Construction of the Transreal Numbers from Rational Numbers via Dedekind Cuts

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***Abstract*—The first constructive definition of the real numbers was in terms of Dedekind cuts. A Dedekind cut is an ordered partition of the rational numbers into two non-empty sets, the lower set and the upper set. However, outlawing empty sets makes the definition partial.**

We totalise the set of ordered partitions by admitting two cuts: the negative infinity cut is the cut with an empty lower set and a full upper set; the positive infinity cut is the cut with a full lower set and an empty upper set. These correspond to the affine infinities of the extended-real numbers. We further admit the nullity cut that has both an empty lower set and an empty upper set. We say that the set of all Trans-Dedekind cuts comprises the set of all Dedekind cuts, together with the three strictly Trans-Dedekind cuts: positive infinity, negative infinity, and nullity.

The arithmetical operations and order relation on Dedekind cuts are usually defined only on the lower or else upper sets, which is incoherent when applied to strictly Trans-Dedekind cuts. We totalise these operations and relation over lower and upper sets. We call our totalised Dedekind arithmetic, Trans-Dedekind arithmetic.

We find that the Trans-Dedekind arithmetic of Trans-Dedekind cuts is isomorphic to transreal arithmetic, which is total. This construction gives transreal arithmetic the same ontological status as real arithmetic.

***Index Terms*—transreal number, Dedekind cut, Trans-Dedekind cut, division by zero.**

I. Introduction

THE first constructive definition of the real numbers was published in the 1900s in terms of the Dedekind cut. The German originals of Dedekind's works are available in English translation [1].

The real numbers, like all of the usual number systems, are partial because they do not allow division by zero. Dedekind notes that division by zero is not defined [1] and goes on to define cuts so that they faithfully describe the real numbers as a partial number system. Dedekind does this by making the cuts partial. A Dedekind cut is an ordered partition of the rational numbers into two non-empty sets, the lower set and the upper set. Outlawing empty sets makes the definition partial.

The computable real numbers were accepted into computer science in the 1930s [2]. However, in general, it is a Turing incomputable problem to determine, at compile time, whether a program will generate a zero at run time. Hence the operation of computable division cannot be guarded, at compile time, from dividing by zero at run time. In the case that a division by zero is instructed, the computer program fails at run time, ultimately because mathematical division of real numbers by zero is not defined. We shall presently say a little more about the computational means by which division by zero is handled as an exception to the closure of the mathematical division operator but, for now, we note that if division is extended to be a closed mathematical operation then no such exceptions occur.

Transmathematics grew out of research in computer science that aims to totalise mathematics by arranging that all functions are total functions. In other words, transmathematics is a research programme that aims to remove all exceptions from mathematics. In this area of research, the prefix *trans* is applied to the name of a mathematical object to warn the reader that it has been totalised and may, therefore, have some unexpected properties.

Transmathematics began, in 1997, with an effort to totalise projective geometry for use in computer vision programs [3]. The point at issue is the contradiction that the position of a camera in Euclidean space is given by its centre of projection but the centre of projection is punctured from projective space. When both spaces are described in homogeneous co-ordinates, the centre of projection has co-ordinates $0/0$. This difficulty was resolved by giving a geometrical construction in which three distinct points, with homogeneous co-ordinates $-1/0$, $1/0$, $0/0$, occur in well defined positions. These three points were named: minus infinity, $-\infty = -1/0$; positive infinity, $\infty = 1/0$; and nullity, $\Phi = 0/0$. We stress that these were recognised as well defined numbers because: they have a well defined geometrical construction, they appear as solutions to an algebraic equation, and they are syntactically identical to rational numbers. This syntactic identity made it easy to extend rational arithmetic packages to solve numerically ill-conditioned problems [4] [5].

Computer science usually treats projective space as a double cover so that the unoriented infinity of projective geometry can be distinguished as the positive and negative affine infinities. Thus the transnumber infinities were identified with their usual projective and affine properties and nullity was identified as an unordered and isolated point that lies outside both projective and extended-real space.

Transrational and transreal arithmetic were developed over a number of years until transreal arithmetic was axiomatised and proved consistent, by machine proof, in 2007 [6]. This publication excited some controversy. After this time, the linear sequence of publications broke down, with results appearing at times that were out of sequence with their development and out of sequence with their foundational role.

There have been many controversies in mathematics, many of which have been settled when a constructive definition is given. With this objective in mind, constructive proofs were given of the consistency of transreal [7] [8] and transcomplex arithmetic [9].

Manuscript received 22 November, 2020. Revised 12 March, 2021.

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Mathematical controversies have also ended when theoretical or practical utility is demonstrated. With this objective in mind, many areas of mathematics have been totalised, in the expectation that the totalised versions will find application in mathematics, computation or physics. The topology of the transreal [10] [11] and transcomplex [12] [13] numbers was developed and lead to both transreal analysis [11] [14] [15] [16] [17] and transcomplex analysis [18] [19] – all of which confirmed transreal nullity, $\Phi = 0/0$, as the uniquely unordered transreal number and the transreal numbers negative infinity, $-\infty = -1/0$, and positive infinity, $\infty = 1/0$, as, respectively, the least and greatest of the ordered numbers. There was some development of paraconsistent logics [20] [21] [22] [23] and Boolean logic was generalised to a wide class of trans-Boolean logics [8]. A number of applications of transreal arithmetic were discussed in computer science [24] [25] [26] [27] [28] [29] and mathematical physics [10] [30]. Some philosophical aspects on transmathematics were also discussed [7] [31].

Perhaps the greatest computational advantage of the transfloating-point numbers is that they remove one binade of exceptional, Not-a-Number (NaN), states [25], which allows twice the numerical accuracy in the same number of floating-point bits and simplifies the semantics of floating-point programs, especially where division by zero occurs [26]. Perhaps the greatest advantages for mathematical physics are that the transreal numbers dissolve the problem of the infinite electron self-energy [10] and explain how convection currents can pass through the singularity in a black hole [30].

Mathematical controversies have also ended when people, other than the original proponents, take up research, as is beginning to happen with transmathematics [32] [33] [34] [35] [36].

We now present a construction of the transreal numbers via Dedekind cuts. Our objective is to put the transreal numbers on the same mathematical foundation as the real numbers so that both number systems have the same validity, whence debate can move on to a comparison or their relative merits.

We define that Trans-Dedekind cuts are made up of the Dedekind cuts, together with three strictly Trans-Dedekind cuts. First, the negative infinity cut is the cut with an empty lower set and a full upper set. Second, the positive infinity cut is the cut with a full lower set and an empty upper set. It is well known that these two cuts correspond to the affine infinities of the extended-real numbers. Finally the nullity cut has both an empty lower set and an empty upper set.

The arithmetical operations and order relation on Dedekind cuts are usually defined only on the lower or else upper sets [37], Chapter 7, and [1]. However if they are applied only to lower sets, they cannot distinguish negative infinity from nullity. If they are, instead, applied only to upper sets, they cannot distinguish positive infinity from nullity. Hence we define the arithmetical operations and order relation so that they apply, simultaneously, to the lower and upper sets so that all Trans-Dedekind cuts are distinguished.

We begin our presentation with two sections of preliminary material. In Section II *Sets of Rational Numbers,* we define certain sets of rational numbers and operations on them. In Section III *Dedekind Cut,* we review the Dedekind Cut and set out some well known theorems. We then present the main work. In Section IV *Equivalent Arithmetic*, we define the additive inverse, multiplication, and multiplicative inverse in a new way so that they apply to all Trans-Dedekind cuts and still produce results identical to Dedekind arithmetic when applied to Dedekind cuts. In Section V *Trans-Dedekind Cut*, we define the Trans-Dedekind cuts and some operations on them. In Section VI *Trans-Dedekind Arithmetic*, we establish that Trans-Dedekind Arithmetic implements transreal arithmetic. In Section VII *Discussion*, we discuss future opportunities for research. In Section VIII *Conclusion,* we state the main consequence of our technical results. In Appendix A *Proofs*, we present the somewhat lengthy proofs that establish the theorems in sections IV, V and VI.

II. Sets of Rational Numbers

Definition 1 (Intervals)**.** *In this paper all intervals are intervals of rational numbers. That is, for all $a, b \in \mathbb{Q}$ it follows that:*

$$
\begin{aligned}
[a,b] &:= \{x \in \mathbb{Q};\ a \leqslant x \leqslant b\},\\
(a,b) &:= \{x \in \mathbb{Q};\ a < x < b\},\\
[a,b) &:= \{x \in \mathbb{Q};\ a \leqslant x < b\},\\
(a,b] &:= \{x \in \mathbb{Q};\ a < x \leqslant b\},\\
(-\infty,b] &:= \{x \in \mathbb{Q};\ x \leqslant b\},\\
(-\infty,b) &:= \{x \in \mathbb{Q};\ x < b\},\\
[a,\infty) &:= \{x \in \mathbb{Q};\ a \leqslant x\},\\
(a,\infty) &:= \{x \in \mathbb{Q};\ a < x\},\\
(-\infty,\infty) &:= \mathbb{Q}.
\end{aligned}
$$

Definition 2 (Negative and Positive Numbers)**.** *We denote $\mathbb{Q}^- := (-\infty, 0)$ and $\mathbb{Q}^+ := (0, \infty)$.*

Definition 3 (Closed Downwards and Closed Upwards Sets)**.** *Let L and U be sets of rational numbers. We say that L is* closed downwards *if and only if L satisfies the property: if $b \in L$ then $(-\infty, b) \subset L$. We say that U is* closed upwards *if and only if U satisfies the property: if $a \in U$ then $(a, \infty) \subset U$.*

Definition 4 (Arithmetical Operations on Sets)**.** *Let A and B be sets of rational numbers. We define:*

a) $A + B := \{x \in \mathbb{Q};$ there are $a \in A$ and $b \in B$ such that $x = a + b\}$

b) $-B := \{x \in \mathbb{Q};$ there is $b \in B$ such that $x = -b\}$

c) $A \cdot B := \{x \in \mathbb{Q};$ there are $a \in A$ and $b \in B$ such that $x = ab\}$

d) $B^{-1} := \{x \in \mathbb{Q};$ there is $b \in B$ such that $x = b^{-1}\}$, *if B does not contain zero.*

Definition 5 (Open Interval of a Set)**.** *Let A be a set of rational numbers. We define $\mathcal{I}(A) := \{x \in \mathbb{Q};$ there are $a, b \in A$ such that $x \in (a, b)\}$.*

III. Dedekind Cut

Definition 6 (Dedekind Cut)**.** *A* Dedekind cut *is an ordered pair, $\langle L, U\rangle$, where L and U are subsets of rational numbers that satisfy:*

0) $L \neq \varnothing$ *and* $L \neq \mathbb{Q}$,

1) $L \cup U = \mathbb{Q}$,

2) $L \cap U = \varnothing$,

3) L is closed downwards,
4) L does not have a greatest element.

Definition 7 (Relation)**.** *Let $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ be Dedekind cuts. We say that $\langle L_1, U_1\rangle < \langle L_2, U_2\rangle$ if and only if L_1 is a proper subset of L_2 and U_2 is a proper subset of U_1. We say that $\langle L_1, U_1\rangle \leqslant \langle L_2, U_2\rangle$ if and only if $\langle L_1, U_1\rangle < \langle L_2, U_2\rangle$ or $\langle L_1, U_1\rangle = \langle L_2, U_2\rangle$.*

Theorem 8 (Total Order Relation)**.** *$\leqslant$ is a total order relation on the set of all Dedekind cuts.*

Definition 9. *Let $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ be Dedekind cuts. We say that:*

a) $\langle L_1, U_1\rangle \geqslant \langle L_2, U_2\rangle$ if and only if $\langle L_2, U_2\rangle \leqslant \langle L_1, U_1\rangle$,
b) $\langle L_1, U_1\rangle > \langle L_2, U_2\rangle$ if and only if $\langle L_2, U_2\rangle < \langle L_1, U_1\rangle$,
c) $\langle L_1, U_1\rangle \nless \langle L_2, U_2\rangle$ if and only if $\langle L_1, U_1\rangle < \langle L_2, U_2\rangle$ does not hold,
d) $\langle L_1, U_1\rangle \ngtr \langle L_2, U_2\rangle$ if and only if $\langle L_1, U_1\rangle > \langle L_2, U_2\rangle$ does not hold.

Theorem 10. *Let $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ be Dedekind cuts. It follows that:*

a) $\langle L_1, U_1\rangle \nless \langle L_2, U_2\rangle$ necessarily implies $\langle L_1, U_1\rangle \geqslant \langle L_2, U_2\rangle$.
b) $\langle L_1, U_1\rangle \ngtr \langle L_2, U_2\rangle$ necessarily implies $\langle L_1, U_1\rangle \leqslant \langle L_2, U_2\rangle$.

Theorem 11. *For all $r \in \mathbb{Q}$, $\langle(-\infty, r), [r, \infty)\rangle$ is a Dedekind cut.*

Definition 12. *We denote the Dedekind cut $\langle(-\infty, 0), [0, \infty)\rangle$ simply as 0 and the Dedekind cut $\langle(-\infty, 1), [1, \infty)\rangle$ simply as 1, that is, $0 = \langle(-\infty, 0), [0, \infty)\rangle$ and $1 = \langle(-\infty, 1), [1, \infty)\rangle$.*

Definition 12 is justified by the fact that $\langle(-\infty, 0), [0, \infty)\rangle$ and $\langle(-\infty, 1), [1, \infty)\rangle$ are, respectively, the additive identity and multiplicative identity of the field of all Dedekind cuts (theorems 15, 21 and 24).

Definition 13 (Addition)**.** *Let $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ be Dedekind cuts. We define $\langle L_1, U_1\rangle + \langle L_2, U_2\rangle := \langle L_1 + L_2, U_1 + U_2\rangle$.*

Theorem 14 (Closure under Addition)**.** *If $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ are Dedekind cuts then $\langle L_1, U_1\rangle + \langle L_2, U_2\rangle$ is a Dedekind cut.*

Theorem 15 (Existence of Additive Identity)**.** *$\langle(-\infty, 0), [0, \infty)\rangle$ is the identity element of the set of all Dedekind cuts with respect to $+$.*

Definition 16 (Additive Inverse and Subtraction)**.** *Let $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ be Dedekind cuts.*

a) We denote $-\langle L_1, U_1\rangle := \langle L_2, U_2\rangle$ if and only if $\langle L_1, U_1\rangle + \langle L_2, U_2\rangle = 0$.
b) We define $\langle L_1, U_1\rangle - \langle L_2, U_2\rangle := \langle L_1, U_1\rangle + (-\langle L_2, U_2\rangle)$.

Theorem 17 (Existence of Additive Inverse)**.** *For all Dedekind cuts $\langle L, U\rangle$, there is $-\langle L, U\rangle$.*

Theorem 18. *Let $\langle L, U\rangle$ be a Dedekind cut. It follows that $\langle L, U\rangle < 0$ if and only if $-\langle L, U\rangle > 0$.*

Definition 19 (Multiplication)**.** *Let $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ be Dedekind cuts. We define:*

a) $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle := \langle L_3, U_3\rangle$ where

$$\begin{aligned} L_3 &= (-\infty, 0) \cup \big((L_1 \cap [0, \infty)) \cdot (L_2 \cap [0, \infty))\big), \\ U_3 &= \mathbb{Q} \backslash L_3, \end{aligned}$$

if $\langle L_1, U_1\rangle \geqslant 0$ and $\langle L_2, U_2\rangle \geqslant 0$,
b) $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle := -\big(\langle L_1, U_1\rangle \times (-\langle L_2, U_2\rangle)\big)$, if $\langle L_1, U_1\rangle \geqslant 0$ and $\langle L_2, U_2\rangle < 0$,
c) $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle := -\big((-\langle L_1, U_1\rangle) \times \langle L_2, U_2\rangle\big)$, if $\langle L_1, U_1\rangle < 0$ and $\langle L_2, U_2\rangle \geqslant 0$,
d) $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle := (-\langle L_1, U_1\rangle) \times (-\langle L_2, U_2\rangle)$, if $\langle L_1, U_1\rangle < 0$ and $\langle L_2, U_2\rangle < 0$.

Theorem 20 (Closure under Multiplication)**.** *If $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ are Dedekind cuts then $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle$ is a Dedekind cut.*

Theorem 21 (Existence of Multiplicative Identity)**.** *$\langle(-\infty, 1), [1, \infty)\rangle$ is the identity element of the set of all Dedekind cuts with respect to $\times$.*

Definition 22 (Multiplicative Inverse and Division)**.** *Let $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ be Dedekind cuts.*

a) We denote $\langle L_1, U_1\rangle^{-1} := \langle L_2, U_2\rangle$ if and only if $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle = 1$.
b) If $\langle L_2, U_2\rangle \neq 0$, we define $\langle L_1, U_1\rangle \div \langle L_2, U_2\rangle := \langle L_1, U_1\rangle \times \big(\langle L_2, U_2\rangle^{-1}\big)$.

