Permanence, Global Mittag-Leffler Stability and Global Asymptotic Periodic Solution for Multi-Species Predator-Prey Model Characterized by Caputo Fractional Differential Equations

Yuntao Liu, Shaobin Rao and Huizhen Qu

***Abstract*—In this paper, a kind of predator-prey system with fraction-order derivative scheme has been proposed and the issue on permanence, global Mittag-Leffler stability and global asymptotic periodic solution for the above system has been investigated. By utilizing comparison principles and fractional calculus theory, some new conditions are established to ensure the permanence, global Mittag-Leffler stability and global asymptotic periodic solution of the above systems. An example is given to demonstrate the effectiveness and feasibility of the proposed theoretical results.**

***Index Terms*—Fractional-order differential equation, Mittag-Leffler function, permanence, global Mittag-Leffler stability.**

I. Introduction

Since 1930s, Lotka and Volterra presented the standard Lotka-Volterra model that the predator and prey permanently oscillate for any positive initial conditions, the dynamic relationship between predator and prey is one of the main subjects in populations dynamics due to its universal existence and importance. The predator-prey systems have been extensively studied, see papers [1]–[7] and the cited is therein. Such as, Lin, Du and Lv [6] studied a delayed multispecies competition predator-prey dynamic system with Beddington-DeAngelis functional response. Some sufficient conditions which guarantee the existence of a positive periodic solution for the system are obtained by applying the Mawhin coincidence theory. Li and Zhao [5] proposed the stability of equilibria and periodic solutions of a non-autonomous Lotka-Volterra competition model with seasonal succession by using the stability analysis of equilibria and the theory of monotone dynamical system. Cai, Huang, Chen [7] were concerned with the existence, uniqueness and global asymptotic stability of positive periodic solutions to a delayed multispecies ecological competition-predator system with Holling-III functional response by using the continuation theorem of coincidence degree theory. By means of comparison theorem and Lyapunov functional, Lin, Du and Lv [6] studied the global asymptotic stability of almost periodic solution for the following system:

$$\begin{cases} \dot{x}_i(t) = x_i(t)\Big[b_i(t) - \sum_{k=1}^{n} a_{ik}(t)x_k(t) \\ \qquad - \sum_{k=1}^{m} \frac{c_{ik}(t)x_i(t)y_k(t)}{x_i^2(t)+f_{ik}(t)}\Big], \\ \dot{y}_j(t) = y_j(t)\Big[-r_j(t) + \sum_{k=1}^{n} \frac{d_{kj}(t)x_k^2(t)}{x_k^2(t)+f_{kj}(t)} \\ \qquad - \sum_{k=1}^{m} e_{jk}(t)y_k(t)\Big], \quad t \ge 0, \end{cases} \tag{1.1}$$

where $x_i(t)$, $y_j(t)$ denote the size of prey and predator population at time t; $b_i(t)$, $a_{il}(t)$, $c_{ik}(t)$, $r_j(t)$, $d_{lj}(t)$, $e_{jk}(t)(i,l = 1,2,\dots,n; j,k = 1,2,\dots,m)$ are all continuous positive almost periodic functions on $\mathbb{R}$ with the ecology meaning as follows: b_i is the prey population grows in the absence of predators, r_j is the predator population decays in the absence of preys, a_{il} is the prey population decays in the competition among the preys, e_{jk} is the predator population decays in the competition among the predator, c_{ik} is the prey is feed upon by the predators, d_{lj} is the coefficient of transformation from preys to predators, $i,l = 1,2,\dots,n$, $j,k = 1,2,\dots,m$.

It is known that, mathematical models, using integer order differential equations, as mentioned above, have been proven valuable in understanding the dynamical behaviors of biological systems. However, lots of systems, such as biological, physical, engineering, have long-range temporal memory and/or long-range space interactions [8], [9]. Modelling of such systems with fractional-order differential equations [10] have more advantages than classical integer-order ones, since fractional-order derivative can provide an excellent instrument for description of memory and hereditary properties of various materials and processes. It seems that, fractional-order differential equations are more consistent with real phenomena than integer-order models, which due to that fractional derivatives and integrals enable the description of the memory and hereditary properties inherent in various materials and process of which exists in most of the biological systems. Thus, in this paper, we combine the fractional-order in the system to describe the complex systems of predator-prey interactions with dynamical characteristics.

Manuscript received Dec 23, 2020; revised Feb 15, 2021. This work is supported by the Scientific Research Fund of Yunnan Provincial Education Department (2020J1223).

