Adaptive Prescribed Performance Control of Uncertain Nonlinear Systems with Input Saturations

Nannan Zhao, Xinyu Ouyang, Libing Wu, and Yilin Ma

Abstract—**This paper is concerned with the problem of prescribed performance control (PPC) for a class of strict-feedback nonlinear systems, in which uncertain nonlinearity and uncertain disturbance exist in the considered nonlinear system at the same time, and the input control signal of the system is constrained by saturation nonlinearity. The adaptive neural network is used to approximate all unknown nonlinearity of the closed-loop system. And a new class of error transformation functions is proposed for the first time, which is superior to other similar documents in that the output error can always be constrained by the prescribed band only according to the properties of the designed error transformation function without any additional conditions. The simulation results finally verify the effectiveness of the proposed algorithm.**

Index Terms—**Adaptive neural control, backstepping design, nonsymmetric input saturation, prescribed performance control (PPC).**

I. Introduction

In recent years, the control problem of uncertain nonlinear system based on backstepping method has been widely concerned, and some important research results have been obtained [1]–[8]. However, due to the existence of parameter uncertainty, dead-zone nonlinearity [9], saturation nonlinearity and unknown external disturbance, the performance of the controller system may be poor or even unstable. To overcome it, for instance, the problem of adaptive tracking control for a class of nonlinear systems with parameter uncertainty and bounded external disturbance was studied in [10], where two kinds of actuator nonlinearity were considered respectively, i.e. symmetrical dead-zone and Bouc-Wen hysteresis. For non-differentiable saturation nonlinearity, Wang *et al.* [11] introduced the smooth nonlinear function to approach it, so that the adaptive fuzzy controller was constructed based on the mean-value theorem and backstepping technology. An adaptive output-feedback fuzzy tracking controller was constructed in [12] for nonlinear systems with unknown external disturbances and uncertain nonlinearities. Further, Long and Zhao [13] extended the results of [12] to the switched system with disturbance and constructed an adaptive neural network tracking controller with output-feedback, which ensured that all signals in the switched signal closed-loop system remained semi-globally uniformly ultimately bounded (SGUUB) and the output error converged to a very small neighborhood of the origin.

From the above literatures, it is not difficult to find that neural networks (NNs) [14]–[16] or fuzzy logic systems (FLSs) [17]–[21], as powerful universal approximation tools, have become one of the essential tools to study the control problem of nonlinear system. Correspondingly, in the past few years, many significant results of adaptive control algorithm based on approximate structure have been successfully applied to practical nonlinear systems [22]–[27]. Based on NNs and mean value theorem, in [23], the problem of output-feedback adaptive control for a class of stochastic non-affine nonlinear systems with actuator dead-zone input was studied, and the output tracking error of the system can converge to a small neighborhood in the sense of quartic mean. Wang *et al.* applied the fuzzy logic system to model the unknown function in [26], and proposed the finite time semi-global practical stability criterion to study the finite time tracking problem of the nonlinear pure-feedback system for the first time .

In addition, in many industrial control systems, the constraints can not be violated in the process of operation, otherwise the control system will be seriously damaged. Therefore, the prescribed performance control (PPC) [28]–[32] of nonlinear system has become an important research topic in recent years. Its main idea is to ensure the transient performance of the system on the premise of ensuring the stability of the system, so as to prevent performance degradation and system damage. Recently, some control strategies based on prescribed performance have been proposed [33]–[35]. Concretely, for the stochastic nonlinear system with non-structured uncertainty, unknown dead-zone and unknown control direction, in [33], the adaptive fuzzy output-feedback prescribed performance controller in the form of non-strict feedback was presented, and it was ensured that all signals of the closed-loop system were SGUUB in the sense of probability. Li and Tong [34] proposed a new decentralized control scheme with output-feedback and prescribed performance with the help of adaptive backstepping technology, assuming that the nonlinear correlation term and nonlinear function of uncertain switched non-strict feedback interconnected nonlinear system were unknown, and the switch signal was

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unknown and arbitrary. In [35], an adaptive controller with prescribed constraint was constructed by using the fuzzy approximation method to ensure the transient and steady-state performance of the nonlinear system, in which a new constraint variable was introduced for error transformation.

Meanwhile, some studies have been concerned with the combination of adaptive PPC and input saturation compensation [36]–[40]. For instance, in [36], the PPC of the teleoperation system with input saturation was addressed. With the same PPC scheme, the observer-based PPC problem for a class of input saturated stochastic nonlinear systems was studied in [38], and an adaptive output-feedback PPC scheme for a class of switched nonlinear systems with input saturation was proposed in [39]. Furthermore, Wang et al. [40] summarized the shortcomings of traditional prescribed performance function, and constructed a new prescribed performance function to circumvent high frequency chattering in control input.

Motivated by the above literatures, for a class of uncertain strict-feedback nonlinear systems, an adaptive prescribed performance tracking controller based on RBF neural network (RBFNN) is proposed. Different from the traditional PPC idea introduced in [36]–[40], a new class of error transformation functions is constructed in this paper, which is a more direct form than that in [36]–[40]. Although the authors in [35] also directly constructed an error transformation function, the constraint of output error can only be realized only if the designed parameters were reasonably selected to satisfy the conditions of Theorem 1 of [35]. The error transformation function proposed in this paper relaxes the condition similar to that in [35]. In addition, the considered strict-feedback nonlinear system not only has parameter uncertainty, but also has unknown disturbance and input signal saturation nonlinearity. On this premise, an adaptive prescribed performance controller based on RBFNN is realized. As far as we know, there are only a limited number of literatures on such issues.

The rest of this paper is as follows: In Section II, the considered nonlinear system model of strict-feedback is given. And a class of new error transformation functions proposed in this note and some necessary assumptions and lemmas are also introduced in Section II. The adaptive tracking controller based on neural network and its stability analysis are presented in Section III. The simulation is carried out in Section IV, and Section V is the conclusion.