Theorem 23 (Existence of Multiplicative Inverse)**.** *For all Dedekind cuts $\langle L, U\rangle$ such that $\langle L, U\rangle \neq 0$, there is $\langle L, U\rangle^{-1}$.*

Theorem 24 (All Dedekind Cuts Form a Complete Ordered Field)**.** *The set of all Dedekind cuts is a complete ordered field with respect to order $\leqslant$, addition $+$, and multiplication $\times$.*

Theorem 25 (Rational Numbers As an Ordered Subfield)**.** *The function $r \mapsto \langle(-\infty, r), [r, \infty)\rangle$ is an isomorphism of ordered fields between $\mathbb{Q}$ and the subset of Dedekind cuts $\{\langle(-\infty, r), [r, \infty)\rangle;\ r \in \mathbb{Q}\}$.*

IV. Equivalent Arithmetic

In this section we define three arithmetical operations on Dedekind cuts that have an identical effect to the usual operations on Dedekind cuts but which support the generalisation to Trans-Dedekind cuts. Thus we generalise the additive inverse, multiplication, and the multiplicative inverse of Dedekind cuts.

Theorem 26 (Equivalent Definition of the Additive Inverse of a Dedekind Cut)**.** *For all Dedekind cuts $\langle L, U\rangle$ it follows that $-\langle L, U\rangle = \langle L_3, U_3\rangle$ where*

$$\begin{aligned} U_3'' &= \{x \in \mathbb{Q};\ x = \inf U\}, \\ U_3 &= -(L \cup U_3''), \\ L_3 &= -(U \backslash U_3''). \end{aligned}$$

Theorem 27 (Equivalent Definition of the Multiplication of Dedekind Cuts)**.** *For all Dedekind cuts $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ such that $\langle L_1, U_1\rangle \geqslant 0$ and $\langle L_2, U_2\rangle \geqslant 0$ it follows* *that* $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle = \langle L_3, U_3\rangle$ *where*

$$\begin{aligned} U_3' &= (U_1 \cap \mathbb{Q}^+)\cdot(U_2 \cap \mathbb{Q}^+),\\ U_3'' &= \{x \in \mathbb{Q};\ x = \inf U_3'\},\\ U_3 &= U_3'' \cup U_3',\\ L_3' &= (L_1 \cap \mathbb{Q}^+)\cdot(L_2 \cap \mathbb{Q}^+),\\ L_3'' &= -(L_3' \cup U_3') \cup L_3',\\ L_3 &= \mathcal{I}(L_3''). \end{aligned}$$

Theorem 28 (Equivalent Definition of the Multiplicative Inverse of a Dedekind Cut)**.** *For all Dedekind cuts* $\langle L, U\rangle$ *such that* $\langle L, U\rangle > 0$ *it follows that* $\langle L, U\rangle^{-1} = \langle L_3, U_3\rangle$ *where*

$$\begin{aligned} U_3' &= (L \cap \mathbb{Q}^+)^{-1},\\ U_3'' &= \{x \in \mathbb{Q};\ x = \inf U_3'\},\\ U_3 &= U_3'' \cup U_3',\\ L_3' &= (U \cap \mathbb{Q}^+)^{-1},\\ L_3'' &= -(L_3' \cup U_3') \cup L_3',\\ L_3 &= \mathcal{I}(L_3''). \end{aligned}$$

Theorems 26, 27 and 28 show that we can define the additive inverse, multiplication and multiplicative inverse in a different way from the usual one but this still gives the same results.

V. Trans-Dedekind Cut

Definition VI (Trans-Dedekind Cut)**.** *A* Trans-Dedekind cut *is an ordered pair,* $\langle L, U\rangle$*, where* L *and* U *are subsets of rational numbers that satisfy:*

I) $L \cup U = \varnothing$ *or* $L \cup U = \mathbb{Q}$*,*
II) $L \cap U = \varnothing$*,*
III) L *is closed downwards,*
IV) L *does not have a greatest element.*

Definition VII (Relation)**.** *Let* $\langle L_1, U_1\rangle$ *and* $\langle L_2, U_2\rangle$ *be Trans-Dedekind cuts. We say that* $\langle L_1, U_1\rangle < \langle L_2, U_2\rangle$ *if and only if* L_1 *is a proper subset of* L_2 *and* U_2 *is a proper subset of* U_1*. We say that* $\langle L_1, U_1\rangle \leqslant \langle L_2, U_2\rangle$ *if and only if* $\langle L_1, U_1\rangle < \langle L_2, U_2\rangle$ *or* $\langle L_1, U_1\rangle = \langle L_2, U_2\rangle$*.*

Theorem VIII (Order Relation)**.** $\leqslant$ *is an order relation in the set of all Trans-Dedekind cuts.*

Definition IX. *Let* $\langle L_1, U_1\rangle$ *and* $\langle L_2, U_2\rangle$ *be Trans-Dedekind cuts. We say that:*

a) $\langle L_1, U_1\rangle \geqslant \langle L_2, U_2\rangle$ *if and only if* $\langle L_2, U_2\rangle \leqslant \langle L_1, U_1\rangle$*,*
b) $\langle L_1, U_1\rangle > \langle L_2, U_2\rangle$ *if and only if* $\langle L_2, U_2\rangle < \langle L_1, U_1\rangle$*,*
c) $\langle L_1, U_1\rangle \nless \langle L_2, U_2\rangle$ *if and only if* $\langle L_1, U_1\rangle < \langle L_2, U_2\rangle$ *does not hold,*
d) $\langle L_1, U_1\rangle \ngtr \langle L_2, U_2\rangle$ *if and only if* $\langle L_1, U_1\rangle > \langle L_2, U_2\rangle$ *does not hold.*

Theorem X. *Let* $\langle L_1, U_1\rangle$ *and* $\langle L_2, U_2\rangle$ *be Trans-Dedekind cuts. It follows that:*

a) $\langle L_1, U_1\rangle \nless \langle L_2, U_2\rangle$ *does not necessarily imply* $\langle L_1, U_1\rangle \geqslant \langle L_2, U_2\rangle$*,*
b) $\langle L_1, U_1\rangle \ngtr \langle L_2, U_2\rangle$ *does not necessarily imply* $\langle L_1, U_1\rangle \leqslant \langle L_2, U_2\rangle$*,*
c) $\langle L_1, U_1\rangle > \langle L_2, U_2\rangle$ *necessarily implies* $\langle L_1, U_1\rangle \nless \langle L_2, U_2\rangle$*,*
d) $\langle L_1, U_1\rangle < \langle L_2, U_2\rangle$ *necessarily implies* $\langle L_1, U_1\rangle \ngtr \langle L_2, U_2\rangle$*.*

Theorem XI. *Every Dedekind cut is a Trans-Dedekind cut.*

Remark XII. *Recall that* 0 *and* 1 *are Dedekind cuts (Definition 12). Thus* 0 *and* 1 *are Trans-Dedekind cuts.*

Definition XIII (Addition)**.** *Let* $\langle L_1, U_1\rangle$ *and* $\langle L_2, U_2\rangle$ *be Trans-Dedekind cuts. We define* $\langle L_1, U_1\rangle + \langle L_2, U_2\rangle := \langle L_1 + L_2, U_1 + U_2\rangle$*.*

Theorem XIV (Closure under Addition)**.** *If* $\langle L_1, U_1\rangle$ *and* $\langle L_2, U_2\rangle$ *are Trans-Dedekind cuts then* $\langle L_1, U_1\rangle + \langle L_2, U_2\rangle$ *is a Trans-Dedekind cut.*

Theorem XV (Existence of Additive Identity)**.** $\langle(-\infty, 0), [0, \infty)\rangle$ *is the identity element of the set of all Trans-Dedekind cuts with respect to* $+$*.*

Definition XVI (Opposite and Subtraction)**.** *Let* $\langle L_1, U_1\rangle$ *and* $\langle L_2, U_2\rangle$ *be Trans-Dedekind cuts. We define:*

a) $-\langle L, U\rangle := \langle L_3, U_3\rangle$ *where*

$$\begin{aligned} U_3'' &= \{x \in \mathbb{Q};\ x = \inf U\},\\ U_3 &= -(L \cup U_3''),\\ L_3 &= -(U \backslash U_3''), \end{aligned}$$

b) $\langle L_1, U_1\rangle - \langle L_2, U_2\rangle := \langle L_1, U_1\rangle + (-\langle L_2, U_2\rangle)$*.*

Theorem XVII (Closure under Opposite and Subtraction)**.** *If* $\langle L, U\rangle$ *is a Trans-Dedekind cut then* $-\langle L, U\rangle$ *is a Trans-Dedekind.*

Theorem XVIII. *Let* $\langle L, U\rangle$ *be a Trans-Dedekind cut. It follows that* $\langle L, U\rangle < 0$ *if and only if* $-\langle L, U\rangle > 0$*.*

Definition XIX (Multiplication)**.** *Let* $\langle L_1, U_1\rangle$ *and* $\langle L_2, U_2\rangle$ *be Trans-Dedekind cuts. We define:*

a) $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle := \langle L_3, U_3\rangle$*, where*

$$\begin{aligned} U_3' &= (U_1 \cap \mathbb{Q}^+)\cdot(U_2 \cap \mathbb{Q}^+),\\ U_3'' &= \{x \in \mathbb{Q};\ x = \inf U_3'\},\\ U_3 &= U_3'' \cup U_3',\\ L_3' &= (L_1 \cap \mathbb{Q}^+)\cdot(L_2 \cap \mathbb{Q}^+),\\ L_3'' &= -(L_3' \cup U_3') \cup L_3',\\ L_3 &= \mathcal{I}(L_3''), \end{aligned}$$

if $\langle L_1, U_1\rangle \nless 0$ *and* $\langle L_2, U_2\rangle \nless 0$*,*
b) $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle := -(\langle L_1, U_1\rangle \times (-\langle L_2, U_2\rangle))$*, if* $\langle L_1, U_1\rangle \nless 0$ *and* $\langle L_2, U_2\rangle < 0$*,*
c) $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle := -((-\langle L_1, U_1\rangle) \times \langle L_2, U_2\rangle)$*, if* $\langle L_1, U_1\rangle < 0$ *and* $\langle L_2, U_2\rangle \nless 0$*,*
d) $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle := (-\langle L_1, U_1\rangle) \times (-\langle L_2, U_2\rangle)$*, if* $\langle L_1, U_1\rangle < 0$ *and* $\langle L_2, U_2\rangle < 0$*.*

Theorem XX (Closure under Multiplication)**.** *If* $\langle L_1, U_1\rangle$ *and* $\langle L_2, U_2\rangle$ *are Trans-Dedekind cuts then* $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle$ *is a Trans-Dedekind cut.*

Theorem XXI (Existence of Multiplicative Identity)**.** $\langle(-\infty, 1), [1, \infty)\rangle$ *is the identity element of the set of all Trans-Dedekind cuts with respect to* $\times$*.*

Definition XXII (Reciprocal and Division)**.** *Let* $\langle L_1, U_1\rangle$ *and* $\langle L_2, U_2\rangle$ *be Trans-Dedekind cuts. We define:*

a) $\langle L_2, U_2\rangle^{-1} := \langle L_3, U_3\rangle$, *where*

$$\begin{aligned} U_3' &= (L_2 \cap \mathbb{Q}^+)^{-1}, \\ U_3'' &= \{x \in \mathbb{Q};\ x = \inf U_3'\}, \\ U_3 &= U_3'' \cup U_3', \\ L_3' &= (U_2 \cap \mathbb{Q}^+)^{-1}, \\ L_3'' &= -(L_3' \cup U_3') \cup L_3', \\ L_3 &= \mathcal{I}(L_3''), \end{aligned}$$

if $\langle L_2, U_2\rangle \nless 0$,
b) $\langle L_2, U_2\rangle^{-1} := -((-\langle L_2, U_2\rangle)^{-1})$, *if* $\langle L_2, U_2\rangle < 0$,
c) $\langle L_1, U_1\rangle \div \langle L_2, U_2\rangle := \langle L_1, U_1\rangle \times \left(\langle L_2, U_2\rangle^{-1}\right)$.

Theorem XXIII (Closure under Reciprocal and Division)**.** *If $\langle L, U\rangle$ is a Trans-Dedekind cut then $\langle L, U\rangle^{-1}$ is a Trans-Dedekind cut.*

Theorem 29 (All Arithmetical Operations are Total on Trans-Dedekind Cuts)**.** *There are no exceptions in any of the arithmetical operations on Trans-Dedekind cuts: addition, opposite, subtraction, multiplication, reciprocal and division. That is, for all Trans-Dedekind cuts $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ it follows that $\langle L_1, U_1\rangle + \langle L_2, U_2\rangle$, $-\langle L_2, U_2\rangle$, $\langle L_1, U_1\rangle - \langle L_2, U_2\rangle$, $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle$, $\langle L_2, U_2\rangle^{-1}$ and $\langle L_1, U_1\rangle \div \langle L_2, U_2\rangle$ are well defined Trans-Dedekind Cuts.*

Theorem 30. *The Trans-Dedekind order relation and all of the Trans-Dedekind arithmetical operations coincide with their Dedekind homologues when applied to Dedekind cuts. That is, for all Dedekind cuts $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$:*

a) $\langle L_1, U_1\rangle \leqslant \langle L_2, U_2\rangle$ *from Definition VII has the same truth value as* $\langle L_1, U_1\rangle \leqslant \langle L_2, U_2\rangle$ *from Definition 7,*
b) $\langle L_1, U_1\rangle + \langle L_2, U_2\rangle$ *from Definition XIII has the same value as* $\langle L_1, U_1\rangle + \langle L_2, U_2\rangle$ *from Definition 13,*
c) $-\langle L_2, U_2\rangle$ *from Definition XVI has the same value as* $-\langle L_2, U_2\rangle$ *from Definition 16,*
d) $\langle L_1, U_1\rangle - \langle L_2, U_2\rangle$ *from Definition XVI has the same value as* $\langle L_1, U_1\rangle - \langle L_2, U_2\rangle$ *from Definition 16,*
e) $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle$ *from Definition XIX has the same value as* $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle$ *from Definition 19,*
f) if $\langle L_2, U_2\rangle \neq 0$ *then* $\langle L_2, U_2\rangle^{-1}$ *from Definition XXII has the same value as* $\langle L_2, U_2\rangle^{-1}$ *from Definition 22,*
g) if $\langle L_2, U_2\rangle \neq 0$ *then* $\langle L_1, U_1\rangle \div \langle L_2, U_2\rangle$ *from Definition XXII has the same value as* $\langle L_1, U_1\rangle \div \langle L_2, U_2\rangle$ *from Definition 22.*

VI. Trans-Dedekind Arithmetic

Trans-Dedekind arithmetic is homologous to transreal arithmetic.

Definition 31. *We denote the set of all Trans-Dedekind Cuts as $\mathbb{R}^T$ and denote $\Phi := \langle \varnothing, \varnothing\rangle$, $-\infty := \langle \varnothing, \mathbb{Q}\rangle$ and $\infty := \langle \mathbb{Q}, \varnothing\rangle$.*

Theorem 32. $\mathbb{R}^T = \mathbb{R} \cup \{\Phi, -\infty, \infty\}$.

Theorem 33. *It follows that:*

a) $x \nless \Phi$ *for all* $x \in \mathbb{R}^T$.
b) $\Phi \nless x$ *for all* $x \in \mathbb{R}^T$.
c) $-\infty < x$ *for all* $x \in \mathbb{R}$.
d) $x < \infty$ *for all* $x \in \mathbb{R}$.
e) $-\Phi = \Phi$.
f) $-(-\infty) = \infty$.
g) $-(\infty) = -\infty$.
h) $0^{-1} = \infty$.
i) $\Phi^{-1} = \Phi$.
j) $(-\infty)^{-1} = 0$.
k) $\infty^{-1} = 0$.
l) $\Phi + x = \Phi$.
m) $\infty + (-\infty) = \Phi$.
n) $\infty + \Phi = \Phi$.
o) $\infty + \infty = \infty$.
p) $\infty + x = \infty$ *for all* $x \in \mathbb{R}$.
q) $-\infty + \infty = \Phi$.
r) $-\infty + \Phi = \Phi$.
s) $-\infty + (-\infty) = -\infty$.
t) $-\infty + x = -\infty$ *for all* $x \in \mathbb{R}$.
u) $\Phi \times x = \Phi$.
v) $\infty \times 0 = \Phi$.
w) $\infty \times \Phi = \Phi$.
x) $\infty \times x = -\infty$ *for all* $x \in \mathbb{R}^T$ *such that* $x < 0$.
y) $\infty \times x = \infty$ *for all* $x \in \mathbb{R}^T$ *such that* $x > 0$.
z) $-\infty \times 0 = \Phi$.
α) $-\infty \times \Phi = \Phi$.
β) $-\infty \times x = \infty$ *for all* $x \in \mathbb{R}^T$ *such that* $x < 0$.
γ) $-\infty \times x = -\infty$ *for all* $x \in \mathbb{R}^T$ *such that* $x > 0$.