Yuntao Liu is a teacher of Oxbridge College, Kunming University of Science and Technology, Kunming 650106, China (e-mail: lythb@126.com).

Shaobin Rao is a teacher of Oxbridge College, Kunming University of Science and Technology, Kunming 650106, China (e-mail: 627894566@qq.com).

Huizhen Qu is a teacher of School of Basic Science, Guilin University of Technology at Nanning, Nanning 530001, China (corresponding author to provide e-mail: quhz0707@163.com).

Set $x_i(t) = \frac{1}{u_i(t)}$, $y_j(t) = \frac{1}{v_j(t)}$, from (1.1) it yields

$$\begin{cases} \dot{u}_i(t) = -b_i(t)u_i(t) + \sum_{k=1}^{n} a_{ik}(t)\frac{u_i(t)}{u_k(t)} \\ \qquad + \sum_{k=1}^{m} \frac{c_{ik}(t)u_i(t)}{v_k(t)[1+u_i^2(t)f_{ik}(t)]}, \\ \dot{v}_j(t) = r_j(t)v_j(t) - \sum_{k=1}^{n} \frac{d_{kj}(t)v_j(t)}{1+u_k^2(t)f_{kj}(t)} \\ \qquad + \sum_{k=1}^{m} e_{jk}(t)\frac{v_j(t)}{v_k(t)}, \quad t \geq 0. \end{cases} \tag{1.2}$$

Based on (1.2), in this paper we consider the fractional order system as follows:

$$\begin{cases} {}^cD_0^\alpha u_i(t) = -b_i(t)u_i(t) + \sum_{k=1}^{n} a_{ik}(t)\frac{u_i(t)}{u_k(t)} \\ \qquad + \sum_{k=1}^{m} \frac{c_{ik}(t)u_i(t)}{v_k(t)[1+u_i^2(t)f_{ik}(t)]}, \\ {}^cD_0^\alpha v_j(t) = r_j(t)v_j(t) - \sum_{k=1}^{n} \frac{d_{kj}(t)v_j(t)}{1+u_k^2(t)f_{kj}(t)} \\ \qquad + \sum_{k=1}^{m} e_{jk}(t)\frac{v_j(t)}{v_k(t)}, \quad t \geq 0, \end{cases} \tag{1.3}$$

where ${}^cD_0^\alpha$ is the Caputo derivative of order $\alpha \in (0,1]$, which is defined as

$${}^cD_0^\alpha f(t) = \frac{1}{\Gamma(\alpha-1)}\int_{t_0}^{t} \frac{\dot{f}(s)}{(t-s)^\alpha}\mathrm{d}s \ \ (0 < \alpha < 1),$$

where $f \in C^1([0,\infty),\mathbb{R})$ and $\Gamma(\cdot)$ is Euler's *Gamma* function. All coefficients in system (1.3) are nonnegative. Obviously, fractional order system (1.3) is transformed to integer order systems (1.1)-(1.2) in case $\alpha = 1$.

The remaining parts of this paper are organized as follows: In Section 2, we will present some definitions and lemmas which will be useful in the proven of the main results of this paper. In Section 3, by using comparison principle, sufficient conditions are built to ensure the permanence of system (1.3). In Section 4, we present that system (1.3) is globally Mittag-Leffler stable. In Section 5, an example is provided to expound the chief results of this paper.

Let $\mathbb{R}$ denote the set of real numbers. $\mathbb{R}^n$ denotes the n-dimensional real vector space and $C^n([t_0,\infty),\mathbb{R}^n)$ denotes the space consisting of n-order continuously differentiable functions. Let f is a continuous bounded function on $\mathbb{R}$ and set $f^M = \sup_{t\in\mathbb{R}} f(t)$, $f^m = \inf_{t\in\mathbb{R}} f(t)$.

II. Preliminaries

A. Caputo fractional derivative and Mittag-Leffler function

Definition II.1. ([11]) The Caputo derivative of fractional order α for $f \in C^n([t_0,\infty),\mathbb{R}^n)$ is defined by

$${}^cD_{t_0}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t} \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s$$

for $0 < n-1 < \alpha < n$.

In this paper, the Mittag-Leffler function is crucial to our main results, so we shall present the definitions and some important properties of two classical Mittag-Leffler functions. More information can be found in the book by Dzhrbashyan [27].