II. System descriptions and preliminaries

A. System descriptions

Choose a class of single-input and single-output (SISO) uncertain nonlinear systems with unknown input saturation and unknown external disturbances as

$$\begin{aligned} &\dot{x}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1} + \lambda_i(t),\ i = 1, \cdots, n-1, \\ &\dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u + \lambda_n(t),\ u = S(v), \\ &\ y = x_1, \end{aligned} \tag{1}$$

where $\bar{x}_i = [x_1, x_2, \cdots, x_i]^T \in \mathbb{R}^i$ $(i = 1, 2, \cdots, n)$ are the system state vectors. $y \in \mathbb{R}$ is the system output. $f_i(\cdot) : \mathbb{R}^i \to \mathbb{R}$ and $g_i(\cdot) : \mathbb{R}^i \to \mathbb{R}$, $i = 1, 2, \cdots n$, are unknown smooth functions. $\lambda_i(t)$, $i = 1, 2, \cdots n$ are unknown external disturbances satisfying $|\lambda_i(t)| \le \bar{\lambda}_i$ with $\bar{\lambda}_i$ being positive constants. $u(t) \in \mathbb{R}$ represents the actuator input of system with input saturation, which is described as:

$$u = S(v) = \begin{cases} u_{max}, & v \ge u_{max}, \\ u_{min}, & v \le u_{min}, \\ v, & otherwise, \end{cases} \tag{2}$$

where $u_{min} < 0$ and $u_{max} > 0$ are unknown constants. Similar to the description in [41], the saturation nonlinearity model $S(\cdot)$ can be represented by the sum of a piecewise smooth continuous function

$$W(v) = \begin{cases} \frac{u_{max} * (e^{v/u_{max}} - e^{-v/u_{max}})}{(e^{v/u_{max}} + e^{-v/u_{max}})}, & v \ge 0, \\ \frac{u_{min} * (e^{v/u_{min}} - e^{-v/u_{min}})}{(e^{v/u_{min}} + e^{-v/u_{min}})}, & v < 0. \end{cases} \tag{3}$$

and a function $w_0(v)$ with an unknown positive constant bound $\bar{w}_0$, such that

$$S(v) = W(v) + w_0(v),\ |w_0(v)| \le \bar{w}_0. \tag{4}$$

Applying the mean-value theorem to $W(v)$ yields

$$W(v) - W(v_0) = W_\delta(v - v_0), \tag{5}$$

where $W_\delta(v) = \frac{\partial W}{\partial v}|_{v=\delta v + (1-\delta)v_0}$ with $0 < \delta < 1$. Choosing $v_0 = 0$ and substituting (2), (4) and (5) into (1) results in

$$\begin{aligned} &\dot{x}_i = f_i + g_i x_{i+1} + \lambda_i,\ i = 1, 2, \cdots, n-1, \\ &\dot{x}_n = f_n + g_n W_\delta v + g_n w_0 + \lambda_n, \\ &y = x_1. \end{aligned} \tag{6}$$

In what follows, to simplify writing, denote $f_i(\cdot)$ and $g_i(\cdot)$ as f_i and g_i, respectively. Also, the time t in some functions is omitted, for example, λ_i for $\lambda_i(t)$, v for $v(t)$, u for $u(t)$ and so on.

B. Prescribed performance control

The state errors are defined as follows:

$$\begin{aligned} &z_1 = x_1 - y_r, \\ &z_i = x_i - \alpha_{i-1},\ i = 2, 3, \cdots, n \end{aligned} \tag{7}$$

where α_{i-1} stands for the virtual control law in step $i-1$, y_r is the desired trajectory, and z_1 denotes the tracking error. In order to achieve PPC, the boundary function $\rho_1(t)$ corresponding to error z_1 is defined as

$$\rho_1(t) = (\rho_0 - \rho_\infty)e^{-\varsigma t} + \rho_\infty \tag{8}$$

with $\rho_0 > \rho_\infty$, where ρ_0, ρ_∞ and ς are predefined positive constants, representing the initial value, the upper bound of steady-state error and the convergence speed of exponential function, respectively.

Traditionally, in order to stabilize (6) and guarantee that z_1 is constrained in the prescribed performance function, the constrained behavior $|z_1(t)| < \rho_1(t)$ is usually transformed into the following equivalent unconstrained behavior [36]–[40]:

$$z_1(t) = \rho_1(t) K_1(\zeta_1(t)) \tag{9}$$

where $K_1(\zeta_1(t)) = \frac{e^{\zeta_1} - e^{-\zeta_1}}{e^{\zeta_1} + e^{-\zeta_1}}$. It is easy to know that the error transformation ζ_1 and its derivation with respect to t are

$$\zeta_1(t) = K_1^{-1}\left(\frac{z_1(t)}{\rho_1(t)}\right) = \frac{1}{2}\ln(\frac{K_1 + 1}{1 - K_1}) \tag{10}$$

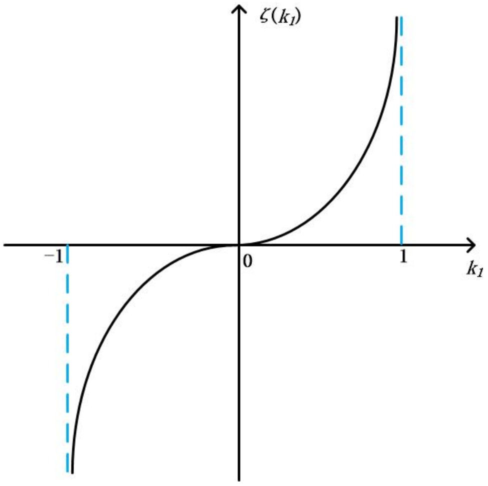

Fig. 1. Illustration of attribute of $\zeta_1(k_1)$.

and

$$\dot{\zeta}_1(t) = \Lambda_1\left(\dot{z}_1 - \frac{z_1\dot{\rho}_1}{\rho_1}dt\right) \tag{11}$$

with $\Lambda_1 = \frac{1}{2\rho_1}(\frac{1}{K_1+1} - \frac{1}{K_1-1})$.

Different from the traditional method, this paper aims to introduce a class of auxiliary functions which can help to realize PPC. First, we introduce a class of smooth functions $\zeta_1(k_1)$, which satisfies

$$\lim_{k_1\to -1^+}\zeta_1(k_1) = -\infty, \quad \lim_{k_1\to 1^-}\zeta_1(k_1) = \infty.$$

The properties of $\zeta_1(k_1)$ are illustrated in Fig. 1. For example, the candidate functions like Fig. 1 can be

$$\zeta_1(k_1) = \tanh^{-1}(k_1) = \frac{1}{2}\ln(\frac{1+k_1}{1-k_1})$$

and

$$\zeta_1(k_1) = \tan(\frac{\pi}{2}k_1).$$

It is not difficult to deduce that $|k_1| < 1$ holds as long as $\zeta_1(k_1)$ is bounded. As mentioned earlier, to fulfill the control objective, the overshoot of $z_1(t)$ should be specified to less than $\rho_1(t)$ for the initial condition $|z_1(0)| < |\rho_1(0)|$, which implies that the inequality $|\frac{z_1(t)}{\rho_1(t)}| < 1$ should be hold for $t > 0$. Naturally, let $k_1 = \frac{z_1(t)}{\rho_1(t)}$. The subsequent work of this study is to construct the appropriate objective function $V_n(\zeta_1^2, \cdot)$. If V_n is proved to be bounded, ζ_1^2 will be bounded accordingly, and $|\frac{z_1(t)}{\rho_1(t)}| < 1$ holds. In this paper, the logarithmic error transformation function is selected as an example, so the following error transformation function is designed:

$$\zeta_1(t) = \ln(\frac{\rho_1 + z_1}{\rho_1 - z_1}). \tag{12}$$

Taking the time derivation of $\zeta_1(t)$ yields

$$\dot{\zeta}_1(t) = 2\Gamma_1(\dot{z}_1 - \frac{\dot{\rho}_1 z_1}{\rho_1}), \tag{13}$$

where $\Gamma_1 = \rho_1/(\rho_1^2 - z_1^2)$, and $\dot{\zeta}_1(t)$ will be used later.