Theorem 34. *It follows that:*

a) $x \div 0 = \infty$ *for all* $x \in \mathbb{R}$ *such that* $x > 0$,
b) $y \div 0 = -\infty$ *for all* $y \in \mathbb{R}$ *such that* $y < 0$,
c) $0 \div 0 = \Phi$.

By Theorem 33, all of the thirty-two axioms of transreal arithmetic [6] follow from Trans-Dedekind arithmetic.

VII. Discussion

The transreal numbers were originally developed geometrically [3]. In the history of mathematics, geometrical definitions of numbers have served for a long time, before being replaced by axiomatic systems. The transreal numbers have been axiomatised and have been proved consistent by machine proof [6]. This is the most detailed kind of proof it is possible to have but mathematicians often prefer human proofs, especially constructive proofs. The transreal numbers have been constructed as ordered pairs of real numbers [8]. This establishes the consistency of the transreal numbers relative to the real numbers. This proof involves side conditions on the arithmetical operations on pairs to force transreal behaviour, however, in the present paper, we present a proof that generalises the Dedekind cut and which unconditionally gives rise to the transreal numbers. This puts transreal numbers on the same mathematical foundation as real numbers.

That real and transreal arithmetic share the same foundation has profound consequences. Given the practical importance of real arithmetic in engineering, science and mathematics itself, mathematics cannot abandon the Dedekind cut but is forced to choose how arithmetic develops from here. Mathematics can continue with the partial definition of the Dedekind cut, which leads to real arithmetic and which, in turn, necessarily leads to partial mathematics with exceptional or error states, such as the inability to divide by zero and the infinitely many consequences of this inability, or it can adopt the total definition of the Trans-Dedekind cut, which leads to transreal arithmetic and which does not necessarily lead to exceptions and error states in any of the applications of transreal arithmetic.

This choice is not entirely straightforward. Mathematicians might continue to prefer the simpler partial system because it leads to the more rapid development of new mathematics but computer scientists might prefer the more complicated total system because it leads to exception-free computing. In the long term, we expect all mathematics to be proved by machines that maintain an encyclopaedia of mathematical knowledge in a machine independent interchange language. We expect these machines to use total systems of computation to describe both total and partial mathematical systems – which should satisfy all parties and lead to the more reliable development of mathematics.

It is instructive to examine the nature of totality in the Trans-Dedekind cut. The Dedekind cut is partial because it excludes the ordered cut with an empty lower set, $-\infty$, and the ordered cut with an empty upper set, ∞. Adding these two cuts totalises the set of ordered cuts but the resulting arithmetic is partial. For example, as usual, $\infty - \infty$, $0 \times \infty$, and ∞/∞ are all undefined. But when the unordered cut nullity is admitted, the arithmetic is totalised and it becomes a theorem that $\infty - \infty = 0 \times \infty = \infty/\infty = \Phi$. We may say that the strictly Trans-Dedekind cuts are totalised over all combinations of empty lower and upper sets of the Dedekind cut but what structural feature does this totalise?

The trivial subsets of the rational numbers are the empty set and the set of rational numbers itself, $\varnothing$ and $\mathbb{Q}$. We say that a Trans-Dedekind cut is a transordered partition of the trivial subsets of the rational numbers. Even so, it is not clear why this particular structural totalisation should lead to a total arithmetic. Perhaps finding useful totalisations is an inherently creative act?

It is interesting to note that we can develop analogues of the Trans-Dedekind cut by replacing the set of rational numbers with any totally ordered set or class, even a proper class, that does not contain the Trans-Dedekind cuts. Trans infinity is then greater than any number in the class. This neatly avoids the Burali-Forti Paradox which shows that, in other circumstances, there is no greatest ordinal number.

VIII. Conclusion

The usual definition of the Dedekind cut is partial and leads to the real numbers, which have exceptional or error states, such as the inability to divide by zero; but the definition of the Trans-Dedekind cut is total and leads to the transreal numbers, which do not have exceptional or error states. This establishes transreal arithmetic as a firmer foundation for mathematics than real arithmetic.

Appendix A
Proofs

Lemma 35. *It follows that:*

a) $\varnothing$ *is closed downwards,*
b) $\mathbb{Q}$ *is closed downwards,*
c) $\varnothing$ *does not have a greatest element,*
d) $\mathbb{Q}$ *does not have a greatest element.*

Proof: The results follow from the vacuity of the empty set and from the properties of the rational numbers. ∎

Lemma 36. *Let A and B be sets of rational numbers. It follows that:*

a) $\varnothing + A = A + \varnothing = \varnothing$,
b) If $A \neq \varnothing$ *then* $\mathbb{Q} + A = A + \mathbb{Q} = \mathbb{Q}$,
c) $-\varnothing = \varnothing$,
d) $-\mathbb{Q} = \mathbb{Q}$,
e) $-(A \backslash B) = (-A) \backslash (-B)$,
f) $-(A \cup B) = (-A) \cup (-B)$,
g) $-(A \cap \mathbb{Q}^-) = (-A) \cap \mathbb{Q}^+$,
h) $-(A \cap \mathbb{Q}^+) = (-A) \cap \mathbb{Q}^-$,
i) $\varnothing \cdot A = A \cdot \varnothing = \varnothing$,
j) If $A \neq \varnothing$ *then* $\mathbb{Q} \cdot A = A \cdot \mathbb{Q} = \mathbb{Q}$,
k) If $A \cap \mathbb{Q}^+ \neq \varnothing$ *then* $\mathbb{Q}^+ \cdot (A \cap \mathbb{Q}^+) = (A \cap \mathbb{Q}^+) \cdot \mathbb{Q}^+ = \mathbb{Q}^+$,
l) $\varnothing^{-1} = \varnothing$,
m) $(\mathbb{Q}^+)^{-1} = \mathbb{Q}^+$,
n) if A and B each one does not contain zero then $A^{-1} \cup B^{-1} = (A \cup B)^{-1}$.

Proof: The results follow from the properties of sets and from the properties of the rational numbers. ∎

We do not present the proofs of the theorems of Section III since Dedekind cuts are widely known.

Lemma 37. *If $\langle L, U \rangle$ is a Dedekind cut then:*

a) $U \neq \varnothing$ *and* $U \neq \mathbb{Q}$,
b) $U = \mathbb{Q} \backslash L$,
c) U *is closed upwards,*
d) $x < y$ *for all* $x \in L$ *and* $y \in U$,
e) $\{x \in \mathbb{Q};\ x = \inf U\}$ *is not empty if and only if* $\{x \in \mathbb{Q};\ x = \sup L\}$ *is not empty and in this case* $\inf U = \sup L$,
f) if $\{x \in \mathbb{Q};\ x = \inf U\}$ *is not empty then* $\inf U \in U$.

Proof: The results follow from the definition of Dedekind cuts (Definition 6) and from the properties of the rational numbers. ∎

Lemma 38. *If $\langle L, U \rangle$ is a Dedekind cut then $\langle L_3, U_3 \rangle$ where*

$$\begin{aligned} U_3'' &= \{x \in \mathbb{Q};\ x = \inf U\}, \\ U_3 &= -(L \cup U_3''), \\ L_3 &= -(U \backslash U_3'') \end{aligned}$$

is a Dedekind cut.

Proof: Let $\langle L, U \rangle$ be a Dedekind cut and $\langle L_3, U_3 \rangle$ where

$$\begin{aligned} U_3'' &= \{x \in \mathbb{Q};\ x = \inf U\}, \\ U_3 &= -(L \cup U_3''), \\ L_3 &= -(U \backslash U_3''). \end{aligned}$$

It is immediate that L_3 and U_3 are sets of rational numbers.

Notice that either U_3'' is the empty set or U_3'' is a singleton set. Since $U \neq \varnothing$ and U is closed upwards it follows that U is not a singleton set whence $U \backslash U_3'' \neq \varnothing$. Hence $L_3 \neq \varnothing$. Since $U \neq \mathbb{Q}$ it follows that $U \backslash U_3'' \neq \mathbb{Q}$. Hence $L_3 \neq \mathbb{Q}$.

Notice that

$$
\begin{aligned}
L_3 \cup U_3 &= (-(U\backslash U_3'')) \cup (-(L \cup U_3''))\\
&= ((-U)\backslash(-U_3'')) \cup ((-L) \cup (-U_3''))\\
&= ((-U)\backslash(-U_3'')) \cup (-L) \cup (-U_3'')\\
&= ((-U)\backslash(-U_3'')) \cup (-U_3'') \cup (-L)\\
&= (-U) \cup (-U_3'') \cup (-L)\\
&= (-L) \cup (-U) \cup (-U_3'')\\
&= (-(L \cup U)) \cup (-U_3'')\\
&= (-\mathbb{Q}) \cup (-U_3'')\\
&= \mathbb{Q} \cup (-U_3'')\\
&= \mathbb{Q}.
\end{aligned}
$$

Notice also that

$$
\begin{aligned}
L_3 \cap U_3 &= (-(U\backslash U_3'')) \cap (-(L \cup U_3''))\\
&= ((-U)\backslash(-U_3'')) \cap ((-L) \cup (-U_3''))\\
&= (((-U)\backslash(-U_3'')) \cap (-L)) \cup (((-U)\backslash(-U_3''))\\
&\quad \cap (-U_3''))\\
&= (((-U) \cap (-L))\backslash(-U_3'')) \cup (((-U) \cap (-U_3''))\backslash\\
&\quad (-U_3''))\\
&= ((-(U \cap L))\backslash(-U_3'')) \cup ((-(U \cap U_3''))\backslash(-U_3''))\\
&= ((-\varnothing)\backslash(-U_3'')) \cup ((-U_3'')\backslash(-U_3''))\\
&= (\varnothing\backslash(-U_3'')) \cup \varnothing\\
&= \varnothing \cup \varnothing\\
&= \varnothing.
\end{aligned}
$$

If there is $x \in (L_3 \cap U_3)$ then let there be an arbitrary $b \in L_3$ and an arbitrary $x \in (-\infty, b)$. Since $x < b$ it follows that $-b < -x$. Since $b \in L_3 = -(U\backslash U_3'') = (-U)\backslash(-U_3'')$ it follows that $b \in (-U)$ whence $-b \in U$. Since $-b \in U$ and $-b < -x$ it follows that $-x$ is not infimum of U whence $-x \notin U_3''$. Since $-b \in U$ and $-b < -x$ and U is closed upwards it follows that $-x \in U$. Since $-x \in U$ and $-x \notin U_3''$ it follows that $-x \in U\backslash U_3''$ whence $x \in (-(U\backslash U_3''))$ that is $x \in (-(U\backslash U_3'')) = L_3$. Since x was taken arbitrarily in $(-\infty, b)$ it follows that $(-\infty, b) \subset L_3$. Since b was taken arbitrarily in L_3 it follows that L_3 is closed downwards.

Let there be an arbitrary $x \in L_3$. Since $x \in L_3 = (-(U\backslash U_3'')) = (-U)\backslash(-U_3'')$ it follows that $x \notin (-U_3'')$ whence $-x \notin U_3''$ that is $-x$ is not infimum of U whence $-x$ is not the least element of U. Hence there is $y \in U$ such that $y < -x$. Since $y < -x$ it follows that $y < \frac{y+(-x)}{2}$. Since $y \in U$ and $y < \frac{y+(-x)}{2}$ and U is closed upwards it follows that $\frac{y+(-x)}{2} \in U$. Since $y \in U$ and $y < \frac{y+(-x)}{2}$ it follows that $\frac{y+(-x)}{2}$ is not infimum of U whence $\frac{y+(-x)}{2} \notin U_3''$. Since $\frac{y+(-x)}{2} \in U$ and $\frac{y+(-x)}{2} \notin U_3''$ it follows that $\frac{y+(-x)}{2} \in U\backslash U_3''$ whence $-\frac{y+(-x)}{2} \in (-(U\backslash U_3'')) = L_3$. Since $y < -x$ it follows that $\frac{y+(-x)}{2} < -x$ whence $x < -\frac{y+(-x)}{2}$. Since $-\frac{y+(-x)}{2} \in L_3$ and $x < -\frac{y+(-x)}{2}$ it follows that x is not the greatest element of L_3. Since x was taken arbitrarily in L_3 it follows that L_3 does not have a greatest element.

Since L_3 and U_3 are sets of rational numbers and $L_3 \neq \varnothing$ and $L_3 \neq \mathbb{Q}$ and $L_3 \cup U_3 = \mathbb{Q}$ and $L_3 \cap U_3 = \varnothing$ and L_3 is closed downwards and L_3 does not have a greatest element, by the Definition 6, $\langle L_3, U_3\rangle$ is a Dedekind cut. ■

Proof of Theorem 26.: Let $\langle L, U\rangle$ be a Dedekind cut and $\langle L_3, U_3\rangle$ where

$$
\begin{aligned}
U_3'' &= \{x \in \mathbb{Q};\ x = \inf U\},\\
U_3 &= -(L \cup U_3''),\\
L_3 &= -(U\backslash U_3'').
\end{aligned}
$$

By Lemma 38, $\langle L_3, U_3\rangle$ is a Dedekind cut whence $\langle L, U\rangle + \langle L_3, U_3\rangle$ is well defined. Denote $\langle L_1, U_1\rangle = \langle L, U\rangle + \langle L_3, U_3\rangle$. By Definition 13, $L_1 = L + L_3$ whence $L_1 = L + L_3 = L + (-(U\backslash U_3'')) = \{w \in \mathbb{Q};$ there are $a \in L$ and $b \in (-(U\backslash U_3''))$ such that $w = a + b\} = \{w \in \mathbb{Q};$ there are $a \in L$ and $c \in U\backslash U_3''$ such that $w = a - c\}$.

For every $a \in L$ and $c \in U\backslash U_3''$ it follows that $a < c$ whence $a - c < 0$. Hence $L_1 \subset (-\infty, 0)$.

Let there be an arbitrary $x \in (-\infty, 0)$. Since $\langle L, U\rangle$ is a Dedekind cut it follows that $L \neq \varnothing$ and $L \neq \mathbb{Q}$ and $L \cup U = \mathbb{Q}$ and U is closed upwards.

- If $U_3'' \neq \varnothing$ then there is $\check{u} \in \mathbb{Q}$ such that $\check{u} = \inf U$ whence $\check{u}$. Denoting $a = \check{u} + \frac{x}{2}$ and $c = \check{u} - \frac{x}{2}$ it follows that $a - c = \check{u} + \frac{x}{2} - \left(\check{u} - \frac{x}{2}\right) = x$. Since $x \in (-\infty, 0)$ it follows that $a = \check{u} + \frac{x}{2} < \check{u}$ whence $a \notin U$. Hence $a \in L$. Since $x \in (-\infty, 0)$ it follows that $c = \check{u} - \frac{x}{2} > \check{u}$. Since $\check{u} \in U$ and $c \in (\check{u}, \infty)$ and U is closed upwards it follows that $c \in U$. Since $c > \check{u}$ it follows that c is not the infimum of U whence $c \notin U_3''$. Since $c \in U$ and $c \notin U_3''$ it follows that $c \in U\backslash U_3''$. Since there are $a \in L$ and $c \in U\backslash U_3''$ such that $x = a - c$ it follows that $x \in L_1$.
- If $U_3'' = \varnothing$ then $U\backslash U_3'' = U$. Since $L \neq \varnothing$ it follows that there is $z \in L$. Since $U \neq \varnothing$ it follows that there is $y \in U$. Since $\mathbb{N}$ is unbounded above, there is $m \in \mathbb{N}$ such that $m > \frac{z-y}{x}$ whence $z - mx > y$. Since $y \in U$ and $z - mx \in (y, \infty)$ and U is closed upwards it follows that $z - mx \in U$. Since $z - mx \in U$ it follows that the set $\{m \in \mathbb{N};\ z - mx \in U\}$ is not empty whence $\{m \in \mathbb{N};\ z - mx \in U\}$ has a least element. Denote n the least element of $\{m \in \mathbb{N};\ z - mx \in U\}$. In this way $n \in \{m \in \mathbb{N};\ z - mx \in U\}$ and $n - 1 \notin \{m \in \mathbb{N};\ z - mx \in U\}$. Denote $a = z - (n-1)x$ and $c = z - nx$. Notice that $a - c = z - (n-1)x - (z - nx) = x$. Since $n \in \{m \in \mathbb{N};\ z - mx \in U\}$ it follows that $c = z - nx \in U$. Since $c \in U$ and $U\backslash U_3'' = U$ it follows that $c \in U\backslash U_3''$. Since $n - 1 \notin \{m \in \mathbb{N};\ z - mx \in U\}$ it follows that $a = z - (n-1)x \notin U$ whence $a \in L$. Since there are $a \in L$ and $c \in U\backslash U_3''$ such that $x = a - c$ it follows that $x \in L_1$.