Definition II.2. ([11]) The definitions of two classical Mittag-Leffler functions:

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)},$$

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z \in \mathbb{C}, \quad \alpha > 0.$$

In particular,

$$(1)\ E_1(z) = e^z; \quad (2)\ E_{\alpha,1}(z) = E_\alpha(z):$$

$$(3)\ E_{1,2}(z) = \frac{e^z - 1}{z}.$$

Lemma II.1. ([11]) $\frac{\mathrm{d}}{\mathrm{d}z}\left[z^\alpha \mathbb{E}_{\alpha,\alpha+1}(\lambda z^\alpha)\right] = z^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda z^\alpha)$, where $\alpha, \lambda, z \in \mathbb{R}$.

Lemma II.2. ([11]) If $0 < \alpha < 2$ and $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$, then

$$\mathbb{E}_{\alpha,\beta}(z) = -\sum_{k=1}^{N} \frac{1}{\Gamma(\beta-\alpha k)}\frac{1}{z^k} + O(\frac{1}{z^{N+1}})$$

$$\text{with } |z| \to \infty, \quad \mu \leq |\arg(z)| \leq \pi, z, \beta \in \mathbb{C}.$$

B. Some important features of Mittag-Leffler function

Lemma II.3. ([12]) Assume $\lambda > 0$ and $\alpha \in (0,1)$, then $\mathbb{E}_\alpha(-\lambda t^\alpha) \geq 0$ and $\mathbb{E}_{\alpha,\alpha}(-\lambda t^\alpha) \geq 0$, $\forall t \geq 0$.

Lemma II.4. ([10], [13]) If $\lambda > 0$ and $\alpha \in (0,1)$, then

(1) $\lim_{t\to\infty} \mathbb{E}_\alpha(-\lambda t^\alpha) = 0$ and $\lim_{t\to\infty} t^\alpha \mathbb{E}_{\alpha,\alpha+1}(-\lambda t^\alpha) = \frac{1}{\lambda}$.

(2) $\lim_{t\to\infty} \int_0^a (t-s)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}[-\lambda(t-s)^\alpha]\mathrm{d}s = 0$.

Lemma II.5. ([10]) If $\lambda > 0$ and $\alpha \in (0,1)$, then $t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha) \leq \frac{1}{\lambda}$ for $t \geq 0$.

C. Some properties for fractional-order differential system

Lemma II.6. ([14], [15]) If $x(t) \in C^1([0,\infty),\mathbb{R})$, then ${}^cD_0^{\alpha^+}|x(t)| \leq \mathrm{sign}(x(t))\,{}^cD_0^\alpha x(t)$, $0 < \alpha < 1$, $\forall t \in [0,\infty)$, where ${}^cD_0^{\alpha^+}$ is defined as that in Ref. [14]. If ${}^cD_0^\alpha x(t) \leq -ax(t) + b$, where $x(t)$ is nonnegative and $a > 0$, then $x(t)$ is bounded.

Lemma II.7. ([16]) The initial value problem ${}^cD_0^\alpha y(x) = Ay(x) + f(x)$ with $y(x)|_{x=0} = y_0$ has a unique solution given by the formula

$$y(x) = y_0 E_\alpha(Ax^\alpha)$$

$$+ \int_0^x (x-t)^{\alpha-1}E_{\alpha,\alpha}(A(x-t)^\alpha)f(t)\mathrm{d}t$$

provided that the integral on the right-hand side converges.

Lemma II.8. ([16]) Considering fractional-order differential inequality: ${}^CD_0^\alpha u(t) \geq -au(t) + b$, $\forall t > 0$, $u(0) \geq u_0$, and the fractional-order differential systems: ${}^CD_0^\alpha v(t) = -av(t) + b$, $\forall t > 0$, $v(0) = u_0$. If $a > 0$, then $u(t) \geq v(t)$, $t \in [0,+\infty)$.

Lemma II.9. ([16]) Considering fractional-order differential inequality: ${}^CD_0^\alpha u(t) \leq -au(t) + b$, $\forall t > 0$, $u(0) \leq u_0$, and the fractional-order differential systems: ${}^CD_0^\alpha v(t) = -av(t) + b$, $\forall t > 0$, $v(0) = u_0$. If $a > 0$, then $u(t) \leq v(t)$, $t \in [0,+\infty)$.

