy of all statistical distributions, five goodness-of-fit statistics are computed. The measures of goodness-of-fit statistics includes the log-likelihood function evaluated at the MLEs, the Kolmogrov-Simnorov (K-S) statistics with their P-values, Akaike information criterion (AIC), Bayesian information criterion (BIC), where AIC$=\ -2\log L + 2p$, BIC$=\ -2\log L + p\log(n)$, p is the number of parameters, and n is the sample size. These statistics are widely used to determine how closely a specific distribution with CDF($\cdot$) fits the associated empirical distribution for given data set.

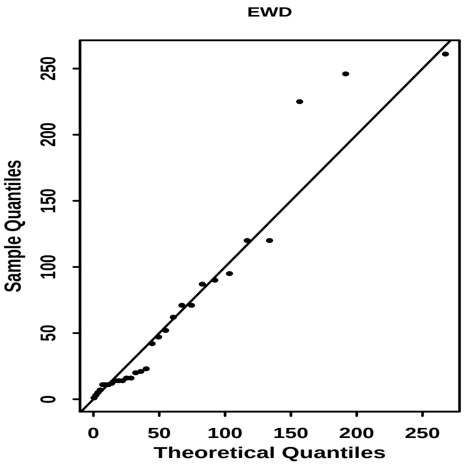
All goodness-of-fit statistics are given in Table II. It is observed that if the comparison is only made with AIC and BIC values, the ED is the best fitted model. If the comparison is made with the K-S statistics, log-likelihood and P-values, the GIEIWD is the best fitted model for the data of the air conditioning system. From the QQ plots in Fig.2, we can see that the GIEIWD outperforms the other models.

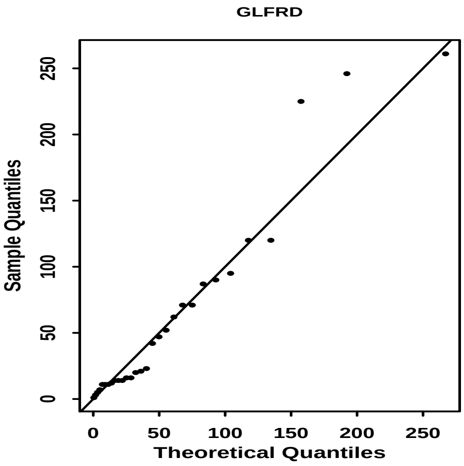
TABLE II: Goodness-of-fit statistics for the data set of the air conditioning system

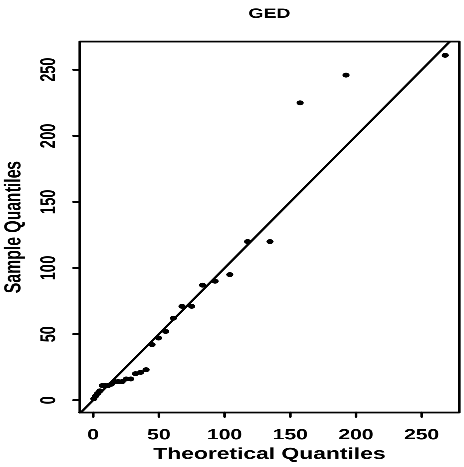
Distribution	log L	AIC	BIC	K-S	P-value
RD	-182.917	367.834	370.449	0.4954	3×10^{-7}
LED	-152.792	309.584	314.814	0.2184	0.0983
ED	-152.630	**307.260**	**309.875**	0.2132	0.1125
GRD	-155.042	314.084	319.314	0.1972	0.1692
WD	-152.226	308.452	313.682	0.1917	0.1935
EWD	-152.167	310.334	318.179	0.1730	0.2954
GLFRD	-152.211	310.422	318.267	0.1722	0.3003
GED	-152.201	308.402	313.632	0.1719	0.3020
GIEIWD	**-151.348**	308.697	314.433	**0.1285**	**0.6578**











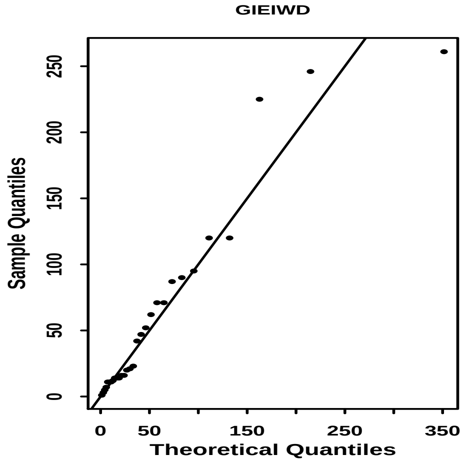


Fig. 2: QQ plots for example 1

B. Example 2

The second data set represents the fatigue time of 101 6061-T6 aluminum coupons cut parallel to the direction of rolling, and was used by Singh and Choudhary [10]. It is listed in Table III.