Since x was taken arbitrarily in $(-\infty, 0)$ it follows that $(-\infty, 0) \subset L_1$.

Since $L_1 \subset (-\infty, 0)$ and $(-\infty, 0) \subset L_1$ it follows that $L_1 = (-\infty, 0)$. Since $U_1 = \mathbb{Q}\backslash L_1$ it follows that $U_1 = [0, \infty)$. Hence $\langle L_1, U_1\rangle = 0$ whence $\langle L, U\rangle + \langle L_3, U_3\rangle = 0$. Thus $\langle L_3, U_3\rangle = -\langle L, U\rangle$. ■

Lemma 39. *Let $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ be Dedekind cuts* *and*

$$\begin{aligned} U_3' &= (U_1 \cap \mathbb{Q}^+) \cdot (U_2 \cap \mathbb{Q}^+), \\ U_3'' &= \{x \in \mathbb{Q};\ x = \inf U_3'\}, \\ U_3 &= U_3'' \cup U_3', \\ L_3' &= (L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+), \\ L_3'' &= -(L_3' \cup U_3') \cup L_3' \\ L_3 &= \mathcal{I}(L_3''). \end{aligned}$$

If $\langle L_1, U_1\rangle = 0$ or $\langle L_2, U_2\rangle = 0$ then $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle = \langle L_3, U_3\rangle$.

Proof: Let $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ be Dedekind cuts and

$$\begin{aligned} U_3' &= (U_1 \cap \mathbb{Q}^+) \cdot (U_2 \cap \mathbb{Q}^+), \\ U_3'' &= \{x \in \mathbb{Q};\ x = \inf U_3'\}, \\ U_3 &= U_3'' \cup U_3', \\ L_3' &= (L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+), \\ L_3'' &= -(L_3' \cup U_3') \cup L_3' \\ L_3 &= \mathcal{I}(L_3''). \end{aligned}$$

By Definition 19, $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle = \langle L_4, U_4\rangle$ where

$$\begin{aligned} L_4 &= (-\infty, 0) \cup \big((L_1 \cap [0,\infty)) \cdot (L_2 \cap [0,\infty))\big), \\ U_4 &= \mathbb{Q}\backslash L_4. \end{aligned}$$

If $\langle L_1, U_1\rangle = 0$ then $L_1 = (-\infty, 0)$ and $U_1 = [0,\infty)$. Hence

$$\begin{aligned} L_4 &= (-\infty, 0) \cup \big((L_1 \cap [0,\infty)) \cdot (L_2 \cap [0,\infty))\big) \\ &= (-\infty, 0) \cup \big(((-\infty,0) \cap [0,\infty)) \cdot (L_2 \cap [0,\infty))\big) \\ &= (-\infty, 0) \cup \big(\varnothing \cdot (L_2 \cap [0,\infty))\big) \\ &= (-\infty, 0) \cup \varnothing \\ &= (-\infty, 0) \end{aligned}$$

and thereby

$$U_4 = \mathbb{Q}\backslash L_4 = \mathbb{Q}\backslash(-\infty,0) = [0,\infty).$$

Furthermore

$$\begin{aligned} U_3' &= (U_1 \cap \mathbb{Q}^+) \cdot (U_2 \cap \mathbb{Q}^+) \\ &= ([0,\infty) \cap \mathbb{Q}^+) \cdot (U_2 \cap \mathbb{Q}^+) \\ &= \mathbb{Q}^+ \cdot (U_2 \cap \mathbb{Q}^+) \\ &= \mathbb{Q}^+ \end{aligned}$$

whence

$$U_3'' = \{x \in \mathbb{Q};\ x = \inf U_3'\} = \{0\}$$

and thereby

$$U_3 = U_3'' \cup U_3' = \{0\} \cup \mathbb{Q}^+ = [0,\infty),$$

and

$$\begin{aligned} L_3' &= (L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+) \\ &= ((-\infty,0) \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+) \\ &= \varnothing \cdot (L_2 \cap \mathbb{Q}^+) \\ &= \varnothing \end{aligned}$$

whence $L_3'' = -(L_3' \cup U_3') \cup L_3' = -(\varnothing \cup U_3') \cup \varnothing = -U_3' = -\mathbb{Q}^+ = \mathbb{Q}^-$ and

$$L_3 = \mathcal{I}(L_3'') = \mathcal{I}(\mathbb{Q}^-) = \mathbb{Q}^- = (-\infty, 0).$$

Therefore $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle = \langle L_4, U_4\rangle = \langle(-\infty,0),[0,\infty)\rangle = \langle L_3, U_3\rangle$.

If $\langle L_2, U_2\rangle = 0$ then in a similar way we prove that $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle = \langle L_3, U_3\rangle$. ■

Lemma 40. *If $\langle L, U\rangle$ is a Dedekind cut such that $\langle L, U\rangle > 0$ then there is $r \in L$ such that $r \in \mathbb{Q}^+$.*

Proof: Let $\langle L, U\rangle$ be a Dedekind cut such that $\langle L, U\rangle > 0$. By Definition 7, $\mathbb{Q}^- \subset L$ and $U \subset \{0\} \cup \mathbb{Q}^+$.

If there would be no $r \in L$ such that $r \in \mathbb{Q}^+$ then $L \subset \mathbb{Q}^- \cup \{0\}$ whence $\mathbb{Q}^- \subset L \subset \mathbb{Q}^- \cup \{0\}$. Hence either $L = \mathbb{Q}^-$ or $L = \mathbb{Q}^- \cup \{0\}$. If $L = \mathbb{Q}^- \cup \{0\}$ then L would have a greatest element, which contradicts Definition 6. If $L = \mathbb{Q}^-$ then, since U is the complement of L, U would be $\{0\} \cup \mathbb{Q}^+$ whence C would be $\langle \mathbb{Q}^-, \{0\} \cup \mathbb{Q}^+\rangle = 0$ which contradicts the fact that C is non zero. Therefore there is $r \in L$ such that $r \in \mathbb{Q}^+$. ■

Lemma 41. *If $\langle L, U\rangle$ is a Dedekind cut such that $\langle L, U\rangle > 0$ then $U \subset \mathbb{Q}^+$.*

Proof: Let $\langle L, U\rangle$ be a Dedekind cut such that $\langle L, U\rangle > 0$. By Definition 7, $\mathbb{Q}^- \subset L$ and $U \subset \{0\} \cup \mathbb{Q}^+$.

If $0 \in U$ then, since U is closed upwards, $U \subset \{0\} \cup \mathbb{Q}^+$. Since $L \cup U = \mathbb{Q}$, it would follow that $L = \mathbb{Q}^-$ and thus $\langle L, U\rangle = \langle \mathbb{Q}^-, \{0\} \cup \mathbb{Q}^+\rangle = 0$ which contradicts the fact that $\langle L, U\rangle > 0$. Therefore $0 \notin U$.

Since $U \subset \{0\} \cup \mathbb{Q}^+$ and $0 \notin U$ it follows that $U \subset \mathbb{Q}^+$. ■

Lemma 42. *If $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ are Dedekind cuts such that $\langle L_1, U_1\rangle > 0$ and $\langle L_2, U_2\rangle > 0$ then $\mathbb{Q}^+\backslash((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+)) = U_1 \cdot U_2$.*

Proof: Let $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ be Dedekind cuts such that $\langle L_1, U_1\rangle > 0$ and $\langle L_2, U_2\rangle > 0$. By Lemma 41, $U_1 \subset \mathbb{Q}^+$ and $U_2 \subset \mathbb{Q}^+$.

Let there be an arbitrary $y \in U_1 \cdot U_2$. It follows that there are $a \in U_1$ and $b \in U_2$ such that $y = ab$. Since $a \in U_1$ and $b \in U_2$ and $U_1 \subset \mathbb{Q}^+$ and $U_2 \subset \mathbb{Q}^+$ it follows that $ab > 0$ whence $y = ab \in \mathbb{Q}^+$. If $y \in ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+))$ then there would be $c \in (L_1 \cap \mathbb{Q}^+)$ and $d \in (L_2 \cap \mathbb{Q}^+)$ such that $y = cd$. Since $c \in L_1$ and $a \in U_1$ it would follow that $c < a$. Since $d \in L_2$ and $b \in U_2$ it would follow that $d < b$. Since $c > 0$ and $d > 0$ and $c < a$ and $d < b$ it would follow that $cd < ab$, that is, $y = cd < ab = y$ which is an absurd. Thus $y \notin ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+))$. Since $y \in \mathbb{Q}^+$ and $y \notin ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+))$ it follows that $y \in \mathbb{Q}^+\backslash((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+))$. Since y was taken arbitrarily in $U_1 \cdot U_2$ it follows that $U_1 \cdot U_2 \subset (\mathbb{Q}^+\backslash((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+)))$.

Let there be an arbitrary $y \in \mathbb{Q}^+\backslash((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+))$. Let there be an arbitrary $b \in L_2 \cap \mathbb{Q}^+$. If there is $a \in L_1 \cap \mathbb{Q}^+$ such that $\frac{y}{b} \leqslant a$ then, since $a \in L_1$ and L_1 is closed downwards, it would follow that $\frac{y}{b} \in L_1$ whence $\frac{y}{b} \in L_1 \cap \mathbb{Q}^+$ and thereby $y = \frac{y}{b} b \in ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+))$ which contradicts the fact that $y \notin ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+))$. Thus $a < \frac{y}{b}$ for all $a \in L_1 \cap \mathbb{Q}^+$ whence $a < \frac{y}{b}$ for all $a \in L_1$. Hence $\frac{y}{b} \notin L_1$ whence $\frac{y}{b} \in U_1$. If there is no $z \in U_1$ such that $z < \frac{y}{b}$ then $\frac{y}{b}$ would be the lest element of U_1 whence $\frac{y}{b}$ would be the infimum of U_1. Since $b \in L_2$ and L_2 does not have a greatest element, there would be $c \in L_2$ such that $b < c$. Hence $c > 0$ and $\frac{b}{c} < 1$ whence $\frac{y}{c} = \frac{b}{c}\frac{y}{b} < \frac{y}{b} = \inf U_1$. Thus $\frac{y}{c} \notin U_1$ whence $\frac{y}{c} \in L_1$. Since $\frac{y}{c} > 0$ it would follow that $\frac{y}{c} \in L_1 \cap \mathbb{Q}^+$. Since $\frac{y}{c} \in L_1 \cap \mathbb{Q}^+$ and $c \in L_2 \cap \mathbb{Q}^+$ it would follow that $y = \frac{y}{c}c \in ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+))$ which contradicts the fact of $y \notin ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+))$. Therefore there is $z \in U_1$ such that $z < \frac{y}{b}$. Since $U_1 \subset \mathbb{Q}^+$ it follows that $z > 0$. Since $z < \frac{y}{b}$ and $0 < z$ it follows that $b < \frac{y}{z}$. Since b was taken arbitrarily in $L_2 \cap \mathbb{Q}^+$ it follows that $b < \frac{y}{z}$ for all $b \in L_2$. Hence $\frac{y}{z} \notin L_2$ whence $\frac{y}{z} \in U_2$. Since $z \in U_1$ and $\frac{y}{z} \in U_2$ it follows that $y = z\frac{y}{z} \in U_1 \cdot U_2$. Since y was taken arbitrarily in $\mathbb{Q}^+ \backslash ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+))$ it follows that $(\mathbb{Q}^+ \backslash ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+))) \subset (U_1 \cdot U_2)$.

Since $U_1 \cdot U_2 \subset (\mathbb{Q}^+ \backslash ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+)))$ and $(\mathbb{Q}^+ \backslash ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+))) \subset (U_1 \cdot U_2)$ it follows that $(\mathbb{Q}^+ \backslash ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+))) = (U_1 \cdot U_2)$. ∎

Lemma 43. *Let $\langle L_1, U_1 \rangle$ and $\langle L_2, U_2 \rangle$ be Dedekind cuts and*

$$\begin{aligned}
U_3' &= (U_1 \cap \mathbb{Q}^+) \cdot (U_2 \cap \mathbb{Q}^+), \\
U_3'' &= \{x \in \mathbb{Q};\ x = \inf U_3'\}, \\
U_3 &= U_3'' \cup U_3', \\
L_3' &= (L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+), \\
L_3'' &= -(L_3' \cup U_3') \cup L_3' \\
L_3 &= \mathcal{I}(L_3'').
\end{aligned}$$

If $\langle L_1, U_1 \rangle > 0$ and $\langle L_2, U_2 \rangle > 0$ then $\langle L_1, U_1 \rangle \times \langle L_2, U_2 \rangle = \langle L_3, U_3 \rangle$.

Proof: Let $\langle L_1, U_1 \rangle$ and $\langle L_2, U_2 \rangle$ be Dedekind cuts such that $\langle L_1, U_1 \rangle > 0$ and $\langle L_2, U_2 \rangle > 0$. Let $\langle L_3, U_3 \rangle$ where

$$\begin{aligned}
U_3' &= (U_1 \cap \mathbb{Q}^+) \cdot (U_2 \cap \mathbb{Q}^+), \\
U_3'' &= \{x \in \mathbb{Q};\ x = \inf U_3'\}, \\
U_3 &= U_3'' \cup U_3', \\
L_3' &= (L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+), \\
L_3'' &= -(L_3' \cup U_3') \cup L_3' \\
L_3 &= \mathcal{I}(L_3'').
\end{aligned}$$

By Definition 19, $\langle L_1, U_1 \rangle \times \langle L_2, U_2 \rangle = \langle L_4, U_4 \rangle$ where

$$\begin{aligned}
L_4 &= (-\infty, 0) \cup \big((L_1 \cap [0, \infty)) \cdot (L_2 \cap [0, \infty))\big), \\
U_4 &= \mathbb{Q} \backslash L_4.
\end{aligned}$$

Since $\langle L_1, U_1 \rangle > 0$ and $\langle L_2, U_2 \rangle > 0$ and the set of all Dedekind cuts is an ordered field, $\langle L_1, U_1 \rangle \times \langle L_2, U_2 \rangle > 0$ whence $\langle L_4, U_4 \rangle > 0$. Hence, by Lemma 40, there is $r \in L_4$ such that $r \in \mathbb{Q}^+$. Since $r \in L_4$ and $0 < r$ and L_4 is closed downwards it follows that $0 \in L_4$. In this way

$$\begin{aligned}
L_4 &= (-\infty, 0) \cup \big((L_1 \cap [0, \infty)) \cdot (L_2 \cap [0, \infty))\big) \\
&= (-\infty, 0) \cup \big((L_1 \cap (\{0\} \cup \mathbb{Q}^+)) \cdot (L_2 \cap (\{0\} \cup \mathbb{Q}^+))\big) \\
&= (-\infty, 0) \cup \big(((L_1 \cap \{0\}) \cup (L_1 \cap \mathbb{Q}^+)) \cdot ((L_2 \cap \{0\}) \\
&\quad \cup (L_2 \cap \mathbb{Q}^+))\big) \\
&= (-\infty, 0) \cup \big((L_1 \cap \{0\}) \cup (L_2 \cap \{0\}) \cup ((L_1 \cap \mathbb{Q}^+) \\
&\quad \cdot (L_2 \cap \mathbb{Q}^+))\big) \\
&= (-\infty, 0) \cup (L_1 \cap \{0\}) \cup (L_2 \cap \{0\}) \cup ((L_1 \cap \mathbb{Q}^+) \\
&\quad \cdot (L_2 \cap \mathbb{Q}^+)) \\
&= (-\infty, 0) \cup \{0\} \cup ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+)) \\
&= (-\infty, 0] \cup ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+))
\end{aligned}$$

and thereby

$$\begin{aligned}
U_4 &= \mathbb{Q} \backslash L_4 \\
&= \mathbb{Q} \backslash ((-\infty, 0] \cup ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+))) \\
&= \mathbb{Q}^+ \backslash ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+)).
\end{aligned}$$

Furthermore, since $\langle L_1, U_1 \rangle > 0$ and $\langle L_2, U_2 \rangle > 0$, by Lemma 41, $U_1 \subset \mathbb{Q}^+$ and $U_2 \subset \mathbb{Q}^+$ whence

$$\begin{aligned}
U_3' &= (U_1 \cap \mathbb{Q}^+) \cdot (U_2 \cap \mathbb{Q}^+) \\
&= U_1 \cdot U_2
\end{aligned}$$

and

$$\begin{aligned}
U_3'' &= \{x \in \mathbb{Q};\ x = \inf U_3'\} \\
&= \{x \in \mathbb{Q};\ x = \inf (U_1 \cdot U_2)\} \\
&= \{x \in \mathbb{Q};\ x = \inf U_1 \inf U_2\}
\end{aligned}$$

and thereby

$$\begin{aligned}
U_3 &= U_3'' \cup U_3' \\
&= \{x \in \mathbb{Q};\ x = \inf U_1 \inf U_2\} \cup (U_1 \cdot U_2) \\
&= (\{x \in \mathbb{Q};\ x = \inf U_1\} \cdot \{x \in \mathbb{Q};\ x = \inf U_2\}) \cup (U_1 \cdot U_2) \\
&= (\{x \in \mathbb{Q};\ x = \inf U_1\} \cup U_1) \cdot (\{x \in \mathbb{Q};\ x = \inf U_2\} \cup U_2) \\
&= U_1 \cdot U_2.
\end{aligned}$$

Further

$$L_3' = (L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+)$$

and

$$\begin{aligned}
L_3'' &= -(L_3' \cup U_3') \cup L_3' \\
&= -(((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+)) \cup (U_1 \cdot U_2)) \cup ((L_1 \cap \mathbb{Q}^+) \\
&\quad \cdot (L_2 \cap \mathbb{Q}^+))
\end{aligned}$$

and $L_3 = \mathcal{I}(L_3'')$.