III. Permanence

Let

$$u_i^m = \frac{a_{ii}^m}{b_i^M},$$

$$v_j^m = \frac{e_{jj}^m}{d_j^*},$$

$$b_i^* = b_i^m - \sum_{k=1,k\neq i}^{n} \frac{a_{ik}^M}{u_i^m},$$

$$d_j^* = \sum_{k=1}^{n} \frac{d_{kj}^M}{1+(u_i^m)^2 f_{kj}^m},$$

$$u_i^M = \sum_{k=1}^{m} \frac{c_{ik}^M}{v_k^m b_i^* u_i^m f_{ik}^m} + \frac{a_{ii}^M}{b_i^*},$$

$$r_j^* = \sum_{k=1}^{n} \frac{d_{kj}^m}{1+(u_k^M)^2 f_{kj}^M} - \sum_{k=1,k\neq j}^{m} \frac{e_{jk}^M}{v_k^m} - r_j^M,$$

$$v_j^M = \frac{e_{jj}^M}{r_j^*},$$

where $i=1,2,\ldots,n$, $j=1,2,\ldots,m$.

Assume that the conditions below hold.

(H_1) $b_i^*>0$, $i=1,2,\ldots,n$.

(H_2) $r_j^*>0$, $j=1,2,\ldots,m$.

A. Upper Bounds of Prey and Predator Population

Proposition III.1. Suppose that system (1.3) satisfies (H_1), then the prey and predator populations of system (1.3) possess upper bounds as follows:

$$\limsup_{t\to\infty} \frac{1}{u_i(t)} \le (u_i^m)^{-1},$$

$$\limsup_{t\to\infty} \frac{1}{v_j(t)} \le (v_j^m)^{-1},$$

where $i=1,2,\ldots,n$, $j=1,2,\ldots,m$.

Proof: It is obvious that $u_i(t)>0$, $v_j(t)>0$, $i=1,2,\ldots,n, j=1,2,\ldots,m$. From the first equation of system (1.3), it has

$$\begin{aligned} {}^cD_0^\alpha u_i(t) \ &\ge -b_i(t)u_i(t) + \sum_{k=1}^{n} a_{ik}(t)\frac{u_i(t)}{u_k(t)} \\ &\ge -b_i^M u_i(t) + a_{ii}^m, \quad i=1,2,\ldots,n. \end{aligned}$$

By Lemmas II.1, II.7 and II.8, it products

$$\begin{aligned} u_i(t) \ &\ge u_{i0}E_\alpha(-b_i^M t^\alpha) \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-b_i^M(t-s)^\alpha)a_{ii}^m \mathrm{d}t \\ &= u_{i0}E_\alpha(-b_i^M t^\alpha) \\ &\quad + a_{ii}^m t^\alpha E_{\alpha,\alpha+1}(-b_i^M t^\alpha), \quad i=1,2,\ldots,n. \end{aligned}$$

From Lemma II.4, it yields

$$\lim_{t\to+\infty} u_i(t) \ge \frac{a_{ii}^m}{b_i^M} := u_i^m, \quad i=1,2,\ldots,n. \tag{3.1}$$

Substituting (3.1) into the second equation of system (1.3), it gets

$$\begin{aligned} {}^cD_0^\alpha v_j(t) \ &\ge -\sum_{k=1}^{n} \frac{d_{kj}(t)v_j(t)}{1+u_k^2(t)f_{kj}(t)} + \sum_{k=1}^{m} e_{jk}(t)\frac{v_j(t)}{v_k(t)} \\ &\ge -\sum_{k=1}^{n} \frac{d_{kj}^M v_j(t)}{1+(u_i^m-\epsilon)^2 f_{kj}^m} \\ &\quad + e_{jj}^m, \quad j=1,2,\ldots,m. \end{aligned}$$

In the same way with (3.1), it obtains

$$\begin{aligned} v_j(t) \ &\ge v_{j0}E_\alpha(-d_j^*(\epsilon)t^\alpha) \\ &\quad + e_{jj}^m t^\alpha E_{\alpha,\alpha+1}(-d_j^*(\epsilon)t^\alpha), \quad j=1,2,\ldots,m. \end{aligned}$$

where $d_j^*(\epsilon) = \sum_{k=1}^{n} \frac{d_{kj}^M}{1+(u_i^m-\epsilon)^2 f_{kj}^m} > 0$. So

$$\begin{aligned} \lim_{t\to+\infty} v_j(t) \ &\ge \frac{e_{jj}^m}{d_j^*(\epsilon)} \\ &\ge \lim_{\epsilon\to 0} \frac{e_{jj}^m}{d_j^*(\epsilon)} \\ &= \frac{e_{jj}^m}{d_j^*} := v_j^m, \quad j=1,2,\ldots,m. \end{aligned} \tag{3.2}$$

The proof is finished. ■

B. Lower Bounds of Prey and Predator Population

Proposition III.2. Suppose that system (1.3) satisfies (H_2), then the prey and predator populations of system (1.3) possess lower bounds as follows:

$$\liminf_{t\to\infty} \frac{1}{u_i(t)} \ge (u_i^M)^{-1},$$

$$\liminf_{t\to\infty} \frac{1}{v_j(t)} \ge (v_j^M)^{-1},$$

where $i=1,2,\ldots,n$, $j=1,2,\ldots,m$.