TABLE III: FATIGUE TIME DATA OF 101 6061-T6 ALUMINUM

70	90	96	97	99	100	103	104	104	105
107	108	108	108	109	109	112	112	113	114
114	114	116	119	120	120	120	121	121	123
124	124	124	124	124	128	128	129	129	130
130	130	131	131	131	131	131	132	132	132
133	134	134	134	134	134	136	136	137	138
138	138	139	139	141	141	142	142	142	142
142	142	144	144	145	146	148	148	149	151
151	152	155	156	157	157	157	157	158	159
162	163	163	164	166	166	168	170	174	196
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Similarly, we compare above five goodness-of-fit statistics for all suggested distributions. The results are listed in Table IV. It can be easily verified from the results that the GIEIWD gives better fitting than other distributions for modeling fatigue time data since statistics (AIC, BIC and K-S values) show smaller values and the P-value and $\log$ L show larger values. The empirical and fitted CDF, PDF, total time on test transforms (TTT-Transforms) and HRF using the PD, EPD, EED and GIEIWD are provided in Fig.3, respectively. It is clear that the GIEIWD provides a better fit to the fatigue time data set than other considered models. From above analysis, it is concluded that the GIEIWD is competitive and alternative.

TABLE IV: GOODNESS-OF-FIT STATISTICS FOR THE DATA SET OF 101 6061-T6 ALUMINUM

Distribution	log L	AIC	BIC	K-S	P-value
PD	-481.38	966.76	971.99	0.1925	0.0011
EPD	-457.91	921.83	929.68	0.0719	0.6731
EED	-463.98	931.97	937.20	0.1051	0.2148
GIEIWD	**-456.41**	**918.82**	**926.67**	**0.0622**	**0.8297**

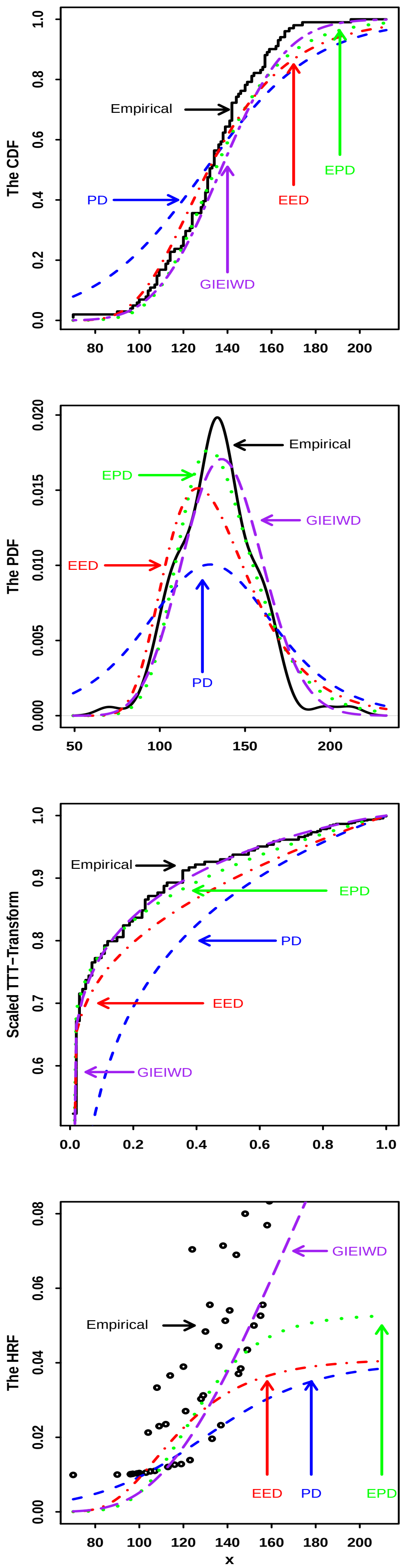

Fig. 3: Plots of the empirical and fitted distribution for example 2 data

C. Example 3

The third data set represents the remission times (in months) of a random sample of 128 bladder cancer patients, and has been used by many authors such as Lemonte and Cordeiroa [11] and Zea et al. [12]. It is listed in Table V.