Remark 1: For the subsequent design of Lyapunov function, it is reasonable that $\zeta_1(t)$ appears on $V_1(t)$ in the form of square. On the one hand, tracking error $z_1(t)$ is expected to be as small as possible, on the other hand, $z_1(t)$ is expected to be as far away from the preset band $\rho_1(t)$ as possible.

Remark 2: For uncertain nonlinear system, an adaptive fuzzy tracking PPC scheme was proposed in [35], but the considered nonlinear system did not involve the input saturation and external disturbance. Different from [35], the PPC method proposed in this paper is for the system (1) with input saturated and external disturbance. In addition, although the error transformation function ξ_1 proposed in [35] can achieve PPC, (51) in [35] has to be satisfied. In this note, a new error transformation function ζ_1 is designed to replace ξ_1 of [35]. It will be proved in the following Theorem 1 that PPC can be realized without the condition like (51) in [35]. Accordingly, the proposed control strategy is more relaxed than that of in [35].

C. Control objectives and preliminaries

The control objectives of this technical note include that:

1) For a class of strict-feedback uncertain nonlinear systems with external disturbance and input saturation, the adaptive RBFNN PPC strategy is constructed, which can track the desired trajectory $y_r(t)$.

2) The tracking error $z_1(t) = y - y_r$ is constrained by $\rho_1(t)$ with the initial condition $|z_1(0)| < \rho_1(0)$.

3) In the sense that all signals in the closed-loop system are bounded, the system is semi-globally stable.

In order to meet the above control objectives, the following lemmas and assumptions are introduced and will be very important in subsequent analysis.

Assumption 1: For the functions $g_i(\bar{x}_i)$ and $W_\delta(v)$, there exist unknown positive constants $\underline{g}$, g_M and $\underline{W}$, such that

$$\begin{aligned} &0 < \underline{g} \le |g_i| \le g_M,\ i = 1, 2, \cdots, n \\ &0 < \underline{W} \le W_\delta \le 1. \end{aligned} \tag{14}$$

Further, denote $g_m = \min\{\underline{g}, \underline{g}\underline{W}\}$, it can assume that

$$g_m \le |g_i| \le g_M,\ g_m \le |g_n W_\delta| \le g_M. \tag{15}$$

Assumption 2: The reference signal $y_r(t)$ and its ith order time derivatives $y_r^{(i)}(t)$, $n \in \{1, 2, \cdots, n\}$, are continuous and bounded.

Lemma 1: [23] If there exists an input vector $\Upsilon \subset \mathbb{R}^n$ defined in a compact set Ω_Υ, a the constant weight vector $\Theta = [\Theta_1, \Theta_2, \ldots, \Theta_l]^T$ and a basis function vector $\Psi(\Upsilon) = [\Psi_1(\Upsilon), \Psi_2(\Upsilon), \cdots, \Psi_l(\Upsilon)]^T$, then radial basic function neural networks (RBFNNs) with l neurons is defined as follows:

$$f_{RBFNN}(\Upsilon) = \Theta^T\Psi(\Upsilon) \tag{16}$$

Generally, the Gauss function is chosen as the basis function defined as:

$$\Psi_i(\Upsilon) = \exp\left[\frac{-(\Upsilon - \xi_i)^T(\Upsilon - \xi_i)}{\omega}\right], \quad i = 1, 2, \cdots, l,$$

where ω is the width of Gaussian function and $\xi_i = [\xi_{i1}, \xi_{i2}, \cdots, \xi_{in}]^T$ denotes the center vector of the basic function.

From the description in [14]–[16], for a smooth function $F(\Upsilon)$ defined in a compact set Ω_Υ, RBFNN can be used to approximate it with a desired level of accuracy such that

$$F(\Upsilon) = \Theta^T\Psi(\Upsilon) + \varepsilon(\Upsilon), \tag{17}$$

where $\varepsilon(\Upsilon)$ is the corresponding approximation error satisfied $|\varepsilon(\Upsilon)| \leq \bar{\varepsilon}$ with $\bar{\varepsilon} > 0$ being an unknown constant. Correspondingly, the optimal weight vector Θ^* can be given as

$$\Theta^* := \arg\min_{\Theta\in\mathbb{R}^l}\left\{\sup_{\Upsilon\in\Omega_\Upsilon}|F(\Upsilon)-\Theta^T\Psi(\Upsilon)|\right\}. \tag{18}$$

Remark 3: In fact, *Assumptions* 1 and 2 and *Lemma* 1 will play important roles in the subsequent stability analysis. Concretely, since the actual control signal and the virtual control signal can not be infinite from the perspective of practical engineering, *Assumption* 1 is reasonable. It follows from *Assumption* 2 that $y_r(t)$ and $y_r^{(i)}(t)$ are all bounded for $i = 1, 2, \cdots, n$. Also, the RBFNN is used to approximate the multidimensional functions due to the existence of uncertain nonlinearities and unknown external disturbances.

III. Main Result

A. Controller Design

In this subsection, the adaptive prescribed performance controller for a class of uncertain nonlinear systems is presented. The virtual controls α_i of the first $n-1$ steps and actual input v are constructed by

$$\begin{aligned}\alpha_1 &= -\frac{\zeta_1}{\Gamma_1}\left(\tau_1+\frac{1}{2}+\frac{1}{2c_1^2}\hat{\theta}\Psi_1^T(\Upsilon_1)\Psi_1(\Upsilon_1)\right),\\ \alpha_i &= -z_i\left(\tau_i+\frac{1}{2}+\frac{1}{2c_i^2}\hat{\theta}\Psi_i^T(\Upsilon_i)\Psi_i(\Upsilon_i)\right),\\ v &= -z_n\left(\tau_n+\frac{1}{2\beta^2}+\frac{1}{2c_n^2}\hat{\theta}\Psi_n^T(\Upsilon_n)\Psi_n(\Upsilon_n)\right),\\ & i = 2,3,\cdots,n-1,\end{aligned} \tag{19}$$

where τ_i, c_i and β, $i = 1, 2, \cdots, n$, are positive design parameters, and $\Upsilon_1 = [x_1, y_r, \dot{y}_r, \rho_1, \dot{\rho}_1]^T \in \mathbb{R}^5$, $\Upsilon_i = [\bar{x}_i^T, \hat{\theta}, \bar{y}_r^{(i)T}, \bar{\rho}_1^{(i)T}]^T \in \mathbb{R}^{(3i+3)}$ with $\bar{y}_r^{(i)} = [y_r, \dot{y}_r, \cdots, y_r^{(i)}]^T$, $\bar{\rho}_1^{(i)} = [\rho_1, \dot{\rho}_1, \cdots, \rho_1^{(i)}]^T$ for $i = 2, \cdots, n$. The unknown constant is defined as