Proof: By (3.1)-(3.2), there exists small enough $\epsilon>0$ and $t_0>0$ such that

$$u_i(t) \ge u_i^m - \epsilon > 0, \ v_j(t) \ge v_i^m - \epsilon > 0, \tag{3.3}$$

where $t\ge t_0$, $i=1,2,\ldots,n$, $j=1,2,\ldots,m$.

Substituting (3.3) into the first equation of system (1.3), it leads to

$$\begin{aligned} {}^cD_0^\alpha u_i(t) \ &\le -b_i^m u_i(t) + \sum_{k=1,k\neq i}^{n} a_{ik}^M \frac{u_i(t)}{u_i^m-\epsilon} \\ &\quad + a_{ii}^M + \sum_{k=1}^{m} \frac{c_{ik}^M}{v_k^m(u_i^m-\epsilon)f_{ik}^m} \\ &= -b_i^*(\epsilon)u_i(t) + a_{ii}^M \\ &\quad + \sum_{k=1}^{m} \frac{c_{ik}^M}{v_k^m(u_i^m-\epsilon)f_{ik}^m}, \quad t\ge t_0, \end{aligned} \tag{3.4}$$

where $b_i^*(\epsilon) = b_i^m - \sum_{k=1,k\neq i}^{n} \frac{a_{ik}^M}{u_i^m-\epsilon}$, $i=1,2,\ldots,n$. By Lemmas II.1, II.7 and II.9, it gets

$$u_i(t) \le u_{i0}E_\alpha(-b_i^*(\epsilon)t^\alpha)$$

$$+\Big[\sum_{k=1}^{m}\frac{c_{ik}^M}{v_k^m(u_i^m-\epsilon)f_{ik}^m}+a_{ii}^M\Big]t^{\alpha}E_{\alpha,\alpha+1}(-b_i^*(\epsilon)t^{\alpha}),\quad t\geq t_0,$$

which implies

$$\lim_{t\to+\infty}u_i(t)\leq\sum_{k=1}^{m}\frac{c_{ik}^M}{v_k^m b_i^*(\epsilon)(u_i^m-\epsilon)f_{ik}^m}+\frac{a_{ii}^M}{b_i^*(\epsilon)},\quad i=1,2,\ldots,n.$$

Letting $\epsilon\to 0$ in the above inequality, it has

$$\lim_{t\to+\infty}u_i(t)\leq\sum_{k=1}^{m}\frac{c_{ik}^M}{v_k^m b_i^* u_i^m f_{ik}^m}+\frac{a_{ii}^M}{b_i^*}:=u_i^M,\quad i=1,2,\ldots,n.\quad (3.5)$$

Substituting (3.5) into the second equation of system (1.3), it has

$$\begin{aligned}{}^cD_0^{\alpha}v_j(t)&\leq r_j^M v_j(t)-\sum_{k=1}^{n}\frac{d_{kj}^m v_j(t)}{1+(u_k^M+\epsilon)^2 f_{kj}^M}\\&\quad+\sum_{k=1,k\neq j}^{m}e_{jk}^M\frac{v_j(t)}{(v_k^m-\epsilon)}+e_{jj}^M\\&\leq -r_j^*(\epsilon)v_j(t)+e_{jj}^M,\quad t\geq t_0,\end{aligned}$$

where $r_j^*(\epsilon)=\sum_{k=1}^{n}\frac{d_{kj}^m}{1+(u_k^M+\epsilon)^2 f_{kj}^M}-\sum_{k=1,k\neq j}^{m}\frac{e_{jk}^M}{(v_k^m-\epsilon)}-r_j^M$, $j=1,2,\ldots,m$. Thus

$$\lim_{t\to+\infty}v_j(t)\leq\frac{e_{jj}^M}{r_j^*}:=v_j^M,\quad (3.6)$$

where $j=1,2,\ldots,m$. The proof is finished. ■

C. Permanence result

Theorem III.1. Suppose that system (1.3) satisfies (H_1)-(H_2), then system (1.3) is permanent. That is, the prey and predator populations of system (1.3) meet

$$(u_i^M)^{-1}\leq\liminf_{t\to\infty}\frac{1}{u_i(t)}\leq\limsup_{t\to\infty}\frac{1}{u_i(t)}\leq(u_i^m)^{-1},$$

$$(v_j^M)^{-1}\leq\liminf_{t\to\infty}\frac{1}{v_j(t)}\leq\limsup_{t\to\infty}\frac{1}{v_j(t)}\leq(v_j^m)^{-1},$$

where $i=1,2,\ldots,n$, $j=1,2,\ldots,m$.