By Lemma 42, $\mathbb{Q}^+ \backslash ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+)) = U_1 \cdot U_2$ whence $U_4 = \mathbb{Q}^+ \backslash ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+)) = U_1 \cdot U_2 = U_3$.

Since $\mathbb{Q}^+ \backslash ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+)) = U_1 \cdot U_2$ it follows that $((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+)) \cup (U_1 \cdot U_2) = \mathbb{Q}^+$ whence

$$\begin{aligned}
L_3'' &= -(((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+)) \cup (U_1 \cdot U_2)) \cup ((L_1 \cap \mathbb{Q}^+) \\
&\quad \cdot (L_2 \cap \mathbb{Q}^+)) \\
&= -\mathbb{Q}^+ \cup ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+)) \\
&= \mathbb{Q}^- \cup ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+)).
\end{aligned}$$

Hence

$$\begin{aligned}
L_3 &= \mathcal{I}(L_3'') \\
&= \mathcal{I}\big(\mathbb{Q}^- \cup ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+))\big) \\
&= (-\infty, 0] \cup ((L_1 \cap \mathbb{Q}^+) \cdot (L_2 \cap \mathbb{Q}^+)) = L_4.
\end{aligned}$$

Therefore $\langle L_1, U_1 \rangle \times \langle L_2, U_2 \rangle = \langle L_4, U_4 \rangle = \langle L_3, U_3 \rangle$ ∎

Proof of Theorem 27.: The results follows from Lemmas 39 and 43. ∎

Lemma 44. *Let $\langle L, U \rangle$ be a Dedekind cut such that $\langle L, U \rangle > 0$ and $\breve{u} \in \mathbb{Q}$. It follows that $\breve{u}$ is the supremum of L if and only if $\breve{u}^{-1}$ is the infimum of $(L \cap \mathbb{Q}^+)^{-1}$.*

Proof: Let $\langle L, U \rangle$ be a Dedekind cut such that $\langle L, U \rangle > 0$ and $\breve{u} \in \mathbb{Q}$.

Suppose that $\breve{u}$ is the supremum of L. Since $\breve{u}$ is the supremum of L, $\breve{u}$ is the infimum of U.

By Lemma 40 there is $r \in L$ such that $r \in \mathbb{Q}^+$. Since $r \in L$ we have that $r < y$ for all $y \in U$ whence r is a lower bound of U. Thus $r \leqslant \breve{u}$. Since $r \in \mathbb{Q}^+$ we have that $0 < r$. Hence $0 < r \leqslant \breve{u}$ that is $0 < \breve{u}$.

Since there is $r \in L \cap \mathbb{Q}^+$ it follows that $(L \cap \mathbb{Q}^+)^{-1}$ is non-empty.

Let there be an arbitrary $x \in (L \cap \mathbb{Q}^+)^{-1}$. Since $x^{-1} \in L$ we have that $x^{-1} < y$ for all $y \in U$ whence x^{-1} is a lower bound of U. Thus $x^{-1} \leqslant \breve{u}$. Since $x > 0$ and $\breve{u} > 0$ it follows that $\breve{u}^{-1} \leqslant x$. Since x was taken arbitrarily in $(L \cap \mathbb{Q}^+)^{-1}$ it follows that $\breve{u}^{-1} \leqslant x$ for all $x \in (L \cap \mathbb{Q}^+)^{-1}$. Thus $\breve{u}^{-1}$ is a lower bound of $(L \cap \mathbb{Q}^+)^{-1}$.

Let there be an arbitrary $x \in \mathbb{Q}$ such that $x > \breve{u}^{-1}$. Since $x > \breve{u}^{-1}$ it follows that $x > 0$ and $x^{-1} < \breve{u}$ whence $x^{-1} < \breve{u} \leqslant y$ for all $y \in U$. Since $x^{-1} \neq y$ for all $y \in U$ it follows that $x^{-1} \notin U$ whence $x^{-1} \in L$. Since L does not have a greatest element, there is $z \in L$ such that $x^{-1} < z$ whence $z > 0$ and $z^{-1} < x$. Since $z \in L$ and $z > 0$ it follows that $z^{-1} \in (L \cap \mathbb{Q}^+)^{-1}$. Since there is $z^{-1} \in (L \cap \mathbb{Q}^+)^{-1}$ such that $z^{-1} < x$ it follows that x is not a lower bound of $(L \cap \mathbb{Q}^+)^{-1}$. Since x satisfying $x > \breve{u}^{-1}$ was taken arbitrarily in $\mathbb{Q}$ it follows that if x is any rational number such that $x > \breve{u}^{-1}$ then x is not a lower bound of $(L \cap \mathbb{Q}^+)^{-1}$. Therefore $\breve{u}^{-1}$ is the greatest lower bound of $(L \cap \mathbb{Q}^+)^{-1}$. That is $\breve{u}^{-1}$ is the infimum of $(L \cap \mathbb{Q}^+)^{-1}$.

If $\breve{u}^{-1}$ is the infimum of $(L \cap \mathbb{Q}^+)^{-1}$ then in a similar way we show that $\breve{u}$ is the supremum of L. ■

Lemma 45. *Let $\langle L, U \rangle$ be a Dedekind cut such that $\langle L, U \rangle > 0$ and $\breve{u} \in \mathbb{Q}$. It follows that $\breve{u}$ is the infimum of U if and only if $\breve{u}^{-1}$ is the supremum of $(U \cap \mathbb{Q}^+)^{-1}$.*

Proof: The proof is similar to the proof of Lemma 44. ■

Lemma 46. *If $\langle L_1, U_1 \rangle$ is a Dedekind cut such that $\langle L_1, U_1 \rangle > 0$ then $\langle L_3, U_3 \rangle$ where*

$$\begin{aligned} U_3' &= (L \cap \mathbb{Q}^+)^{-1}, \\ U_3'' &= \{x \in \mathbb{Q};\ x = \inf U_3'\}, \\ U_3 &= U_3'' \cup U_3', \\ L_3' &= (U \cap \mathbb{Q}^+)^{-1}, \\ L_3'' &= -(L_3' \cup U_3') \cup L_3', \\ L_3 &= \mathcal{I}(L_3'') \end{aligned}$$

is a Dedekind cut.

Proof: Let $\langle L_1, U_1 \rangle$ be a Dedekind cut such that $\langle L_1, U_1 \rangle > 0$ and $\langle L_3, U_3 \rangle$ where

$$\begin{aligned} U_3' &= (L_1 \cap \mathbb{Q}^+)^{-1}, \\ U_3'' &= \{x \in \mathbb{Q};\ x = \inf U_3'\}, \\ U_3 &= U_3'' \cup U_3', \\ L_3' &= (U_1 \cap \mathbb{Q}^+)^{-1}, \\ L_3'' &= -(L_3' \cup U_3') \cup L_3', \\ L_3 &= \mathcal{I}(L_3''), \end{aligned}$$

It is immediate that L_3 and U_3 are sets of rational numbers.

Since $\langle L_1, U_1 \rangle > 0$, by Lemma 41, $U_1 \subset \mathbb{Q}^+$ whence $U_1 \cap \mathbb{Q}^+ = U_1$. Hence

$$\begin{aligned} L_3'' &= -(L_3' \cup U_3') \cup L_3' \\ &= -((U_1 \cap \mathbb{Q}^+)^{-1} \cup (L_1 \cap \mathbb{Q}^+)^{-1}) \cup (U_1 \cap \mathbb{Q}^+)^{-1} \\ &= -(((U_1 \cap \mathbb{Q}^+) \cup (L_1 \cap \mathbb{Q}^+))^{-1}) \cup (U_1 \cap \mathbb{Q}^+)^{-1} \\ &= -(((U_1 \cup L_1) \cap \mathbb{Q}^+)^{-1}) \cup U_1^{-1} \\ &= -((\mathbb{Q} \cap \mathbb{Q}^+)^{-1}) \cup U_1^{-1} \\ &= -((\mathbb{Q}^+)^{-1}) \cup U_1^{-1} \\ &= -(\mathbb{Q}^+) \cup U_1^{-1} \\ &= \mathbb{Q}^- \cup U_1^{-1}. \end{aligned}$$

Since $L_3'' = \mathbb{Q}^- \cup U_1^{-1}$ it follows that $\mathbb{Q}^- \subset L_3''$ whence $\mathbb{Q}^- \subset \mathcal{I}(L_3'') = L_3$. Thus $L_3 \neq \varnothing$. By Lemma 40 there is $r \in L_1$ such that $r \in \mathbb{Q}^+$. Thus $(0, r) \cap U_1 = \varnothing$ whence $\left(r^{-1}, \infty\right) \cap U_1^{-1} = \varnothing$. Hence $r^{-1} \notin \mathcal{I}(\mathbb{Q}^- \cup U_1^{-1})$ whence $L_3 = \mathcal{I}(L_3'') = \mathcal{I}(\mathbb{Q}^- \cup U_1^{-1}) \neq \mathbb{Q}$.

Let there be an arbitrary $r \in \mathbb{Q}$. Suppose $r \notin U_3$.

- If $r < 0$ then $r \in \left(2r, \frac{r}{2}\right)$. Since $2r, \frac{r}{2} \in \mathbb{Q}^-$ it follows that $2r, \frac{r}{2} \in \mathbb{Q}^- \cup U_1^{-1} = L_3''$. Since $r \in \left(2r, \frac{r}{2}\right)$ and $2r, \frac{r}{2} \in L_3''$ it follows that $r \in \{x \in \mathbb{Q};\ \text{there are } a, b \in L_3'' \text{ such that } x \in (a, b)\} = \mathcal{I}(L_3'') = L_3$.
- If $r = 0$ then, since U_1 is non-empty and $U_1 \subset \mathbb{Q}^+$, it follows that U_1^{-1} is non-empty and $U_1^{-1} \subset \mathbb{Q}^+$. Hence there is $s \in U_1^{-1}$. Since $U_1^{-1} \subset \mathbb{Q}^+$ it follows that $s \in \mathbb{Q}^+$. Since $r = 0$ and $s \in \mathbb{Q}^+$ it follows that $r \in (-1, s)$. Since $r \in (-1, s)$ and $-1, s \in \mathbb{Q}^- \cup U_1^{-1} = L_3''$ it follows that $r \in \{x \in \mathbb{Q};\ \text{there are } a, b \in L_3'' \text{ such that } x \in (a, b)\} = \mathcal{I}(L_3'') = L_3$.
- If $r > 0$ then, since $r \notin U_3$, it follows that $r \notin U_3'$ whence $r \notin (L_1 \cap \mathbb{Q}^+)^{-1}$. Since $r \in \mathbb{Q}^+$ and $r \notin (L_1 \cap \mathbb{Q}^+)^{-1}$ it follows that $r^{-1} \notin L_1$. Hence $r^{-1} \in U_1$ whence $x < r^{-1}$ for all $x \in L_1$. Thus r^{-1} is a upper bound of L_1.
  Since $r \notin U_3$ it follows that r is not the infimum of U_3' that is r is not the infimum of $(L_1 \cap \mathbb{Q}^+)^{-1}$. Hence, by Lemma 44, r^{-1} is not the supremum of L_1.
  Since r^{-1} is a upper bound of L_1 but not the supremum of L_1 there is $s \in \mathbb{Q}$ such that $s < r^{-1}$ and s is a upper bound of L_1.
  If $s \in L_1$ then, since s is a upper bound of L_1, s would be the greatest element of L_1 which contradicts the fact that L_1 does not have a greatest element. Thus $s \notin L_1$. Since $s \notin L_1$ it follows that $s \in U_1$ whence $s^{-1} \in U_1^{-1}$. Since $s < r^{-1}$ it follows that $s^{-1} > r$.
  Since $s^{-1} > r$ and $r > 0$ it follows that $r \in (-1, s^{-1})$. Since $r \in (-1, s^{-1})$ and $-1, s^{-1} \in \mathbb{Q}^- \cup U_1^{-1} = L_3''$ it follows that $r \in \{x \in \mathbb{Q};\ \text{there are } a, b \in L_3'' \text{ such that } x \in (a, b)\} = \mathcal{I}(L_3'') = L_3$.

That is, in any case if $r \notin U_3$ then $r \in L_3$. Therefore $L_3 \cup U_3 = \mathbb{Q}$.

Let there be an arbitrary $x \in U_3$. It follows that $x \in (U_3'' \cup U_3')$. If $x \in U_3''$ then $x = \inf U_3'$ that is $x = \inf(L_1 \cap \mathbb{Q}^+)^{-1}$. By Lemma 44, $x^{-1} = \sup L_1$ whence $x^{-1} = \inf U_1$. By Lemma 45, $x = \sup(U_1 \cap \mathbb{Q}^+)^{-1}$ whence $x = \sup L_3'$. Thus $x \notin \mathcal{I}(-(L_3' \cup U_3') \cup L_3') = \mathcal{I}(L_3'') = L_3$. If $x \in U_3'$ then $x \in (L_1 \cap \mathbb{Q}^+)^{-1}$ whence $x^{-1} \in L_1$. Hence $x^{-1} \notin U_1$ that is $x^{-1} \notin (U_1 \cap \mathbb{Q}^+)$ whence $x \notin (U_1 \cap \mathbb{Q}^+)^{-1}$ that is $x \notin L_3'$. Since $x \in \mathbb{Q}^+$ it follows that $x \notin (-\infty, 0]$. Since $x \notin (-\infty, 0]$ and $x \notin L_3'$ it follows that $x \notin \mathcal{I}(-(L_3' \cup U_3') \cup L_3') =$

$\mathcal{I}(L_3'') = L_3$. Since x was taken arbitrarily in U_3 it follows that $L_3 \cap U_3 = \varnothing$.

Let $x, y \in \mathbb{Q}$ such that $x < y$ and $y \in L_3$. Since $y \in L_3 = \mathcal{I}(L_3'') = \{x \in \mathbb{Q};$ there are $a, b \in L_3''$ such that $x \in (a,b)\}$, there are $w, z \in L_3''$ such that $y \in (w,z)$. Since $x < y$ and $y \in (w,z)$ it follows that $x < y < z$. Since $\mathbb{Q}^-$ is not bounded below there is $v \in \mathbb{Q}^-$ such that $v < x$. Since $v \in \mathbb{Q}^-$ it follows that $v \in \mathbb{Q}^- \cup U_1^{-1} = L_3''$. Since $v < x$ and $x < z$ it follows that $x \in (v,z)$. Since $x \in (v,z)$ and $v, z \in L_3''$ it follows that $x \in \{x \in \mathbb{Q};$ there are $a, b \in L_3''$ such that $x \in (a,b)\} = \mathcal{I}(L_3'') = L_3$. Therefore L_3 is closed downwards.