$$\theta = \max_{1\leq i\leq n}\{\frac{1}{g_m}||\Theta_i||^2\} \tag{20}$$

where Θ_i will be specified later, and g_m is defined in (15). Denote $\hat{\theta}$ as the estimation of θ, which is updated by

$$\dot{\hat{\theta}}(t) = \frac{\gamma}{2c_1^2}\zeta_1^2\Psi_1^T\Psi_1+\sum_{i=2}^{n}\frac{\gamma}{2c_i^2}z_i^2\Psi_i^T\Psi_i-\sigma\hat{\theta} \tag{21}$$

with $\gamma > 0$ and $\sigma > 0$ being design parameters. $\tilde{\theta} = \theta - \hat{\theta}$ denotes the estimated error.

In next, the design process of the standard backstepping scheme is divided into n steps.

Step 1: A positive definite Lyapunov function is

$$V_1 = \frac{1}{4}\zeta_1^2+\frac{g_m}{2\gamma}\tilde{\theta}^2 \tag{22}$$

By $\dot{z}_1 = f_1 + g_1x_2 + \lambda_1 - \dot{y}_r$, the time derivative of $V_1(t)$ can be given by

$$\dot{V}_1 = \zeta_1\Gamma_1(f_1+g_1x_2+\lambda_1-\dot{y}_r-\frac{\dot{\rho}_1 z_1}{\rho_1})-\frac{g_m}{\gamma}\tilde{\theta}\dot{\hat{\theta}} \tag{23}$$

Invoking completion of square gets

$$\zeta_1\Gamma_1\lambda_1 \leq \frac{\zeta_1^2\Gamma_1^2}{2}+\frac{\bar{\lambda}_1^2}{2} \tag{24}$$

Now, let $F_1(\Upsilon_1) = \Gamma_1(f_1 - \dot{y}_r - \frac{\dot{\rho}_1 z_1}{\rho_1} + \frac{\zeta_1\Gamma_1}{2}) + \frac{\zeta_1}{2}$, and then substituting (24) into (23) yields

$$\dot{V}_1 \leq \zeta_1\Gamma_1 g_1x_2+\zeta_1F_1-\frac{\zeta_1^2}{2}+\frac{\bar{\lambda}_1^2}{2}-\frac{g_m}{\gamma}\tilde{\theta}\dot{\hat{\theta}} \tag{25}$$

The RBFNN is used to approximate $F_1(\Upsilon_1)$ such that

$$F_1(\Upsilon_1) = \Theta_1^T\Psi_1(\Upsilon_1)+\varepsilon_1(\Upsilon_1),\ |\varepsilon_1(\Upsilon_1)| \leq \bar{\varepsilon}_1, \tag{26}$$

where $\varepsilon_1(\Upsilon_1)$ denotes the approximation error with $\bar{\varepsilon}_1 > 0$ being a bounded constant. Using the completion of square again, based on (20) and (26), one has

$$\begin{aligned}\zeta_1F_1(\Upsilon_1) &\leq \frac{1}{2c_1^2}\zeta_1^2||\Theta||^2\Psi_1^T\Psi_1+\frac{c_1^2}{2}+\frac{\zeta_1^2}{2}+\frac{\bar{\varepsilon}_1^2}{2}\\ &\leq \frac{g_m}{2c_1^2}\zeta_1^2\theta\Psi_1^T\Psi_1+\frac{c_1^2}{2}+\frac{\zeta_1^2}{2}+\frac{\bar{\varepsilon}_1^2}{2}.\end{aligned} \tag{27}$$

Next, through choosing the virtual control law α_1 in (19) and using the completion of square, the following inequalities can be obtained:

$$\begin{aligned}\zeta_1\Gamma_1 g_1 z_2 &\leq \frac{g_1\zeta_1^2}{2}+\frac{g_1\Gamma_1^2z_2^2}{2}\\ \zeta_1\Gamma_1 g_1\alpha_1 &\leq -\tau_1 g_m\zeta_1^2-\frac{g_1\zeta_1^2}{2}-\frac{g_m\zeta_1^2\hat{\theta}\Psi_1^T\Psi_1}{2c_1^2}\end{aligned} \tag{28}$$

Combining (27) and (28) with (25) produces

$$\begin{aligned}\dot{V}_1 \leq & -\eta_1\zeta_1^2+\frac{g_M\Gamma_1^2z_2^2}{2}+\Delta_1\\ &+\frac{g_m}{\gamma}\tilde{\theta}(\frac{\gamma\zeta_1^2\Psi_1^T\Psi_1}{2c_1^2}-\dot{\hat{\theta}}),\end{aligned} \tag{29}$$

where $\eta_1 = \tau_1 g_m$ and $\Delta_1 = \frac{c_1^2}{2} + \frac{\bar{\lambda}_1^2}{2} + \frac{\bar{\varepsilon}_1^2}{2}$. The term $\frac{g_M\Gamma_1^2z_2^2}{2}$ will be dealt with in Step 2.

Step 2: Consider the following Lyapunov function candidate:

$$V_2 = V_1+\frac{1}{2}z_2^2 \tag{30}$$

From $\dot{z}_2 = f_2 + g_2x_3 + \lambda_2 - \dot{\alpha}_1$ and $\dot{\alpha}_1 = \frac{\partial\alpha_1}{\partial x_1}(f_1 + g_1x_2 + \lambda_1) + \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} + \sum_{k=0}^{1}\frac{\partial\alpha_1}{\partial\rho_1^{(k)}}\rho_1^{(k+1)} + \sum_{k=0}^{1}\frac{\partial\alpha_1}{\partial y_r^{(k)}}y_r^{(k+1)}$, the time derivative of V_2 is

$$\begin{aligned}\dot{V}_2 =& \dot{V}_1+z_2\Big(f_2+g_2x_3+\lambda_2\\ &-\frac{\partial\alpha_1}{\partial x_1}(f_1+g_1x_2+\lambda_1)-\frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}\\ &-\sum_{k=0}^{1}\frac{\partial\alpha_1}{\partial\rho_1^{(k)}}\rho_1^{(k+1)}-\sum_{k=0}^{1}\frac{\partial\alpha_1}{\partial y_r^{(k)}}y_r^{(k+1)}\Big)\end{aligned} \tag{31}$$