IV. Global Mittag-Leffler Stability

Let

$$\lambda_1=\min_{1\leq i\leq n}\Big[b_i^m+\sum_{k=1}^{m}\frac{c_{ik}^m}{v_k^M}\frac{(u_i^m)^2 f_{ik}^m}{[1+(u_i^M)^2 f_{ik}^M]^2}-\sum_{k=1}^{m}\frac{c_{ik}^M}{v_k^m}\frac{1}{[1+(u_i^m)^2 f_{ik}^m]^2}-\sum_{k=1,k\neq i}^{n}\frac{a_{ik}^M}{u_i^m}\Big],$$

$$\lambda_2=\max_{1\leq i\leq n}\sum_{k=1,k\neq i}^{n}\frac{a_{ik}^M u_i^M}{(u_k^m)^2},$$

$$\lambda_3=\max_{1\leq i\leq n}\sum_{k=1}^{m}\frac{c_{ik}^M}{(v_k^m)^2}\frac{u_i^M}{[1+(u_i^m)^2 f_{ik}^m]^2}.$$

A. Two crucial inequalities

Let $(u_i(t),v_j(t))$ and $(\bar u_i(t),\bar v_j(t))$ be any two positive solutions of system (1.3), $z_i(t)=u_i(t)-\bar u_i(t)$, and $w_j(t)=v_j(t)-\bar v_j(t)$, $i=1,2,\ldots,n$, $j=1,2,\ldots,m$.

Proposition IV.1. There exists $t_1>0$ ensuring that

$${}^cD_0^{\alpha}|z_i(t)|\leq-\lambda_1|z_i(t)|+\lambda_2\max_{1\leq i\leq n}|z_i(t)|+\lambda_3\max_{1\leq j\leq m}|w_j(t)|,\quad (4.1)$$

where $t\geq t_1$, $i=1,2,\ldots,n$.

Proof: By Theorem III.1, there exists small enough $\epsilon>0$ and $t_1>0$ such that

$$0<u_i^m-\epsilon\leq u_i(t)\leq u_i^M+\epsilon,$$

$$0<v_i^m-\epsilon\leq v_j(t)\leq v_i^M+\epsilon,\quad t\geq t_1,$$

where $i=1,2,\ldots,n$, $j=1,2,\ldots,m$. Then it has

$$\begin{aligned}{}^cD_0^{\alpha}z_i(t)&=-b_i(t)z_i(t)+\sum_{k=1,k\neq i}^{n}a_{ik}(t)\Big[\frac{u_i(t)}{u_k(t)}-\frac{\bar u_i(t)}{\bar u_k(t)}\Big]\\&\quad+\sum_{k=1}^{m}c_{ik}(t)\Big[\frac{u_i(t)}{v_k(t)[1+u_i^2(t)f_{ik}(t)]}\\&\quad-\frac{\bar u_i(t)}{\bar v_k(t)[1+\bar u_i^2(t)f_{ik}(t)]}\Big]\\&=-b_i(t)z_i(t)+\sum_{k=1,k\neq i}^{n}a_{ik}(t)\Big[\frac{z_i(t)}{u_k(t)}-\frac{\bar u_i(t)z_k(t)}{u_k(t)\bar u_k(t)}\Big]\\&\quad+\sum_{k=1}^{m}\frac{c_{ik}(t)}{\theta_k}\frac{[1-\xi_i^2 f_{ik}(t)]z_i(t)}{[1+\xi_i^2 f_{ik}(t)]^2}\\&\quad-\sum_{k=1}^{m}\frac{c_{ik}(t)}{\theta_k^2}\frac{\xi_i w_k(t)}{[1+\xi_i^2 f_{ik}(t)]^2}\\&\leq-\Big[b_i(t)-\sum_{k=1}^{m}\frac{c_{ik}(t)}{\theta_k}\frac{[1-\xi_i^2 f_{ik}(t)]}{[1+\xi_i^2 f_{ik}(t)]^2}\Big]z_i(t)\\&\quad+\sum_{k=1,k\neq i}^{n}\frac{a_{ik}^M}{u_i^m-\epsilon}|z_i(t)|\\&\quad+\sum_{k=1,k\neq i}^{n}\frac{a_{ik}^M(u_i^M+\epsilon)}{(u_k^m-\epsilon)^2}|z_k(t)|\\&\quad+\sum_{k=1}^{m}\frac{c_{ik}(t)}{\theta_k^2}\frac{\xi_i}{[1+\xi_i^2 f_{ik}(t)]^2}|w_k(t)|,\end{aligned}$$