TABLE V: THE REMISSION TIMES (IN MONTHS) OF BLADDER CANCER PATIENTS

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63
0.20	2.23	3.52	4.98	6.97	9.02	13.29	0.40
2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50
2.46	3.64	5.09	7.26	9.47	14.24	25.82	0.51
2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81
2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64
3.88	5.32	7.39	10.34	14.83	34.26	0.90	2.69
4.18	5.34	7.59	10.66	15.96	36.66	1.05	2.69
4.23	5.41	7.62	10.75	16.62	43.01	1.19	2.75
4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33
5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62
7.87	11.64	17.36	1.40	3.02	4.34	5.71	7.93
11.79	18.10	1.46	4.40	5.85	8.26	11.98	19.13
1.76	3.25	4.50	6.25	8.37	12.02	2.02	3.31
4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76
12.07	21.73	2.07	3.36	6.93	8.65	12.63	22.69

We compare five goodness-of-fit statistics including the $Cram\acute{e}r$-von Mises (W^*) and Anderson-Darling (A^*). Here $W^* = W^2(1 + \frac{0.5}{n})$, $A^* = A^2(1 + \frac{0.75}{n} + \frac{2.25}{n^2})$, $W^2 = \sum_{i=1}^{n}\left[u_i - \frac{2i-1}{2n}\right]^2 + \frac{1}{12}$, $A^2 = -n - \frac{1}{n}\Sigma_{i=1}^{n}[(2i-1)\log(u_i) + (2n+1-2i)\log(1-u_i)]$, $u_i = \Phi[(y_i - \bar{y})/s_y]$, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$, $s_y = \sqrt{(n-1)^{-1}\sum_{i=1}^{n}(y_i - \bar{y})^2}$, $y_i = \Phi^{-1}(v_i)$, $\Phi(\cdot)$ is the standard normal CDF and $\Phi^{-1}(\cdot)$ is its inverse, $v_i = F(x_i;\hat{\theta})$, $F(x;\theta)$ is the CDF, $\hat{\theta}$ is MLEs of unknown parameter vector θ, the x_i's are in ascending order, and n is the sample size. In general, the smaller the values of the statistics W^* and A^*, the better the fit to the data. The values of all statistics for all considered models are given in Table VI. These results show that the GIEIWD has the lowest K-S, W^* and A^* values and the largest log L and P-value among all the fitted models, and so it could be chosen as the best model.

TABLE VI: GOODNESS-OF-FIT STATISTICS FOR THE DATA SET OF BLADDER CANCER PATIENTS

Distribution	log L	K-S	P-value	W^*	A^*
EMaD	-412.1556	0.0598	0.7505	0.0965	0.5962
GMaD	-426.6019	0.1420	0.0115	0.7577	3.9113
PLD	-413.3538	0.0682	0.5904	0.1273	1.3320
GLD	-416.2859	0.0928	0.2204	0.2479	1.3320
GED	-413.0776	0.0725	0.5113	0.1284	0.7182
WD	-414.0869	0.0700	0.5569	0.1543	0.9635
GD	-413.3678	0.0733	0.4974	0.1361	0.7763
FWD	-460.2659	0.2084	3.0e-05	1.5977	8.2057
WLD	-416.4422	0.0926	0.2227	0.2518	1.3490
IWD	-444.0008	0.1408	0.0125	0.9825	6.1549
GIED	-457.2024	0.2067	4.0e-05	1.8419	9.4491
GIEIWD	**-411.1114**	**0.0506**	**0.8978**	**0.0532**	**0.3499**

IV. CONCLUDING REMARKS

In this paper, we propose a three-parameter GIEIWD and use the new model to fit three real data sets. The results of the comparisons show that the proposed GIEIWD agrees well with the complex systems lifetime data and can provide a better candidate.