For the terms $z_2\lambda_2$ and $z_2\frac{\partial\alpha_1}{\partial x_1}\lambda_1$, the following inequalities hold

$$\begin{cases} z_2\lambda_2 & \leq \frac{z_2^2}{2}+\frac{\bar{\lambda}_2^2}{2}\\ -z_2\frac{\partial\alpha_1}{\partial x_1}\lambda_1 & \leq \left(\frac{\partial\alpha_1}{\partial x_1}\right)^2 z_2^2+\frac{\bar{\lambda}_1^2}{4}\end{cases} \tag{32}$$

Denoting $F_2(\Upsilon_2) = \frac{g_M\Gamma_1^2z_2}{2} + f_2 + z_2 - \frac{\partial\alpha_1}{\partial x_1}(f_1 + g_1x_2) + z_2(\frac{\partial\alpha_1}{\partial x_1})^2 - \frac{\partial\alpha_1}{\partial\hat{\theta}}\frac{\gamma}{2c_1^2}\zeta_1^2\Psi_1^T\Psi_1 - \frac{\partial\alpha_1}{\partial\hat{\theta}}\frac{\gamma}{2c_2^2}z_2^2\Psi_2^T\Psi_2 + \frac{\partial\alpha_1}{\partial\hat{\theta}}\sigma\hat{\theta} -$

$\sum_{k=0}^{1}\frac{\partial\alpha_1}{\partial\rho_1^{(k)}}\rho_1^{(k+1)}-\sum_{k=0}^{1}\frac{\partial\alpha_1}{\partial y_r^{(k)}}y_r^{(k+1)}$, and substituting (32) into (31) result in

$$\begin{aligned}\dot{V}_2\leq&-\eta_1\zeta_1^2+\frac{g_m}{\gamma}\tilde{\theta}(\frac{\gamma\zeta_1^2\Psi_1^T\Psi_1}{2c_1^2}-\dot{\hat{\theta}})+\Delta_1\\&-\frac{\partial\alpha_1}{\partial\hat{\theta}}z_2\sum_{\ell=3}^{n}\frac{\gamma}{2c_\ell^2}z_\ell^2\Psi_\ell^T\Psi_\ell\\&+z_2(g_2z_3+g_2\alpha_2+F_2)-\frac{z_2^2}{2}+\frac{\bar{\lambda}_2^2}{2}+\frac{\bar{\lambda}_1^2}{4}\end{aligned}\tag{33}$$

Here, $F_2(\Upsilon_2)$ can be approximated by RBFNN $\Theta_2^T\Psi_2(\Upsilon_2)$ for a desired level of accuracy ε_2 such that

$$F_2(\Upsilon_2)=\Theta_2^T\Psi_2(\Upsilon_2)+\varepsilon_2(\Upsilon_2),\ |\varepsilon_2(\Upsilon_2)|\leq\bar{\varepsilon}_2,\tag{34}$$

with $\bar{\varepsilon}_2$ being a positive constant. Similar to Step 1, by using the completion of square, (19), (20) and Assumption 1, it is easy to obtain the following three inequalities

$$\begin{cases}z_2F_2&\leq\frac{g_mz_2^2\theta\Psi_2^T\Psi_2}{2c_2^2}+\frac{c_2^2}{2}+\frac{z_2^2}{2}+\frac{\bar{\varepsilon}_2^2}{2}\\g_2z_2z_3&\leq\frac{g_2z_2^2}{2}+\frac{g_2z_3^2}{2}\\g_2z_2\alpha_2&\leq-\tau_2g_mz_2^2-\frac{g_2z_2^2}{2}-\frac{g_mz_2^2\hat{\theta}\Psi_2^T\Psi_2}{2c_2^2}\end{cases}\tag{35}$$

Then, it follows from (33) and (35) that

$$\begin{aligned}\dot{V}_2\leq&-\eta_1\zeta_1^2-\eta_2z_2^2+\frac{g_Mz_3^2}{2}\\&+\frac{g_m}{\gamma}\tilde{\theta}(\frac{\gamma\zeta_1^2\Psi_1^T\Psi_1}{2c_1^2}+\frac{\gamma z_2^2\Psi_2^T\Psi_2}{2c_2^2}-\dot{\hat{\theta}})\\&-\frac{\partial\alpha_1}{\partial\hat{\theta}}z_2\sum_{\ell=3}^{n}\frac{\gamma}{2c_\ell^2}z_\ell^2\Psi_\ell^T\Psi_\ell+\Delta_1+\Delta_2\end{aligned}\tag{36}$$

where $\eta_2=\tau_2g_m$ and $\Delta_2=\frac{c_2^2}{2}+\frac{\bar{\lambda}_2^2}{2}+\frac{\bar{\varepsilon}_2^2}{2}+\frac{\bar{\lambda}_1^2}{4}$.

Step i ($i=3,\cdots,n-1$): Consider the following Lyapunov function:

$$V_i=V_{i-1}+\frac{1}{2}z_i^2\tag{37}$$

By combining $\dot{z}_i=f_i+g_ix_{i+1}+\lambda_i-\dot{\alpha}_{i-1}$, and $\dot{\alpha}_{i-1}=\sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}(f_k+g_kx_{k+1}+\lambda_k)+\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}+\sum_{k=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\rho_1^{(k)}}\rho_1^{(k+1)}+\sum_{k=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_r^{(k)}}y_r^{(k+1)}$, the time derivative of $V_i(t)$ is

$$\begin{aligned}\dot{V}_i=&\dot{V}_{i-1}+z_i\left(f_i+g_ix_{i+1}+\lambda_i\right.\\&-\sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}(f_k+g_kx_{k+1}+\lambda_k)-\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\\&\left.-\sum_{k=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\rho_1^{(k)}}\rho_1^{(k+1)}-\sum_{k=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_r^{(k)}}y_r^{(k+1)}\right)\end{aligned}\tag{38}$$

Recursively, it can be deduced that

$$\begin{aligned}\dot{V}_{i-1}\leq&-\eta_1\zeta_1^2-\sum_{k=2}^{i-1}\eta_kz_k^2+\sum_{k=1}^{i-1}\Delta_k+\frac{g_Mz_i^2}{2}\\&+\frac{g_m}{\gamma}\tilde{\theta}(\frac{\gamma\zeta_1^2\Psi_1^T\Psi_1}{2c_1^2}+\sum_{k=2}^{i-1}\frac{\gamma z_k^2\Psi_k^T\Psi_k}{2c_k^2}-\dot{\hat{\theta}})\\&-\sum_{k=1}^{i-2}\frac{\partial\alpha_k}{\partial\hat{\theta}}z_{k+1}\sum_{\ell=i}^{n}\frac{\gamma}{2c_\ell^2}z_\ell^2\Psi_\ell^T\Psi_\ell\end{aligned}\tag{39}$$

where $\Delta_i=\frac{c_i^2}{2}+\frac{\bar{\lambda}_i^2}{2}+\frac{\bar{\varepsilon}_i^2}{2}+\sum_{k=1}^{i-1}\sum_{\ell=1}^{k}\frac{\bar{\lambda}_\ell^2}{4}$. Combining (39), the following two inequalities