where ξ_i is between u_i and $\bar u_i$, θ_k is between v_k and $\bar v_k$.

From Lemma II.6, it yields that

$$\begin{aligned}&{}^cD_0^{\alpha}|z_i(t)|\\&\leq-\Big[b_i^m+\sum_{k=1}^{m}\frac{c_{ik}^m}{v_k^M+\epsilon}\frac{(u_i^m-\epsilon)^2 f_{ik}^m}{[1+(u_i^M+\epsilon)^2 f_{ik}^M]^2}\\&\quad-\sum_{k=1}^{m}\frac{c_{ik}^M}{v_k^m-\epsilon}\frac{1}{[1+(u_i^m-\epsilon)^2 f_{ik}^m]^2}\\&\quad-\sum_{k=1,k\neq i}^{n}\frac{a_{ik}^M}{u_i^m-\epsilon}\Big]|z_i(t)|\\&\quad+\sum_{k=1,k\neq i}^{n}\frac{a_{ik}^M(u_i^M+\epsilon)}{(u_k^m-\epsilon)^2}|z_k(t)|\end{aligned}$$

$$+\sum_{k=1}^{m}\frac{c_{ik}^M}{(v_k^m-\epsilon)^2}\frac{u_i^M+\epsilon}{[1+(u_i^m-\epsilon)^2 f_{ik}^m]^2}|w_k(t)|, \quad (4.2)$$

where $t \geq t_1$, $i = 1, 2, \ldots, n$. Enabling $\epsilon \to 0$ in (4.2), it derives (4.1). The proof is finished. ■

Let

$$\beta_1 = \min_{1\leq j\leq m}\left[\sum_{k=1}^{n}\frac{d_{kj}^m}{1+(u_k^M)^2 f_{kj}^M} - \sum_{k=1,k\neq j}^{m}\frac{e_{jk}^M}{v_k^m} - r_j^M\right],$$

$$\beta_2 = \max_{1\leq j\leq m}\sum_{k=1,k\neq j}^{m}\frac{e_{jk}^M \bar{v}_j^M}{(v_k^m)^2},$$

$$\beta_3 = \max_{1\leq j\leq m}\sum_{k=1}^{n}\frac{2d_{kj}^M f_{kj}^M u_k^M v_j^M}{[1+(u_k^m)^2 f_{kj}^m]^2}.$$

Proposition IV.2. There exists $t_1 > 0$ ensuring that

$$\begin{aligned} {}^cD_0^\alpha|w_j(t)| \leq & -\beta_1|w_j(t)| + \beta_2\max_{1\leq j\leq m}|w_j(t)| \\ & +\beta_3\max_{1\leq i\leq n}|z_i(t)|, \end{aligned} \quad (4.3)$$

where $t \geq t_1$, $j = 1, 2, \ldots, m$.

Proof: By the second equation of system (1.3), it has

$$\begin{aligned} {}^cD_0^\alpha w_j(t) =& r_j(t)w_j(t) - \sum_{k=1}^{n}\frac{d_{kj}(t)w_j(t)}{1+\mu_k^2 f_{kj}(t)} \\ & +\sum_{k=1}^{n}\frac{2\mu_k\nu_j d_{kj}(t)f_{kj}(t)z_k(t)}{[1+\mu_k^2 f_{kj}(t)]^2} \\ & +\sum_{k=1,k\neq j}^{m}e_{jk}(t)\left[\frac{w_j(t)}{v_k(t)} - \frac{\bar{v}_j(t)w_k(t)}{v_k(t)\bar{v}_k(t)}\right] \\ \leq & -\left[\sum_{k=1}^{n}\frac{d_{kj}(t)}{1+\mu_k^2 f_{kj}(t)}\right. \\ & \left.-\sum_{k=1,k\neq j}^{m}\frac{e_{jk}(t)}{v_k(t)} - r_j(t)\right]w_j(t) \\ & -\sum_{k=1,k\neq j}^{m}\frac{e_{jk}(t)\bar{v}_j(t)}{v_k(t)\bar{v}_k(t)}w_k(t) \\ & +\sum_{k=1}^{n}\frac{2\mu_k\kappa_j d_{kj}(t)f_{kj}(t)}{[1+\mu_k^2 f_{kj}(t)]^2}z_k(t), \end{aligned}$$