If L_3 would have a greatest element u_3 then $u_3 \in L_3 = \{x \in \mathbb{Q};$ there are $a, b \in L_3''$ such that $x \in (a,b)\}$ whence there would be $w, z \in L_3''$ such that $u_3 \in (w,z)$. Since $w < u_3 < \frac{u_3+z}{2} < z$, it would follow that $\frac{u_3+z}{2} \in (w,z)$ whence $\frac{u_3+z}{2} \in \{x \in \mathbb{Q};$ there are $a, b \in L_3''$ such that $x \in (a,b)\} = L_3$. But, $\frac{u_3+z}{2} > u_3$ which contradicts the fact that u_3 is the greatest element of L_3. Therefore L_3 does not have a greatest element.

Since L_3 and U_3 are sets of rational numbers and $L_3 \neq \varnothing$ and $L_3 \neq \mathbb{Q}$ and $L_3 \cup U_3 = \mathbb{Q}$ and $L_3 \cap U_3 = \varnothing$ and L_3 is closed downwards and L_3 does not have a greatest element, by the Definition 6, $\langle L_3, U_3\rangle$ is a Dedekind cut. ■

Proof of Theorem 28.: Let $\langle L, U\rangle$ be a Dedekind cut such that $\langle L, U\rangle > 0$ and $\langle L_3, U_3\rangle$ where

$$\begin{aligned}
U_3' &= (L \cap \mathbb{Q}^+)^{-1},\\
U_3'' &= \{x \in \mathbb{Q};\ x = \inf U_3'\},\\
U_3 &= U_3'' \cup U_3',\\
L_3' &= (U \cap \mathbb{Q}^+)^{-1},\\
L_3'' &= -(L_3' \cup U_3') \cup L_3',\\
L_3 &= \mathcal{I}(L_3'').
\end{aligned}$$

By Lemma 46, $\langle L_3, U_3\rangle$ is a Dedekind cut whence $\langle L, U\rangle \times \langle L_3, U_3\rangle$ is well defined. Denote $\langle L_4, U_4\rangle = \langle L, U\rangle \times \langle L_3, U_3\rangle$. So, as it was seen in the proof of Lemma 43, $U_4 = U \cdot U_3$.

Let there be an arbitrary $y \in U_4$. Since $y \in U \cdot U_3$ it follows that there are $x \in U$ and $x_3 \in U_3$ such that $y = xx_3$.

- If x_3 is the infimum of U_3' then x_3 is the infimum of $(L \cap \mathbb{Q}^+)^{-1}$ whence, by Lemma 44, x_3^{-1} is the supremum of L. Hence x_3^{-1} is the infimum of U whence $x_3^{-1} \leqslant x$. Thus $1 = x_3^{-1}x_3 \leqslant xx_3 = y$.
- If x_3 is not the infimum of U_3' then $x_3 \in U_3' = (L \cap \mathbb{Q}^+)^{-1}$ whence $x_3^{-1} \in L$. Since $x_3^{-1} \in L$ and $x \in U$ it follows that $x_3^{-1} < x$ whence $1 = x_3^{-1}x_3 < xx_3 = y$.

Since y was taken arbitrarily in U_4 it follows that $U_4 \subset [1, \infty)$.

Since $U_4 \subset [1,\infty)$ it follows that 1 is a lower bound of U_4. Now, let there be an arbitrary $w \in (1,\infty)$. Let $y \in (1,w)$. Since $\langle L, U\rangle > 0$, by Lemma 40, there is $r \in L$ such that $r \in \mathbb{Q}^+$ whence $r \in L \cap \mathbb{Q}^+$. Since $U \neq \varnothing$, there is $z \in U$. Since $y > 1$ there is $m \in \mathbb{N}$ such that $y^m > \frac{z}{r}$ whence $ry^m > z$. Since $z \in U$ and $ry^m > z$ and U is closed upwards it follows that $ry^m \in U$. Since $ry^m \in U$ it follows that the set $\{m \in \mathbb{N};\ ry^m \in U\}$ is not empty whence $\{m \in \mathbb{N};\ ry^m \in U\}$ has a least element. Denote n the least element of $\{m \in \mathbb{N};\ ry^m \in U\}$. In this way $n \in \{m \in \mathbb{N};\ ry^m \in U\}$ and $n - 1 \notin \{m \in \mathbb{N};\ ry^m \in U\}$. Since $n \in \{m \in \mathbb{N};\ ry^m \in U\}$ it follows that $ry^n \in U$. Since $n - 1 \notin \{m \in \mathbb{N};\ ry^m \in U\}$ it follows that $ry^{n-1} \notin U$ whence $ry^{n-1} \in L$. Hence $ry^{n-1} \in L \cap \mathbb{Q}^+$ whence $r^{-1}y^{1-n} \in (L \cap \mathbb{Q}^+)^{-1} = U_3' \subset (U_3'' \cup U_3') = U_3$ Since $ry^n \in U$ and $r^{-1}y^{1-n} \in U_3$ it follows that $y = ry^n r^{-1}y^{1-n} \in U \cdot U_3$. Since $y < w$ and $y \in U \cdot U_3 = U_4$ it follows that w is not a lower bound of U_4. Since $y < w$ and $y \in U_4$ and U_4 is closed upwards it follows that $w \in U_4$. Since w was taken arbitrarily in $(1,\infty)$ it follows that, for all $w \in (1,\infty)$, w is not a lower bound of U_4 and $w \in U_4$. Since $w \in U_4$ for all $w \in (1,\infty)$ it follows that $(1,\infty) \subset U_4$. Since 1 is a lower bound of U_4 and, for all $w \in (1,\infty)$, w is not a lower bound of U_4 it follows that 1 is the greatest lower bound of U_4 that is $1 = \inf U_4$. Since $\langle L_4, U_4\rangle$ is a Dedekind cut it follows that $1 = \inf U_4 \in U_4$. Since $(1,\infty) \subset U_4$ and $1 \in U_4$ it follows that $[1,\infty) \subset U_4$.

Since $U_4 \subset [1,\infty)$ and $[1,\infty) \subset U_4$ it follows that $U_4 = [1,\infty)$.

Since $\langle L_4, U_4\rangle$ is a Dedekind cut it follows that $L_4 = \mathbb{Q}\backslash U_4$. Thus, since $U_4 = [1,\infty)$, it follows that $L_4 = (-\infty, 1)$. Hence $\langle L, U\rangle \times \langle L_3, U_3\rangle = \langle L_4, U_4\rangle = \langle(-\infty,1),[1,\infty)\rangle = 1$. Therefore $\langle L, U\rangle^{-1} = \langle L_3, U_3\rangle$. ■

Proof of Theorem VIII: The properties reflexivity, antisymmetry and transitivity of $\leqslant$ on Trans-Dedekind cuts follow, respectively, from the properties reflexivity, antisymmetry and transitivity of the inclusion relation, $\subset$, on sets. ■

Lemma 47. *It follows that:*

a) $\langle L, U\rangle \nless \langle\varnothing,\varnothing\rangle$ *for all Trans-Dedekind cut* $\langle L, U\rangle$,
b) $\langle\varnothing,\varnothing\rangle \nless \langle L, U\rangle$ *for all Trans-Dedekind cut* $\langle L, U\rangle$,
c) $\langle\varnothing,\mathbb{Q}\rangle < \langle L, U\rangle$ *for all Dedekind cuts* $\langle L, U\rangle$,
d) $\langle L, U\rangle < \langle\mathbb{Q},\varnothing\rangle$ *for all Dedekind cuts* $\langle L, U\rangle$.

Proof:

a) Let $\langle L, U\rangle$ be a Trans-Dedekind cut. Since L is not a proper subset of $\varnothing$ it follows that $\langle L, U\rangle < \langle\varnothing,\varnothing\rangle$ does not hold. Thus $\langle L, U\rangle \nless \langle\varnothing,\varnothing\rangle$.
b) Let $\langle L, U\rangle$ be a Trans-Dedekind cut. Since U is not a proper subset of $\varnothing$ it follows that $\langle\varnothing,\varnothing\rangle < \langle L, U\rangle$ does not hold. Thus $\langle\varnothing,\varnothing\rangle \nless \langle L, U\rangle$.
c) Let $\langle L, U\rangle$ be a Dedekind cut. It follows that $L \neq \varnothing$ and $U \neq \mathbb{Q}$ whence $\varnothing$ is a proper subset of L and U is a proper subset of $\mathbb{Q}$. Thus $\langle\varnothing,\mathbb{Q}\rangle < \langle L, U\rangle$.
d) Let $\langle L, U\rangle$ be a Dedekind cut. It follows that $L \neq \mathbb{Q}$ and $U \neq \varnothing$ whence L is a proper subset of $\mathbb{Q}$ and $\varnothing$ is a proper subset of U. Thus $\langle L, U\rangle < \langle\mathbb{Q},\varnothing\rangle$.

■

Proof of Theorem X:

a) By item a of Lemma 47, $\langle\mathbb{Q},\varnothing\rangle \nless \langle\varnothing,\varnothing\rangle$ but $\langle\mathbb{Q},\varnothing\rangle \geqslant \langle\varnothing,\varnothing\rangle$ does not hold.
b) The proof is similar to item a.
c) Let $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ be Trans-Dedekind cuts such that $\langle L_1, U_1\rangle > \langle L_2, U_2\rangle$. It follows that $\langle L_2, U_2\rangle < \langle L_1, U_1\rangle$ whence $L_2 \subset L_1$ and $L_2 \neq L_1$. Hence $L_1 \subset L_2$ does not hold whence $\langle L_1, U_1\rangle \leqslant \langle L_2, U_2\rangle$ does not hold. Thus $\langle L_1, U_1\rangle < \langle L_2, U_2\rangle$ does not hold whence $\langle L_1, U_1\rangle \nless \langle L_2, U_2\rangle$.
d) The proof is similar to item c.

■

Proof of Theorem XI: The result is immediate from the definitions of Trans-Dedekind cuts (Definition VI) and Dedekind cuts (Definition 6). ■

Lemma 48. *The set of all Trans-Dedekind cuts is of all Dedekind cuts together with* $\langle\varnothing,\varnothing\rangle$*,* $\langle\varnothing,\mathbb{Q}\rangle$ *and* $\langle\mathbb{Q},\varnothing\rangle$*.*

Proof: Let $\langle L,U\rangle$ be a Trans-Dedekind cut. By the Definition VI,

I) $L\cup U=\varnothing$ or $L\cup U=\mathbb{Q}$,
II) $L\cap U=\varnothing$,
III) L is closed downwards,
IV) L does not have a greatest element.

- If $L=\varnothing$ then either $U=\varnothing$ or $U=\mathbb{Q}$. If $U=\varnothing$ then $\langle L,U\rangle=\langle\varnothing,\varnothing\rangle$. If $U=\mathbb{Q}$ then $\langle L,U\rangle=\langle\varnothing,\mathbb{Q}\rangle$.
- If $L=\mathbb{Q}$ then, by item II, $U=\varnothing$ whence $\langle L,U\rangle=\langle\mathbb{Q},\varnothing\rangle$.
- If $L\neq\varnothing$ and $L\neq\mathbb{Q}$ then $L\cup U\neq\varnothing$ whence, by item I, $L\cup U=\mathbb{Q}$. Thus
  0) $L\neq\varnothing$ and $L\neq\mathbb{Q}$,
  1) $L\cup U=\mathbb{Q}$,
  2) $L\cap U=\varnothing$,
  3) L is closed downwards,
  4) L does not have a greatest element

  whence, by the Definition 6, $\langle L,U\rangle$ is a Dedekind cut.

■

Lemma 49. *It follows that:*

a) $\langle\varnothing,\varnothing\rangle+\langle L,U\rangle=\langle\varnothing,\varnothing\rangle$ *for all Trans-Dedekind cut* $\langle L,U\rangle$*.*
b) $\langle\varnothing,\mathbb{Q}\rangle+\langle\mathbb{Q},\varnothing\rangle=\langle\varnothing,\varnothing\rangle$*.*
c) $\langle\varnothing,\mathbb{Q}\rangle+\langle\varnothing,\varnothing\rangle=\langle\varnothing,\varnothing\rangle$*.*
d) $\langle\varnothing,\mathbb{Q}\rangle+(\langle\varnothing,\mathbb{Q}\rangle)=\langle\varnothing,\mathbb{Q}\rangle$*.*
e) $\langle\varnothing,\mathbb{Q}\rangle+\langle L,U\rangle=\langle\varnothing,\mathbb{Q}\rangle$ *for all Dedekind cuts* $\langle L,U\rangle$*.*
f) $\langle\mathbb{Q},\varnothing\rangle+\langle\varnothing,\mathbb{Q}\rangle=\langle\varnothing,\varnothing\rangle$*.*
g) $\langle\mathbb{Q},\varnothing\rangle+\langle\varnothing,\varnothing\rangle=\langle\varnothing,\varnothing\rangle$*.*
h) $\langle\mathbb{Q},\varnothing\rangle+\langle\mathbb{Q},\varnothing\rangle=\langle\mathbb{Q},\varnothing\rangle$*.*
i) $\langle\mathbb{Q},\varnothing\rangle+\langle L,U\rangle=\langle\mathbb{Q},\varnothing\rangle$ *for all Dedekind cuts* $\langle L,U\rangle$*.*

Proof: Let $\langle L,U\rangle$ be a Trans-Dedekind cut.

a) Since $\varnothing+L=\varnothing$ and $\varnothing+U=\varnothing$, by the Definition XIII, $\langle\varnothing,\varnothing\rangle+\langle L,U\rangle=\langle\varnothing,\varnothing\rangle$.
b) Since $\varnothing+\mathbb{Q}=\varnothing$ and $\mathbb{Q}+\varnothing=\varnothing$, by the Definition XIII, $\langle\varnothing,\mathbb{Q}\rangle+\langle\mathbb{Q},\varnothing\rangle=\langle\varnothing,\varnothing\rangle$.
c) Since $\varnothing+\varnothing=\varnothing$ and $\mathbb{Q}+\varnothing=\varnothing$, by the Definition XIII, $\langle\varnothing,\mathbb{Q}\rangle+\langle\varnothing,\varnothing\rangle=\langle\varnothing,\varnothing\rangle$.
d) Since $\varnothing+\varnothing=\varnothing$ and $\mathbb{Q}+\mathbb{Q}=\mathbb{Q}$, by the Definition XIII, $\langle\varnothing,\mathbb{Q}\rangle+\langle\varnothing,\mathbb{Q}\rangle=\langle\varnothing,\mathbb{Q}\rangle$.
e) If $\langle L,U\rangle$ is a Dedekind cut then $L\neq$ whence $\mathbb{Q}+U=\mathbb{Q}$. Thus, by the Definition XIII, $\langle\varnothing,\mathbb{Q}\rangle+\langle L,U\rangle=\langle\varnothing,\mathbb{Q}\rangle$.
f) Since $\mathbb{Q}+\varnothing=\varnothing$ and $\varnothing+\mathbb{Q}=\varnothing$, by the Definition XIII, $\langle\mathbb{Q},\varnothing\rangle+\langle\varnothing,\mathbb{Q}\rangle=\langle\varnothing,\varnothing\rangle$.
g) Since $\mathbb{Q}+\varnothing=\varnothing$ and $\varnothing+\varnothing=\varnothing$, by the Definition XIII, $\langle\mathbb{Q},\varnothing\rangle+\langle\varnothing,\varnothing\rangle=\langle\varnothing,\varnothing\rangle$.
h) Since $\mathbb{Q}+\mathbb{Q}=\mathbb{Q}$ and $\varnothing+\varnothing=\varnothing$, by the Definition XIII, $\langle\mathbb{Q},\varnothing\rangle+\langle\mathbb{Q},\varnothing\rangle=\langle\mathbb{Q},\varnothing\rangle$.
i) If $\langle L,U\rangle$ is a Dedekind cut then $L\neq$ whence $\mathbb{Q}+L=\mathbb{Q}$. Thus, by the Definition XIII, $\langle\mathbb{Q},\varnothing\rangle+\langle L,U\rangle=\langle\mathbb{Q},\varnothing\rangle$.

■

Proof of Theorem XIV: Let $\langle L_1,U_1\rangle$ and $\langle L_2,U_2\rangle$ be Trans-Dedekind cuts. By Lemma 48, either $\langle L_1,U_1\rangle$ and $\langle L_2,U_2\rangle$ are both Dedekind cuts or $\langle L_1,U_1\rangle=\langle\varnothing,\varnothing\rangle$ or $\langle L_1,U_1\rangle=\langle\varnothing,\mathbb{Q}\rangle$ or $\langle L_1,U_1\rangle=\langle\mathbb{Q},\varnothing\rangle$.