$$\begin{cases}z_i\lambda_i&\leq\frac{z_i^2}{2}+\frac{\bar{\lambda}_i^2}{2},\\-z_i\sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}\lambda_k&\leq z_i^2\sum_{k=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_k}\right)^2\\&\quad+\sum_{k=1}^{i-1}\frac{\bar{\lambda}_k^2}{4},\end{cases}\tag{40}$$

and $F_i(\Upsilon_i)=-\frac{\gamma}{2c_i^2}z_i\Psi_i^T\Psi_i\sum_{k=1}^{i-2}\frac{\partial\alpha_k}{\partial\hat{\theta}}z_{k+1}+\frac{g_Mz_i}{2}+f_i+z_i-\sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}(f_k+g_kx_{k+1})+z_i\sum_{k=1}^{i-1}(\frac{\partial\alpha_{i-1}}{\partial x_k})^2-\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\frac{\gamma}{2c_1^2}\zeta_1^2\Psi_1^T\Psi_1-\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\sum_{k=2}^{i}\frac{\gamma}{2c_k^2}z_k^2\Psi_k^T\Psi_k+\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\sigma\hat{\theta}-\sum_{k=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\rho_1^{(k)}}\rho_1^{(k+1)}-\sum_{k=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_r^{(k)}}y_r^{(k+1)}$ with (38) results in

$$\begin{aligned}\dot{V}_i\leq&-\eta_1\zeta_1^2-\sum_{k=2}^{i-1}\eta_kz_k^2+\sum_{k=1}^{i-1}\Delta_k\\&+\frac{g_m}{\gamma}\tilde{\theta}\left(\frac{\gamma\zeta_1^2\Psi_1^T\Psi_1}{2c_1^2}+\sum_{k=2}^{i-1}\frac{\gamma z_k^2\Psi_k^T\Psi_k}{2c_k^2}-\dot{\hat{\theta}}\right)\\&-\sum_{k=1}^{i-1}\frac{\partial\alpha_k}{\partial\hat{\theta}}z_{k+1}\sum_{\ell=i+1}^{n}\frac{\gamma}{2c_\ell^2}z_\ell^2\Psi_\ell^T\Psi_\ell\\&+g_iz_i(z_{i+1}+\alpha_i)+z_iF_i-\frac{z_i^2}{2}\\&+\frac{\bar{\lambda}_i^2}{2}+\sum_{k=1}^{i-1}\frac{\bar{\lambda}_k^2}{4}.\end{aligned}\tag{41}$$

Using RBFNN, for given positive constants $\bar{\varepsilon}_i$, we have

$$F_i(\Upsilon_i)=\Theta_i^T\Psi_i(\Upsilon_i)+\varepsilon_i(\Upsilon_i),\ |\varepsilon_i(\Upsilon_i)|\leq\bar{\varepsilon}_i\tag{42}$$

Next, substituting the following three inequalities

$$\begin{cases}z_iF_i&\leq\frac{g_mz_i^2\theta\Psi_i^T\Psi_i}{2c_i^2}+\frac{c_i^2}{2}+\frac{z_i^2}{2}+\frac{\bar{\varepsilon}_i^2}{2}\\g_iz_iz_{i+1}&\leq\frac{g_iz_i^2}{2}+\frac{g_iz_{i+1}^2}{2}\\g_iz_i\alpha_i&\leq-\tau_ig_mz_i^2-\frac{g_iz_i^2}{2}-\frac{g_mz_i^2\hat{\theta}\Psi_i^T\Psi_i}{2c_i^2}\end{cases}\tag{43}$$

into (41), it yields

$$\begin{aligned}\dot{V}_i\leq&-\eta_1\zeta_1^2-\sum_{k=2}^{i}\eta_kz_k^2+\sum_{k=1}^{i}\Delta_k+\frac{g_Mz_{i+1}^2}{2}\\&+\frac{g_m}{\gamma}\tilde{\theta}(\frac{\gamma\zeta_1^2\Psi_1^T\Psi_1}{2c_1^2}+\sum_{k=2}^{i}\frac{\gamma z_k^2\Psi_k^T\Psi_k}{2c_k^2}-\dot{\hat{\theta}})\\&-\sum_{k=1}^{i-1}\frac{\partial\alpha_k}{\partial\hat{\theta}}z_{k+1}\sum_{\ell=i+1}^{n}\frac{\gamma}{2c_\ell^2}z_\ell^2\Psi_\ell^T\Psi_\ell.\end{aligned}\tag{44}$$

where $\eta_i=\tau_ig_m$.

Remark 4: Inspired by [11], a general adaptive control law $\hat{\theta}$ is designed to be updated online. So, the weight vector Θ_i of each step does not need to be estimated, and the calculation amount is greatly reduced. However, the fuzzy logic system $\Theta_i^T\Psi_i(\Upsilon_i)$ can not be directly used to approximate the term $\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}$ in (38), because it contains the terms $z_{i+1},\cdots,z_n$. To overcome it, $\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}$ has to be divided into two parts: $\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}(\frac{\gamma}{2c_1^2}\zeta_1^2\Psi_1^T\Psi_1+\sum_{k=2}^{i}\frac{\gamma}{2c_k^2}z_k^2\Psi_k^T\Psi_k-\sigma\hat{\theta})$ and $\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}(\sum_{k=i+1}^{n}\frac{\gamma}{2c_k^2}z_k^2\Psi_k^T\Psi_k)$. Among them, the first part can be summed up in the packaged function $F_i(\Upsilon_i)$, and the second part will be dealt with in steps $i+1$ to n.

Step n: The actual controller $v(t)$ will be given in the last step. Now, choose the following the Lyapunov function:

$$V_n = V_{n-1} + \frac{1}{2}z_n^2 \tag{45}$$

and then taking the time derivative of $V_n(t)$ yields

$$\begin{aligned}\dot{V}_n \leq &\dot{V}_{n-1} + z_n \Big(f_n + g_n W_\delta v + g_n w_0 + \lambda_n \\ &- \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k}(f_k + g_k x_{k+1} + \lambda_k) - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} \\ &- \sum_{k=0}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \rho_1^{(k)}}\rho_1^{(k+1)} - \sum_{k=0}^{n-1}\frac{\partial \alpha_{n-1}}{\partial y_r^{(k)}} y_r^{(k+1)}\Big)\end{aligned} \tag{46}$$