Appendix A

(1) Rayleigh distribution (RD) with the PDF

$$f(x;\beta) = 2\beta x e^{-\beta x^2}, \quad x>0, \beta>0.$$

$$\hat{\beta} = 7.3104 \times 10^{-5}.$$

(2) Linear exponential distribution (LED) with the PDF

$$f(x;\lambda,\beta) = (\lambda + 2\beta x)e^{-(\lambda x+\beta x^2)}, \quad x>0, \lambda>0, \beta>0.$$

$$\hat{\lambda} = 0.0164, \hat{\beta} = 3.4644 \times 10^{-6}.$$

(3) Exponential distribution (ED) with the PDF

$$f(x;\lambda) = \lambda e^{-\lambda x}, \quad x>0, \lambda>0.$$

$$\hat{\lambda} = 0.0168.$$

(4) Generalized Rayleigh distribution (GRD) with the PDF

$$f(x;\alpha,\beta) = 2\alpha\beta x e^{-\beta x^2}(1-e^{-\beta x^2})^{\alpha-1},$$
$$x>0, \alpha>0, \beta>0.$$

$$\hat{\alpha} = 0.2486, \hat{\beta} = 2.3674 \times 10^{-5}.$$

(5) Weibull distribution (WD) with the PDF

$$f(x;\beta,\gamma) = \beta\gamma x^{\gamma-1} e^{-\beta x^\gamma}, \quad x>0, \beta>0, \gamma>0.$$

$\hat{\beta} = 0.0215, \hat{\gamma} = 0.9468$. (Example 1)

$\hat{\beta} = 1.0478, \hat{\gamma} = 0.0938$. (Example 3)

(6) Exponential Weibull distribution (EWD) with the PDF

$$f(x;\alpha,\beta,\gamma) = \alpha\beta\gamma x^{\gamma-1} e^{-\beta x^\gamma}(1-e^{-\beta x^\gamma})^{\alpha-1},$$
$$x>0, \alpha>0, \beta>0, \gamma>0.$$

$$\hat{\alpha} = 0.8384, \hat{\beta} = 0.0161, \hat{\gamma} = 0.9832.$$

(7) Generalized linear failure rate distribution (GLFRD) with the PDF

$$f(x;\alpha,\lambda,\beta) = \alpha(\lambda+2\beta x)e^{-(\lambda x+\beta x^2)}(1-e^{-(\lambda x+\beta x^2)})^{\alpha-1},$$
$$x>0, \alpha>0, \lambda>0, \beta>0.$$

$$\hat{\alpha} = 0.8064, \hat{\lambda} = 0.0144, \hat{\beta} = 4.9116 \times 10^{-7}.$$

(8) Generalized exponential distribution (GED) with the PDF

$$f(x;\alpha,\lambda) = \alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}, \quad x>0, \alpha>0, \lambda>0.$$

$\hat{\alpha} = 0.8089, \hat{\lambda} = 0.0145$. (Example 1)

$\hat{\alpha} = 1.2179, \hat{\lambda} = 0.1211$. (Example 3)

(9) Perks distribution (PD) with the CDF

$$F(x;\alpha,\lambda) = 1 - \frac{1+\alpha}{1+\alpha e^{\lambda x}}, \quad x>0, \alpha>0, \lambda>0,$$

PDF

$$f(x;\alpha,\lambda) = \alpha\lambda e^{\lambda x}\frac{1+\alpha}{(1+\alpha e^{\lambda x})^2},$$

HRF

$$h(x;\alpha,\lambda) = \frac{\alpha\lambda e^{\lambda x}}{1+\alpha e^{\lambda x}}.$$

$$\hat{\alpha} = 0.0056, \hat{\lambda} = 0.040.$$

(10) Exponentiated Perks distribution (EPD) with the CDF

$$F(x;\alpha,\beta,\lambda) = [1 - \frac{1+\alpha}{1+\alpha e^{\lambda x}}]^\beta,$$
$$x>0, \alpha>0, \beta>0, \lambda>0,$$

PDF

$$f(x;\alpha,\beta,\lambda) = \alpha\beta\lambda(1+\alpha)\frac{e^{\lambda x}}{(1+\alpha e^{\lambda x})^2}[1 - \frac{1+\alpha}{1+\alpha e^{\lambda x}}]^{(\beta-1)},$$

HRF

$$h(x;\alpha,\beta,\lambda) = \frac{\alpha^\beta\beta\lambda(1+\alpha)e^{\lambda x}(e^{\lambda x}-1)^{\beta-1}}{(1+\alpha e^{\lambda x})[(1+\alpha e^{\lambda x})^\beta - \alpha^\beta(e^{\lambda x}-1)^\beta]}.$$

$$\hat{\alpha} = 0.0052, \hat{\beta} = 4.79, \hat{\lambda} = 0.053.$$

(11) Exponentiated exponential distribution (EED) with the CDF

$$F(x;\alpha,\lambda) = (1-e^{-\lambda x})^\alpha, \quad x>0, \alpha>0, \lambda>0,$$