If $\langle L_1,U_1\rangle$ and $\langle L_2,U_2\rangle$ are both Dedekind cuts then, by Theorem 14, the Dedekind sum between $\langle L_1,U_1\rangle$ and $\langle L_2,U_2\rangle$ is a Dedekind cut. Since the addition of Trans-Dedekind cuts (Definition XIII) is identical to the addition of Dedekind cuts (Definition 13) it follows that the Trans-Dedekind sum between $\langle L_1,U_1\rangle$ and $\langle L_2,U_2\rangle$, $\langle L_1,U_1\rangle+\langle L_2,U_2\rangle$, is the Dedekind sum between $\langle L_1,U_1\rangle$ and $\langle L_2,U_2\rangle$. Hence $\langle L_1,U_1\rangle+\langle L_2,U_2\rangle$ is a Dedekind cut. Thus, by Theorem XI, $\langle L_1,U_1\rangle+\langle L_2,U_2\rangle$ is a Trans-Dedekind cut.

If $\langle L_1,U_1\rangle=\langle\varnothing,\varnothing\rangle$ then, by item a of Lemma 49, $\langle L_1,U_1\rangle+\langle L_2,U_2\rangle$ is a Trans-Dedekind cut.

If $\langle L_1,U_1\rangle=\langle\varnothing,\mathbb{Q}\rangle$ then, by items b, c, d and e of Lemma 49, $\langle L_1,U_1\rangle+\langle L_2,U_2\rangle$ is a Trans-Dedekind cut.

If $\langle L_1,U_1\rangle=\langle\mathbb{Q},\varnothing\rangle$ then, by items f, g, h and i of Lemma 49, $\langle L_1,U_1\rangle+\langle L_2,U_2\rangle$ is a Trans-Dedekind cut. ■

Proof of Theorem XV: Let $\langle L,U\rangle$ be a Trans-Dedekind cut. By Lemma 48, either $\langle L,U\rangle$ is a Dedekind cut or $\langle L,U\rangle=\langle\varnothing,\varnothing\rangle$ or $\langle L,U\rangle=\langle\varnothing,\mathbb{Q}\rangle$ or $\langle L,U\rangle=\langle\mathbb{Q},\varnothing\rangle$.

If $\langle L,U\rangle$ is a Dedekind cut then, by Theorem 15, $\langle L,U\rangle+\langle(-\infty,0),[0,\infty)\rangle=\langle L,U\rangle$.

If $\langle L,U\rangle=\langle\varnothing,\varnothing\rangle$ then, by item a of Lemma 49, $\langle L,U\rangle+\langle(-\infty,0),[0,\infty)\rangle=\langle L,U\rangle$.

If $\langle L,U\rangle=\langle\varnothing,\mathbb{Q}\rangle$ then, by item e of Lemma 49, $\langle L,U\rangle+\langle(-\infty,0),[0,\infty)\rangle=\langle L,U\rangle$.

If $\langle L,U\rangle=\langle\mathbb{Q},\varnothing\rangle$ then, by item i of Lemma 49, $\langle L,U\rangle+\langle(-\infty,0),[0,\infty)\rangle=\langle L,U\rangle$.

Therefore $\langle(-\infty,0),[0,\infty)\rangle$ is the identity element of the set of Trans-Dedekind cuts with respect to $+$. ■

Lemma 50. *It follows that:*

a) $-\langle\varnothing,\varnothing\rangle=\langle\varnothing,\varnothing\rangle$*.*
b) $-\langle\varnothing,\mathbb{Q}\rangle=\langle\mathbb{Q},\varnothing\rangle$*.*
c) $-\langle\mathbb{Q},\varnothing\rangle=\langle\varnothing,\mathbb{Q}\rangle$*.*

Proof:

a) It follows that $-\langle\varnothing,\varnothing\rangle=\langle L_3,U_3\rangle$ where

$$\begin{aligned}U_3''&=\{x\in\mathbb{Q};\ x=\inf\varnothing\}=\varnothing,\\ U_3&=-(\varnothing\cup\varnothing)=-\varnothing=\varnothing,\\ L_3&=-(\varnothing\backslash\varnothing)=-\varnothing=\varnothing.\end{aligned}$$

Thus $-\langle\varnothing,\varnothing\rangle=\langle L_3,U_3\rangle=\langle\varnothing,\varnothing\rangle$.

b) It follows that $-\langle\varnothing,\mathbb{Q}\rangle=\langle L_3,U_3\rangle$ where

$$\begin{aligned}U_3''&=\{x\in\mathbb{Q};\ x=\inf\mathbb{Q}\}=\varnothing,\\ U_3&=-(\varnothing\cup\varnothing)=-\varnothing=\varnothing,\\ L_3&=-(\mathbb{Q}\backslash\varnothing)=-\mathbb{Q}=\mathbb{Q},\end{aligned}$$

Thus $-\langle\varnothing,\mathbb{Q}\rangle=\langle L_3,U_3\rangle=\langle\mathbb{Q},\varnothing\rangle$.

c) It follows that $-\langle\mathbb{Q},\varnothing\rangle=\langle L_3,U_3\rangle$ where

$$\begin{aligned}U_3''&=\{x\in\mathbb{Q};\ x=\inf\varnothing\}=\varnothing,\\ U_3&=-(\mathbb{Q}\cup\varnothing)=-\mathbb{Q}=\mathbb{Q},\\ L_3&=-(\varnothing\backslash\varnothing)=-\varnothing=\varnothing,\end{aligned}$$

Thus $-\langle\mathbb{Q},\varnothing\rangle=\langle L_3,U_3\rangle=\langle\varnothing,\mathbb{Q}\rangle$.

■

Proof of Theorem XVII: Let $\langle L,U\rangle$ be a Trans-Dedekind cut. By Lemma 48, either $\langle L,U\rangle$ is a Dedekind cut or $\langle L,U\rangle = \langle\varnothing,\varnothing\rangle$ or $\langle L,U\rangle = \langle\varnothing,\mathbb{Q}\rangle$ or $\langle L,U\rangle = \langle\mathbb{Q},\varnothing\rangle$.

If $\langle L,U\rangle$ is a Dedekind cut then, by the Definition XVI and Theorem 26, $-\langle L,U\rangle$ is the additive inverse of $\langle L,U\rangle$. Hence, by Theorem 17, $-\langle L,U\rangle$ is a Dedekind cut. Thus, by Theorem XI, $-\langle L,U\rangle$ is a Trans-Dedekind cut.

If $\langle L,U\rangle = \langle\varnothing,\varnothing\rangle$ then, by item a of Lemma 50, $-\langle L,U\rangle$ is a Trans-Dedekind cut.

If $\langle L,U\rangle = \langle\varnothing,\mathbb{Q}\rangle$ then, by item b of Lemma 50, $-\langle L,U\rangle$ is a Trans-Dedekind cut.

If $\langle L,U\rangle = \langle\mathbb{Q},\varnothing\rangle$ then, by item c of Lemma 50, $-\langle L,U\rangle$ is a Trans-Dedekind cut. ■

Proof of Theorem XVIII: Let $\langle L,U\rangle$ be a Trans-Dedekind cut.

Suppose $\langle L,U\rangle < 0$. By item b of Lemma 47, $\langle L,U\rangle \neq \langle\varnothing,\varnothing\rangle$. By item d of Lemma 47, $\langle L,U\rangle \neq \langle\mathbb{Q},\varnothing\rangle$. Thus, by Lemma 48, either $\langle L,U\rangle$ is a Dedekind cut or $\langle L,U\rangle = \langle\varnothing,\mathbb{Q}\rangle$. If $\langle L,U\rangle$ is a Dedekind cut then, by the Definition XVI and Theorem 26, $-\langle L,U\rangle$ is the additive inverse of $\langle L,U\rangle$. Hence, by Theorem 24, $-\langle L,U\rangle$ is greater than 0 with respect to the Dedekind order relation. Thus, since the Trans-Dedekind order relation (Definition VII) is identical to the Dedekind order relation (Definition 7) it follows that $-\langle L,U\rangle$ is greater than 0 with respect to the Trans-Dedekind order relation, that is, $-\langle L,U\rangle > 0$. If $\langle L,U\rangle = \langle\varnothing,\mathbb{Q}\rangle$ then, by item b of Lemma 50 and item d of Lemma 47, $-\langle L,U\rangle = -\langle\varnothing,\mathbb{Q}\rangle = \langle\mathbb{Q},\varnothing\rangle > 0$.

Suppose $-\langle L,U\rangle > 0$. In a similar way we prove that $\langle L,U\rangle < 0$. ■

Lemma 51. *It follows that:*

a) $\langle\varnothing,\varnothing\rangle \times \langle L,U\rangle = \langle\varnothing,\varnothing\rangle$ *for all Trans-Dedekind cut* $\langle L,U\rangle$.
b) $\langle\mathbb{Q},\varnothing\rangle \times \langle(-\infty,0),[0,\infty)\rangle = \langle\varnothing,\varnothing\rangle$.
c) $\langle\mathbb{Q},\varnothing\rangle \times \langle\varnothing,\varnothing\rangle = \langle\varnothing,\varnothing\rangle$.
d) $\langle\mathbb{Q},\varnothing\rangle \times \langle L,U\rangle = \langle\mathbb{Q},\varnothing\rangle$ *for all Trans-Dedekind cut* $\langle L,U\rangle$ *such that* $\langle L,U\rangle > 0$.
e) $\langle\mathbb{Q},\varnothing\rangle \times \langle L,U\rangle = \langle\varnothing,\mathbb{Q}\rangle$ *for all Trans-Dedekind cut* $\langle L,U\rangle$ *such that* $\langle L,U\rangle < 0$.
f) $\langle\varnothing,\mathbb{Q}\rangle \times \langle(-\infty,0),[0,\infty)\rangle = \langle\varnothing,\varnothing\rangle$.
g) $\langle\varnothing,\mathbb{Q}\rangle \times \langle\varnothing,\varnothing\rangle = \langle\varnothing,\varnothing\rangle$.
h) $\langle\varnothing,\mathbb{Q}\rangle \times \langle L,U\rangle = \langle\varnothing,\mathbb{Q}\rangle$ *for all Trans-Dedekind cut* $\langle L,U\rangle$ *such that* $\langle L,U\rangle > 0$.
i) $\langle\varnothing,\mathbb{Q}\rangle \times \langle L,U\rangle = \langle\mathbb{Q},\varnothing\rangle$ *for all Trans-Dedekind cut* $\langle L,U\rangle$ *such that* $\langle L,U\rangle < 0$.

Proof:

a) Let $\langle L,U\rangle$ be a Trans-Dedekind cut. It follows that $\langle\varnothing,\varnothing\rangle \times \langle L,U\rangle = \langle L_3,U_3\rangle$ where

$$\begin{aligned}
U_3' &= (\varnothing\cap\mathbb{Q}^+)\cdot(U\cap\mathbb{Q}^+) = \varnothing\cdot(U\cap\mathbb{Q}^+) = \varnothing,\\
U_3'' &= \{x\in\mathbb{Q};\ x = \inf\varnothing\} = \varnothing,\\
U_3 &= \varnothing\cup\varnothing = \varnothing,\\
L_3' &= (\varnothing\cap\mathbb{Q}^+)\cdot(L\cap\mathbb{Q}^+) = \varnothing\cdot(L\cap\mathbb{Q}^+) = \varnothing,\\
L_3'' &= -(\varnothing\cup\varnothing)\cup\varnothing = -\varnothing\cup\varnothing = \varnothing\cup\varnothing = \varnothing\\
L_3 &= \mathcal{I}(\varnothing) = \varnothing.
\end{aligned}$$

Thus $\langle\varnothing,\varnothing\rangle \times \langle L,U\rangle = \langle L_3,U_3\rangle = \langle\varnothing,\varnothing\rangle$.

b) It follows that $\langle\mathbb{Q},\varnothing\rangle \times \langle(-\infty,0),[0,\infty)\rangle = \langle L_3,U_3\rangle$ where

$$\begin{aligned}
U_3' &= (\varnothing\cap\mathbb{Q}^+)\cdot([0,\infty)\cap\mathbb{Q}^+) = \varnothing\cdot([0,\infty)\cap\mathbb{Q}^+) = \varnothing,\\
U_3'' &= \{x\in\mathbb{Q};\ x = \inf\varnothing\} = \varnothing,\\
U_3 &= \varnothing\cup\varnothing = \varnothing,\\
L_3' &= (\mathbb{Q}\cap\mathbb{Q}^+)\cdot((-\infty,0)\cap\mathbb{Q}^+) = (\mathbb{Q}\cap\mathbb{Q}^+)\cdot\varnothing = \varnothing,\\
L_3'' &= -(\varnothing\cup\varnothing)\cup\varnothing = -\varnothing\cup\varnothing = \varnothing\cup\varnothing = \varnothing\\
L_3 &= \mathcal{I}(\varnothing) = \varnothing.
\end{aligned}$$

Thus $\langle\mathbb{Q},\varnothing\rangle \times \langle(-\infty,0),[0,\infty)\rangle = \langle L_3,U_3\rangle = \langle\varnothing,\varnothing\rangle$.

c) It follows that $\langle\mathbb{Q},\varnothing\rangle \times \langle\varnothing,\varnothing\rangle = \langle L_3,U_3\rangle$ where

$$\begin{aligned}
U_3' &= (\varnothing\cap\mathbb{Q}^+)\cdot(\varnothing\cap\mathbb{Q}^+) = \varnothing\cdot\varnothing = \varnothing,\\
U_3'' &= \{x\in\mathbb{Q};\ x = \inf\varnothing\} = \varnothing,\\
U_3 &= \varnothing\cup\varnothing = \varnothing,\\
L_3' &= (\mathbb{Q}\cap\mathbb{Q}^+)\cdot(\varnothing\cap\mathbb{Q}^+) = (\mathbb{Q}\cap\mathbb{Q}^+)\cdot\varnothing = \varnothing,\\
L_3'' &= -(\varnothing\cup\varnothing)\cup\varnothing = -\varnothing\cup\varnothing = \varnothing\cup\varnothing = \varnothing\\
L_3 &= \mathcal{I}(\varnothing) = \varnothing.
\end{aligned}$$

Thus $\langle\mathbb{Q},\varnothing\rangle \times \langle\varnothing,\varnothing\rangle = \langle L_3,U_3\rangle = \langle\varnothing,\varnothing\rangle$.

d) Let $\langle L,U\rangle$ be a Trans-Dedekind cut such that $\langle L,U\rangle > 0$. By Lemma 40, there is $r\in L$ such that $r\in\mathbb{Q}^+$ whence $L\cap\mathbb{Q}^+ \neq \varnothing$. Thus $\langle\mathbb{Q},\varnothing\rangle\times\langle L,U\rangle = \langle L_3,U_3\rangle$ where

$$\begin{aligned}
U_3' &= (\varnothing\cap\mathbb{Q}^+)\cdot(U_2\cap\mathbb{Q}^+) = \varnothing\cdot(U_2\cap\mathbb{Q}^+) = \varnothing,\\
U_3'' &= \{x\in\mathbb{Q};\ x = \inf\varnothing\} = \varnothing,\\
U_3 &= \varnothing\cup\varnothing = \varnothing,\\
L_3' &= (\mathbb{Q}\cap\mathbb{Q}^+)\cdot(L\cap\mathbb{Q}^+) = \mathbb{Q}^+\cdot(L\cap\mathbb{Q}^+) = \mathbb{Q}^+,\\
L_3'' &= -(\mathbb{Q}^+\cup\varnothing)\cup\mathbb{Q}^+ = -\mathbb{Q}^+\cup\mathbb{Q}^+ = \mathbb{Q}^-\cup\mathbb{Q}^+ = \mathbb{Q}\backslash\{0\}\\
L_3 &= \mathcal{I}(\mathbb{Q}\backslash\{0\}) = \mathbb{Q},
\end{aligned}$$

Thus $\langle\mathbb{Q},\varnothing\rangle \times \langle L,U\rangle = \langle L_3,U_3\rangle = \langle\mathbb{Q},\varnothing\rangle$.