Denote $F_n(\Upsilon_n) = -\frac{\gamma}{2c_n^2} z_n \Psi_n^T \Psi_n \sum_{k=1}^{n-2} \frac{\partial \alpha_k}{\partial \hat{\theta}} z_{k+1} + f_n + \frac{g_M z_n}{2} + z_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k}(f_k + g_k x_{k+1}) + z_n \sum_{k=1}^{n-1}\left(\frac{\partial \alpha_{n-1}}{\partial x_k}\right)^2 - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} - \sum_{k=0}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \rho_1^{(k)}}\rho_1^{(k+1)} - \sum_{k=0}^{n-1} \frac{\partial \alpha_{n-1}}{\partial y_r^{(k)}} y_r^{(k+1)}$, we have

$$\begin{aligned}\dot{V}_n \leq &-\eta_1 \zeta_1^2 - \sum_{k=2}^{n-1} \eta_k z_k^2 + \sum_{k=1}^{n-1}\Delta_k \\ &+ \frac{g_m}{\gamma}\tilde{\theta}\Big(\frac{\gamma \zeta_1^2 \Psi_1^T \Psi_1}{2c_1^2} + \sum_{k=2}^{n-1}\frac{\gamma z_k^2 \Psi_k^T \Psi_k}{2c_k^2} - \dot{\hat{\theta}}\Big) \\ &+ z_n(g_n W_\delta v + g_n w_0 + F_n) - \frac{z_n^2}{2} \\ &+ \frac{\bar{\lambda}_n^2}{2} + \sum_{k=1}^{n-1}\frac{\bar{\lambda}_k^2}{4}\end{aligned} \tag{47}$$

By using RBFNN, for a given positive constant $\bar{\varepsilon}_n$, we have

$$F_n(\Upsilon_n) = \Theta_n^T \Psi_n(\Upsilon_n) + \varepsilon_n(\Upsilon_n), \ |\varepsilon_n(\Upsilon_n)| \leq \bar{\varepsilon}_n \tag{48}$$

Based on the same process as Step i, one has

$$z_n F_n \leq \frac{g_m}{2c_n^2} z_n^2 \theta \Psi_n^T \Psi_n + \frac{c_n^2}{2} + \frac{z_n^2}{2} + \frac{\bar{\varepsilon}_n^2}{2} \tag{49}$$

According to *Assumption* 1 and the completion of square, it produces

$$z_n g_n w_0 \leq g_n\Big(\frac{\underline{W} z_n^2}{2\beta} + \frac{\beta \bar{w}_0^2}{2\underline{W}}\Big) \leq \frac{g_n \underline{W} z_n^2}{2\beta} + \frac{g_M \beta \bar{w}_0^2}{2\underline{W}} \tag{50}$$

Invoking the actual control signal v in (19) and Assumption 1, one has

$$\begin{aligned}z_n g_n W_\delta v &\leq -g_n \underline{W}\Big(\tau_n z_n^2 + \frac{z_n^2}{2\beta} + \frac{z_n^2 \hat{\theta} \Psi_n^T \Psi_n}{2c_n^2}\Big) \\ &\leq -\tau_n g_m z_n^2 - \frac{g_n \underline{W} z_n^2}{2\beta} - \frac{g_m z_n^2 \hat{\theta}\Psi_n^T \Psi_n}{2c_n^2}\end{aligned} \tag{51}$$

Considering (49)-(51), it follows from (47) that

$$\dot{V}_n \leq -\eta_1 \zeta_1^2 - \sum_{k=2}^{n} \eta_k z_k^2 + \sum_{k=1}^{n}\Delta_k + \frac{\sigma g_m}{\gamma}\tilde{\theta}\hat{\theta} + \frac{g_M \beta \bar{w}_0^2}{2\underline{W}} \tag{52}$$

where $\eta_n = \tau_n g_m$.

B. Stability analysis

In this subsection, the stability analysis will be completed by Theorem 1.

Theorem 1: Supposed that the unknown compound functions $F_i(\Upsilon_i)$ can be approximated by RBFNNs $\Theta_i^T \psi_i(\Upsilon_i)$ with a bounded error $\varepsilon_i(\Upsilon_i)$ for $i = 1, 2, \cdots, n$. Under *Assumptions* 1 and 2, and by using the designed prescribed performance tracking controller (19) and adaptive parameter update law (21), all signals in the closed-loop system can be guaranteed to be semi-globally bounded. And then, for the reference signal $y_r(t)$, the output tracking error $z_1(t)$ is constrained by $\rho_1(t)$ with the initial condition $|z_1(0)| < \rho_1(0)$ for all $t > 0$.

Proof: The proof is divided into two parts. In the first part, the semi-global stability of the closed-loop system is proved, and the second part is to prove that the inequality $|z_1(t)| < \rho_1(t)$ holds.

1) In order to stabilize system (1), the Lyapunov function is constructed as:

$$V_n(\chi(t)) = \frac{\zeta_1^2}{4} + \sum_{i=2}^{n}\frac{z_i^2}{2} + \frac{g_m}{2\gamma}\tilde{\theta}^2, \tag{53}$$

where $\chi(t) = [\zeta_1, z_2, \cdots, z_n, \tilde{\theta}]^T$. For the term $\tilde{\theta}\hat{\theta}$ in (52), the inequality $\tilde{\theta}\hat{\theta} \leq -\frac{\tilde{\theta}^2}{2} + \frac{\theta^2}{2}$ holds, and it follows from (52) and (53) that

$$\begin{aligned}\dot{V}_n(\chi(t)) \leq &-\eta_1 \zeta_1^2 - \sum_{k=2}^{n}\eta_k z_k^2 - \frac{\sigma g_m \tilde{\theta}^2}{2\gamma} \\ &+ \frac{\sigma g_m \theta^2}{2\gamma} + \sum_{k=1}^{n}\Delta_k + \frac{g_M \beta \bar{w}^2}{2\underline{W}} \\ \leq &-\mu_0 V_n(\chi(t)) + \vartheta_0,\end{aligned} \tag{54}$$

where $\mu_0 = \min\{4\eta_1, 2\eta_i, \sigma, \ i = 2, 3, \cdots, n\}$ and $\vartheta_0 = \frac{\sigma g_m \theta^2}{2\gamma} + \sum_{k=1}^{n}\Delta_k + \frac{g_M \beta \bar{w}^2}{2\underline{W}}$, which implies that

$$\begin{aligned}V_n(\chi(t)) &\leq e^{-\mu_0 t} V_n(0) + \frac{\vartheta_0}{\mu_0}(1 - e^{-\mu_0 t}) \\ &\leq e^{-\mu_0 t} V_n(0) + \frac{\vartheta_0}{\mu_0}.\end{aligned} \tag{55}$$

By (55), we can conclude that ζ_1, z_i, $i = 2, 3, \cdots, n$ and $\tilde{\theta}$ are bounded. $\hat{\theta}$ is bounded since θ is a constant. Further, by the definition of (7), it easy to deduce that x_i, $i = 1, 2, \cdots, n$ are bounded. Hence, all the signals in the closed-loop system (1), (19) and (21) are bounded.