PDF

$$f(x;\alpha,\lambda) = \alpha\lambda(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x},$$

HRF

$$h(x;\alpha,\lambda) = \frac{\alpha\lambda(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x}}{1-(1-e^{-\lambda x})^\alpha}.$$

$$\hat{\alpha} = 151.12, \hat{\lambda} = 0.041.$$

(12) The extended Maxwell distribution (EMaD) with the PDF

$$f(x;\kappa,\theta) = \frac{\kappa}{\sqrt{2\pi\theta^3}} x^{\frac{3}{2}\kappa-1} e^{-\frac{x^\kappa}{2\theta^2}}, \quad x>0, \kappa>0, \theta>0.$$

$$\hat{\kappa} = 0.8446, \hat{\theta} = 1.4431.$$

(13) Generalized Maxwell distribution (GMaD) with the PDF

$$f(x;\kappa,\theta) = \frac{2}{\Gamma(\kappa/2)}\frac{x^{\kappa-1}}{\theta^{k/2}}e^{-\frac{x^2}{\theta}}, \quad x>0, \kappa>0, \theta>0.$$

$$\hat{\kappa} = 0.7483, \hat{\theta} = 527.2314.$$

(14) Power Lindley distribution (PLD) with the PDF

$$f(x;\alpha,\beta) = \frac{\alpha\beta^2}{1+\beta}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha},$$
$$x>0, \alpha>0, \beta>0.$$

$$\hat{\alpha} = 0.8302, \hat{\beta} = 0.2943.$$

(15) Generalized Lindley distribution (GLD) with the PDF

$$f(x;\alpha,\lambda) = \frac{\alpha\lambda^2}{1+\lambda}(1+x)\Big[1 - \frac{1+\lambda+\lambda x}{1+\lambda}e^{-\lambda x}\Big]^{\alpha-1}e^{-\lambda x},$$
$$x>0, \alpha>0, \lambda>0.$$

$$\hat{\alpha} = 0.7336, \hat{\lambda} = 0.1648.$$

(16) Gamma distribution (GD) with the PDF

$$f(x;\alpha,\theta) = \frac{\theta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\theta x}, \quad x>0, \alpha>0, \theta>0.$$

$$\hat{\alpha} = 1.1725, \hat{\theta} = 0.1252.$$

(17) Flexible Weibull distribution (FWD) with the PDF

$$f(x;\alpha,\beta) = (\alpha + \frac{\beta}{x^2})\exp(\alpha x - \frac{\beta}{x})\exp(-\exp(\alpha x - \frac{\beta}{x})),$$
$$x > 0, \alpha > 0, \beta > 0.$$

$$\hat{\alpha} = 0.0325, \hat{\beta} = 2.1547.$$

(18) Weighted Lindley distribution (WLD) with the PDF

$$f(x;\theta,\sigma) = \frac{\theta^\sigma}{(\theta+\sigma)\Gamma(\sigma)}x^{\sigma-1}(1+x)e^{-\theta x},$$
$$x > 0, \theta > 0, \sigma > 0.$$

$$\hat{\theta} = 0.1594, \hat{\sigma} = 0.6827.$$

(19) Inverse Weibull distribution (IWD) with the PDF

$$f(x;\alpha,\lambda) = \alpha\lambda x^{-\alpha-1}e^{-\lambda x^{-\alpha}}, \quad x > 0, \alpha > 0, \lambda > 0.$$

$$\hat{\alpha} = 0.7521, \hat{\lambda} = 2.4311.$$

(20) Generalized inverted exponential distribution (GIED) with the PDF

$$f(x;\gamma,\lambda) = \Big(\frac{\gamma\lambda}{x^2}\Big)\exp\Big(-\frac{\lambda}{x}\Big)\Big[1-\exp\Big(-\frac{\lambda}{x}\Big)\Big]^{\gamma-1},$$
$$x > 0, \gamma > 0, \lambda > 0.$$

$$\hat{\gamma} = 0.7462, \hat{\lambda} = 1.9944.$$

(21) GIEIWD

$\hat{\alpha} = 0.1562, \hat{\beta} = 9.6737, \hat{\gamma} = 194.6121.$ (Example 1)

$\hat{\alpha} = 1.25, \hat{\beta} = 2522.93, \hat{\gamma} = 150.687.$ (Example 2)

$\hat{\alpha} = 0.1564, \hat{\beta} = 9.4225, \hat{\gamma} = 808.7305.$ (Example 3)

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