e) Let $\langle L,U\rangle$ be a Trans-Dedekind cut such that $\langle L,U\rangle < 0$. It follows that $-\langle L,U\rangle > 0$ and $\langle\mathbb{Q},\varnothing\rangle \times \langle L,U\rangle = -(\langle\mathbb{Q},\varnothing\rangle \times (-\langle L,U\rangle))$. Thus

$$\begin{aligned}
\langle\mathbb{Q},\varnothing\rangle \times \langle L,U\rangle &= -(\langle\mathbb{Q},\varnothing\rangle \times (-\langle L,U\rangle))\\
&= -\langle\mathbb{Q},\varnothing\rangle\\
&= \langle\varnothing,\mathbb{Q}\rangle.
\end{aligned}$$

f) By item b of Lemma 50, $-\langle\varnothing,\mathbb{Q}\rangle = \langle\mathbb{Q},\varnothing\rangle$. It follows that $\langle\varnothing,\mathbb{Q}\rangle \times \langle(-\infty,0),[0,\infty)\rangle = -((-\langle\varnothing,\mathbb{Q}\rangle) \times \langle(-\infty,0),[0,\infty)\rangle)$. Thus

$$\begin{aligned}
\langle\varnothing,\mathbb{Q}\rangle\times\langle(-\infty,0),[0,\infty)\rangle &= -((-\langle\varnothing,\mathbb{Q}\rangle)\times\langle(-\infty,0),[0,\infty)\rangle)\\
&= -(\langle\mathbb{Q},\varnothing\rangle \times \langle(-\infty,0),[0,\infty)\rangle)\\
&= -\langle\varnothing,\varnothing\rangle\\
&= \langle\varnothing,\varnothing\rangle.
\end{aligned}$$

g) By item b of Lemma 50, $-\langle\varnothing,\mathbb{Q}\rangle = \langle\mathbb{Q},\varnothing\rangle$. It follows that $\langle\varnothing,\mathbb{Q}\rangle \times \langle\varnothing,\varnothing\rangle = -((-\langle\varnothing,\mathbb{Q}\rangle) \times \langle\varnothing,\varnothing\rangle)$. Thus

$$\begin{aligned}
\langle\varnothing,\mathbb{Q}\rangle \times \langle\varnothing,\varnothing\rangle &= -((-\langle\varnothing,\mathbb{Q}\rangle) \times \langle\varnothing,\varnothing\rangle)\\
&= -(\langle\mathbb{Q},\varnothing\rangle \times \langle\varnothing,\varnothing\rangle)\\
&= -\langle\varnothing,\varnothing\rangle\\
&= \langle\varnothing,\varnothing\rangle.
\end{aligned}$$

h) Let $\langle L, U\rangle$ be a Trans-Dedekind cut such that $\langle L, U\rangle > 0$. By item b of Lemma 50, $-\langle\varnothing, \mathbb{Q}\rangle = \langle\mathbb{Q}, \varnothing\rangle$. It follows that $\langle\varnothing, \mathbb{Q}\rangle \times \langle L, U\rangle = -((-\langle\varnothing, \mathbb{Q}\rangle) \times \langle L, U\rangle)$. Thus

$$\begin{aligned}\langle\varnothing, \mathbb{Q}\rangle \times \langle L, U\rangle &= -((-\langle\varnothing, \mathbb{Q}\rangle) \times \langle L, U\rangle)\\ &= -(\langle\mathbb{Q}, \varnothing\rangle \times \langle L, U\rangle)\\ &= -\langle\mathbb{Q}, \varnothing\rangle\\ &= \langle\varnothing, \mathbb{Q}\rangle.\end{aligned}$$

i) Let $\langle L, U\rangle$ be a Trans-Dedekind cut such that $\langle L, U\rangle < 0$. It follows that $-\langle L, U\rangle > 0$ and $\langle\varnothing, \mathbb{Q}\rangle \times \langle L, U\rangle = (-\langle\varnothing, \mathbb{Q}\rangle) \times (-\langle L, U\rangle)$. Thus

$$\begin{aligned}\langle\varnothing, \mathbb{Q}\rangle \times \langle L, U\rangle &= (-\langle\varnothing, \mathbb{Q}\rangle) \times (-\langle L, U\rangle)\\ &= \langle\mathbb{Q}, \varnothing\rangle \times (-\langle L, U\rangle)\\ &= \langle\mathbb{Q}, \varnothing\rangle.\end{aligned}$$

■

Proof of Theorem XX: Let $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ be Trans-Dedekind cuts. By Lemma 48, either $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ are both Dedekind cuts or $\langle L_1, U_1\rangle = \langle\varnothing, \varnothing\rangle$ or $\langle L_1, U_1\rangle = \langle\varnothing, \mathbb{Q}\rangle$ or $\langle L_1, U_1\rangle = \langle\mathbb{Q}, \varnothing\rangle$.

If $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ are both Dedekind cuts then, by Theorem 20, the Dedekind product between $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ is a Dedekind cut. By Theorem 27, the Trans-Dedekind product between $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$, $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle$, is the Dedekind product between $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$. Hence $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle$ is a Dedekind cut. Thus, by Theorem XI, $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle$ is a Trans-Dedekind cut.

If $\langle L_1, U_1\rangle = \langle\varnothing, \varnothing\rangle$ then, by item a of Lemma 51, $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle$ is a Trans-Dedekind cut.

If $\langle L_1, U_1\rangle = \langle\varnothing, \mathbb{Q}\rangle$ then, by items f, g, i and h of Lemma 51, $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle$ is a Trans-Dedekind cut.

If $\langle L_1, U_1\rangle = \langle\mathbb{Q}, \varnothing\rangle$ then, by items b, c, e and d of Lemma 51, $\langle L_1, U_1\rangle \times \langle L_2, U_2\rangle$ is a Trans-Dedekind cut. ■

Proof of Theorem XXI: Let $\langle L, U\rangle$ be a Trans-Dedekind cut. By Lemma 48, either $\langle L, U\rangle$ is a Dedekind cut or $\langle L, U\rangle = \langle\varnothing, \varnothing\rangle$ or $\langle L, U\rangle = \langle\varnothing, \mathbb{Q}\rangle$ or $\langle L, U\rangle = \langle\mathbb{Q}, \varnothing\rangle$.

If $\langle L, U\rangle$ is a Dedekind cut then, by Theorem 21, $\langle L, U\rangle \times \langle(-\infty, 1), [1, \infty)\rangle = \langle L, U\rangle$.

If $\langle L, U\rangle = \langle\varnothing, \varnothing\rangle$ then, by item a of Lemma 51, $\langle L, U\rangle \times \langle(-\infty, 1), [1, \infty)\rangle = \langle L, U\rangle$.

If $\langle L, U\rangle = \langle\varnothing, \mathbb{Q}\rangle$ then, by item h of Lemma 51, $\langle L, U\rangle \times \langle(-\infty, 1), [1, \infty)\rangle = \langle L, U\rangle$.

If $\langle L, U\rangle = \langle\mathbb{Q}, \varnothing\rangle$ then, by item d of Lemma 51, $\langle L, U\rangle \times \langle(-\infty, 1), [1, \infty)\rangle = \langle L, U\rangle$.

Therefore $\langle(-\infty, 1), [1, \infty)\rangle$ is the identity element of the set of all Trans-Dedekind cuts with respect to $\times$. ■

Lemma 52. *It follows that:*

a) $\langle(-\infty, 0), [0, \infty)\rangle^{-1} = \langle\mathbb{Q}, \varnothing\rangle$.
b) $\langle\varnothing, \varnothing\rangle^{-1} = \langle\varnothing, \varnothing\rangle$.
c) $\langle\mathbb{Q}, \varnothing\rangle^{-1} = \langle(-\infty, 0), [0, \infty)\rangle$.
d) $\langle\varnothing, \mathbb{Q}\rangle^{-1} = \langle(-\infty, 0), [0, \infty)\rangle$.

Proof:

a) Let $L_2 = (-\infty, 0)$ and $U_2 = [0, \infty)$. It follows that $\langle L_2, U_2\rangle^{-1} = \langle L_3, U_3\rangle$ where

$$\begin{aligned}U_3' &= (L_2 \cap \mathbb{Q}^+)^{-1} = ((-\infty, 0) \cap \mathbb{Q}^+)^{-1} = \varnothing,\\ U_3'' &= \{x \in \mathbb{Q};\ x = \inf U_3'\} = \varnothing,\\ U_3 &= U_3'' \cup U_3' = \varnothing,\\ L_3' &= (U_2 \cap \mathbb{Q}^+)^{-1} = ([0, \infty) \cap \mathbb{Q}^+)^{-1} = \mathbb{Q}^+,\\ L_3'' &= -(L_3' \cup U_3') \cup L_3' = -(\mathbb{Q}^+ \cup \varnothing) \cup \mathbb{Q}^+ = \mathbb{Q}\backslash\{0\},\\ L_3 &= \mathcal{I}(L_3'') = \mathcal{I}(\mathbb{Q}\backslash\{0\}) = \mathbb{Q}.\end{aligned}$$

Thus $\langle(-\infty, 0), [0, \infty)\rangle^{-1} = \langle L_2, U_2\rangle^{-1} = \langle L_3, U_3\rangle = \langle\mathbb{Q}, \varnothing\rangle$.

b) Let $L_2 = \varnothing$ and $U_2 = \varnothing$. It follows that $\langle L_2, U_2\rangle^{-1} = \langle L_3, U_3\rangle$ where

$$\begin{aligned}U_3' &= (L_2 \cap \mathbb{Q}^+)^{-1} = (\varnothing \cap \mathbb{Q}^+)^{-1} = \varnothing,\\ U_3'' &= \{x \in \mathbb{Q};\ x = \inf U_3'\} = \varnothing,\\ U_3 &= U_3'' \cup U_3' = \varnothing,\\ L_3' &= (U_2 \cap \mathbb{Q}^+)^{-1} = (\varnothing \cap \mathbb{Q}^+)^{-1} = \varnothing,\\ L_3'' &= -(L_3' \cup U_3') \cup L_3' = -(\varnothing \cup \varnothing) \cup \varnothing = \varnothing,\\ L_3 &= \mathcal{I}(L_3'') = \mathcal{I}(\varnothing) = \varnothing.\end{aligned}$$

Thus $\langle\varnothing, \varnothing\rangle^{-1} = \langle L_2, U_2\rangle^{-1} = \langle L_3, U_3\rangle = \langle\varnothing, \varnothing\rangle$.

c) Let $L_2 = \mathbb{Q}$ and $U_2 = \varnothing$. It follows that $\langle L_2, U_2\rangle^{-1} = \langle L_3, U_3\rangle$ where

$$\begin{aligned}U_3' &= (L_2 \cap \mathbb{Q}^+)^{-1} = (\mathbb{Q} \cap \mathbb{Q}^+)^{-1} = \mathbb{Q}^+,\\ U_3'' &= \{x \in \mathbb{Q};\ x = \inf U_3'\} = \{x \in \mathbb{Q};\ x = \inf \mathbb{Q}^+\} = \{0\},\\ U_3 &= U_3'' \cup U_3' = \{0\} \cup \mathbb{Q}^+ = [0, \infty),\\ L_3' &= (U_2 \cap \mathbb{Q}^+)^{-1} = (\varnothing \cap \mathbb{Q}^+)^{-1} = \varnothing,\\ L_3'' &= -(L_3' \cup U_3') \cup L_3' = -(\varnothing \cup \mathbb{Q}^+) \cup \varnothing = \mathbb{Q}^-,\\ L_3 &= \mathcal{I}(L_3'') = \mathcal{I}(\mathbb{Q}^-) = \mathbb{Q}^- = (-\infty, 0).\end{aligned}$$

Thus $\langle\mathbb{Q}, \varnothing\rangle^{-1} = \langle L_2, U_2\rangle^{-1} = \langle L_3, U_3\rangle = \langle(-\infty, 0), [0, \infty)\rangle$.

d) By item c of Lemma 47, $\langle\varnothing, \mathbb{Q}\rangle < \langle(-\infty, 0), [0, \infty)\rangle = 0$. Hence $\langle\varnothing, \mathbb{Q}\rangle^{-1} = -((-\langle\varnothing, \mathbb{Q}\rangle)^{-1})$. Thus

$$\begin{aligned}\langle\varnothing, \mathbb{Q}\rangle^{-1} &= -((-\langle\varnothing, \mathbb{Q}\rangle)^{-1})\\ &= -(\langle\mathbb{Q}, \varnothing\rangle^{-1})\\ &= -\langle(-\infty, 0), [0, \infty)\rangle\\ &= \langle(-\infty, 0), [0, \infty)\rangle.\end{aligned}$$

■

Proof of Theorem XXIII: Let $\langle L, U\rangle$ be a Trans-Dedekind cut. By Lemma 48, either $\langle L, U\rangle$ is a Dedekind cut or $\langle L, U\rangle = \langle\varnothing, \varnothing\rangle$ or $\langle L, U\rangle = \langle\varnothing, \mathbb{Q}\rangle$ or $\langle L, U\rangle = \langle\mathbb{Q}, \varnothing\rangle$.

If $\langle L, U\rangle$ is a Dedekind cut and $\langle L, U\rangle \neq \langle(-\infty, 0), [0, \infty)\rangle$ then, by the Definitions XXII and Theorem 28, $\langle L, U\rangle^{-1}$ is the multiplicative inverse of $\langle L, U\rangle$. Hence, by Theorem 23, $\langle L, U\rangle^{-1}$ is a Dedekind cut. Thus, by Theorem XI, $\langle L, U\rangle^{-1}$ is a Trans-Dedekind cut.

If $\langle L, U\rangle = \langle(-\infty, 0), [0, \infty)\rangle$ then, by item a of Lemma 52, $\langle L, U\rangle^{-1}$ is a Trans-Dedekind cut.

If $\langle L, U\rangle = \langle\varnothing, \varnothing\rangle$ then, by item b of Lemma 52, $\langle L, U\rangle^{-1}$ is a Trans-Dedekind cut.

If $\langle L, U\rangle = \langle \varnothing, \mathbb{Q}\rangle$ then, by item d of Lemma 52, $\langle L, U\rangle^{-1}$ is a Trans-Dedekind cut.

If $\langle L, U\rangle = \langle \mathbb{Q}, \varnothing\rangle$ then, by item c of Lemma 52, $\langle L, U\rangle^{-1}$ is a Trans-Dedekind cut. ■

Proof of Theorem 29: The result follows from theorems XIV, XVII, XX and XXIII. ■

Proof of Theorem 30:

a) The result is immediate since the order relation from Definition VII is identical to the order relation from Definition 7.
b) The result is immediate since the addition from Definition XIII is identical to the addition from Definition 13.
c) The result follows from Definition XVI and Theorem 26.
d) The result is immediate since the subtraction from Definition XVI is identical to the subtraction from Definition 16.
e) The result follows from Definition XIX and Theorem 27.
f) The result follows from Definition XXII and Theorem 28.
g) The result is immediate since the division from Definition XXII is identical to the division from Definition 22.

■

Proof of Theorem 32: The results follows from Lemma 48 and Theorem 24 and the Definition 31. ■

Proof of Theorem 33:

a) The results follows from the Definition 31 and item a of Lemma 47.
b) The results follows from the Definition 31 and item b of Lemma 47.
c) The results follows from the Definition 31 and Theorem 24 and item c of Lemma 47.
d) The results follows from the Definition 31 and Theorem 24 and item d of Lemma 47.
e) The results follows from the Definition 31 and item a of Lemma 50.
f) The results follows from the Definition 31 and item b of Lemma 50.
g) The results follows from the Definition 31 and item c of Lemma 50.
h) The results follows from the Definition 31 and item a of Lemma 52.
i) The results follows from the Definition 31 and item b of Lemma 52.
j) The results follows from the Definition 31 and item d of Lemma 52.
k) The results follows from the Definition 31 and item c of Lemma 52.
l) The results follows from the Definition 31 and item a of Lemma 49.
m) The results follows from the Definition 31 and item f of Lemma 49.
n) The results follows from the Definition 31 and item g of Lemma 49.
o) The results follows from the Definition 31 and item h of Lemma 49.
p) The results follows from the Definition 31 and Theorem 24 and item i of Lemma 49.
q) The results follows from the Definition 31 and item b of Lemma 49.
r) The results follows from the Definition 31 and item c of Lemma 49.
s) The results follows from the Definition 31 and item d of Lemma 49.
t) The results follows from the Definition 31 and Theorem 24 and item e of Lemma 49.
u) The results follows from the Definition 31 and item a of Lemma 51.
v) The results follows from the Definition 31 and item b of Lemma 51.
w) The results follows from the Definition 31 and item c of Lemma 51.
x) The results follows from the Definition 31 and item e of Lemma 51.
y) The results follows from the Definition 31 and item d of Lemma 51. $x \in \mathbb{R}^T$ such that $x > 0$.
z) The results follows from the Definition 31 and item f of Lemma 51.
α) The results follows from the Definition 31 and item g of Lemma 51.
β) The results follows from the Definition 31 and item i of Lemma 51.
γ) The results follows from the Definition 31 and item h of Lemma 51.

■

Proof of Theorem 34: Let $x, y \in \mathbb{R}$ where $x > 0$ and $y < 0$. It follows that:

a) $x \div 0 = x \times 0^{-1} = x \times \infty = \infty$,
b) $y \div 0 = y \times 0^{-1} = y \times \infty = -\infty$,
c) $0 \div 0 = 0 \times 0^{-1} = 0 \times \infty = \Phi$.

■

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