2) From (53) and (55), it means that

$$\frac{\zeta_1^2}{4} = \frac{1}{4}\ln^2\left(\frac{1 + \frac{z_1}{\rho_1}}{1 - \frac{z_1}{\rho_1}}\right) \leq e^{-\mu_0 t} V_n(0) + \frac{\vartheta_0}{\mu_0}. \tag{56}$$

Based on the property of the function $(\tanh^{-1}(x))^2 = \frac{1}{4}\ln^2(\frac{1+x}{1-x})$, $|x| < 1$ holds as long as $\frac{1}{4}\ln^2(\frac{1+x}{1-x}) \leq \phi_0$ for any given constant $\phi_0 > 0$. So, it is not difficult to deduce that the item z_1/ρ_1 in (56) satisfies $|z_1/\rho_1| < 1$, which means that $|z_1(t)| < \rho_1(t)$ for all $t > 0$. □

IV. Simulation

In this section, the effectiveness of the proposed control method is verified by the following numerical simulation example. A third-order nonlinear system with unknown external disturbance, input saturation and unknown nonlinearity is considered as follows:

$$\begin{aligned} \dot{x}_1 &= (1+\cos(x_1))\,x_2 + 0.3x_1^2\sin(2x_1) + 0.2\sin(t) \\ \dot{x}_2 &= (1-\sin(x_1x_2))x_3 + 0.5\left(0.8x_2 + e^{-x_1^2}\right) \\ \dot{x}_3 &= 12u(t) + e^{-2x_1^2x_2^2}\cos(x_3) + \sin(2t) \\ y &= x_1. \end{aligned} \tag{57}$$

The tracking target of (57) is $y_r = 0.6\sin(t)$. Noting that $\lambda_1(t) = 0.2\sin(t)$, $\lambda_2(t) = 0$ and $\lambda_3(t) = \sin(2t)$ are the external disturbances of each subsystem respectively. $u(t)$ denotes the input control signal, which is constrained by the saturation nonlinear function $S(\cdot)$ with $u_{min} = -3$ and $u_{max} = 3$.

In order to stabilize (57), adaptive controller (19) and parameter update law (21) are used, and the design parameters are chosen as: $\tau_1 = 10$, $\tau_2 = 10$, $\tau_3 = 10$, $c_1 = 30$, $c_2 = 30$, $c_3 = 30$, $\gamma = 10$, $\sigma = 0.3$, and $\beta = 1$. Performance prescribed function is designed as $\rho_1 = (1-0.08)e^{-1.5t}+0.08$. Meanwhile, the initial conditions of simulation are $\theta(0) = 1$ and $[x_1(0), x_2(0), x_3(0)]^T = [0, 0, 0]^T$.

The simulation time is set to 30 seconds, and the simulation results of the proposed control scheme are shown in Figs. 2-7. It can be seen from Fig. 2 that the system output y can track the reference signal y_r well despite the existence of external disturbances. Other states x_2 and x_3 of the system are shown in Fig. 3. The output error z_1 and the performance prescribed function ρ_1 are shown in Fig. 4. And then, Figs. 5 and 6 show the actual control signal $v(t)$ of the system and the function $u(t)$ constrained by the saturation function $S(\cdot)$, respectively. Finally, the adaptive rate $\hat{\theta}$ is shown in Fig. 7. It is no hard to find from these simulation figures that PPC is implemented in the case of input saturation nonlinearity, unknown nonlinearity and unknown external disturbance.

V. Conclusion

An adaptive prescribed performance controller is designed for a class of uncertain strict-feedback nonlinear systems with input saturation constraints and external disturbances in this note. Different from other related researches, a new class of error transformation functions is proposed for the first time, which can be used to realize the performance constraint of output error and guarantee the transient and steady characteristics of nonlinear system. In addition, neural network is introduced to eliminate the influence of uncertain disturbance and uncertain nonlinearity on the system. For the future, this research achievement is ready to be used in the output-feedback control of switching system.

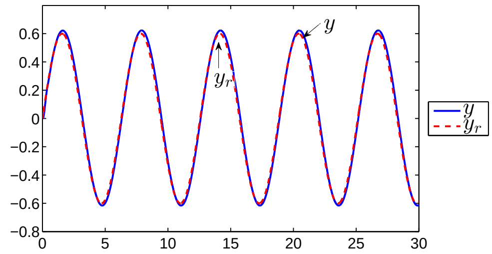

Fig. 2. System out $y(t)$ and reference signal $y_r(t)$.

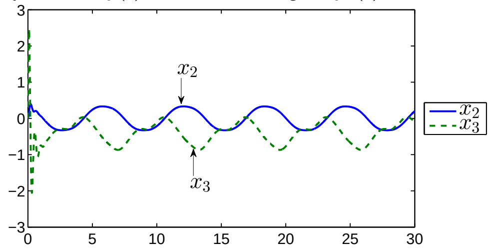

Fig. 3. System states x_2 and x_3.

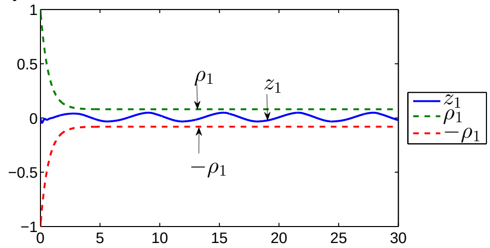

Fig. 4. State error z_1 under the prescribed performance constraint $\rho_1(t)$.

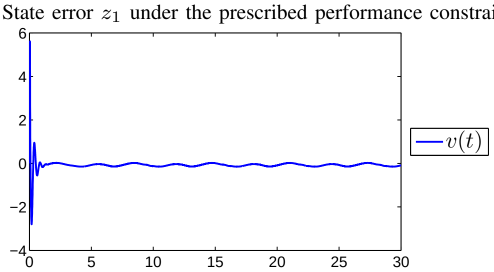

Fig. 5. Actual control input signal $v(t)$.

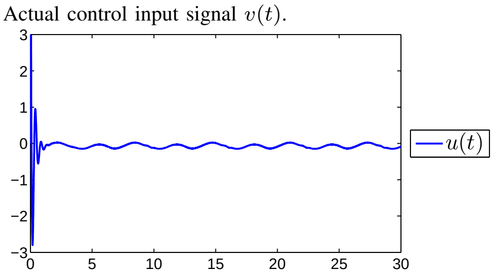

Fig. 6. Saturation output signal $u(t) = S(v(t))$.

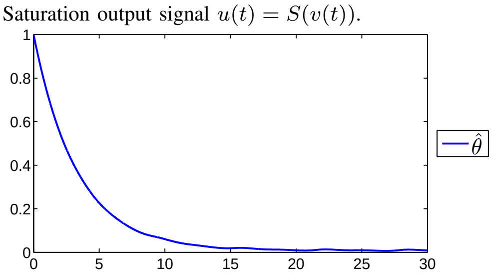

Fig. 7. Adaptive law $\hat{\theta}(t)$.

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