where μ_k is between u_k and $\bar{u}_k$, κ_j is between v_j and $\bar{v}_j$. It products from Lemma II.6 that

$$\begin{aligned} & {}^cD_0^\alpha|w_j(t)| \\ \leq & -\left[\sum_{k=1}^{n}\frac{d_{kj}^m}{1+(u_k^M+\epsilon)^2 f_{kj}^M}\right. \\ & \left.-\sum_{k=1,k\neq j}^{m}\frac{e_{jk}^M}{v_k^m-\epsilon} - r_j^M\right]|w_j(t)| \\ & +\sum_{k=1,k\neq j}^{m}\frac{e_{jk}^M(\bar{v}_j^M+\epsilon)}{(v_k^m-\epsilon)^2}|w_k(t)| \\ & +\sum_{k=1}^{n}\frac{2(u_k^M+\epsilon)(v_j^M+\epsilon)d_{kj}^M f_{kj}^M}{[1+(u_k^m-\epsilon)^2 f_{kj}^m]^2}|z_k(t)|, \end{aligned} \quad (4.4)$$

where $t \geq t_1$, $j = 1, 2, \ldots, m$. Enabling $\epsilon \to 0$ in (4.4), it derives (4.3). The proof is finished. ■

B. Result of Global Mittag-Leffler Stability

Definition IV.1. Supposing that $(u_i(t), v_j(t))$ and $(\bar{u}_i(t), \bar{v}_j(t))$ are any two positive solutions of system (1.3), $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. System (1.3) is said to be globally Mittag-Leffler stable, if there exist $M > 0$ and $\gamma > 0$ ensuring that

$$\begin{aligned} & \max_{1\leq i\leq n, 1\leq j\leq m}\{|u_i(t)-\bar{u}_i(t)|, |v_i(t)-\bar{v}_i(t)|\} \\ & \leq ME_\alpha(-\gamma t^\alpha), \quad t \geq 0. \end{aligned}$$

The number γ is called the convergent rate, which presents the convergent speed of positive solutions for system (1.3).

Theorem IV.1. System (1.3) is globally Mittag-Leffler stable in case

(H_3) $\lambda_1 > \lambda_2 + \lambda_3$.
(H_4) $\beta_1 > \beta_2 + \beta_3$.

Further, the convergent rate $\gamma = \min\{\lambda_1 - \lambda_2 - \lambda_3, \beta_1 - \beta_2 - \beta_3\}$.

Proof: Let $|p(t)|_{\max} = \max_{1\leq i\leq n, 1\leq j\leq m}\{|z_i(t)|, |w_j(t)|\}$. For any $t' \geq t_1$, there exists $i_0 \in \{1, 2, \ldots, n\}$ or $j_0 \in \{1, 2, \ldots, m\}$ such that $|p(t')|_{\max} = |z_{i_0}(t')|$ or $|p(t')|_{\max} = |w_{j_0}(t')|$, respectively.

In case $|p(t')|_{\max} = |z_{i_0}(t')|$, it follows from (4.1) that

$$\begin{aligned} {}^cD_0^\alpha|p(t')|_{\max} =& {}^cD_0^\alpha|z_{i_0}(t')| \\ \leq & -\lambda_1|z_{i_0}(t')| + \lambda_2\max_{1\leq i\leq n}|z_i(t')| \\ & +\lambda_3\max_{1\leq j\leq m}|w_j(t')| \\ \leq & -(\lambda_1-\lambda_2-\lambda_3)|p(t')|_{\max}. \end{aligned} \quad (4.5)$$

In case $|p(t')|_{\max} = |w_{j_0}(t')|$, it follows from (4.3) that

$$\begin{aligned} {}^cD_0^\alpha|p(t')|_{\max} =& {}^cD_0^\alpha|w_{j_0}(t')| \\ \leq & -\beta_1|w_{j_0}(t')| \\ & +\beta_2\max_{1\leq j\leq m}|w_j(t')| + \beta_3\max_{1\leq i\leq n}|z_i(t')| \\ \leq & -(\beta_1-\beta_2-\beta_3)|p(t')|_{\max}. \end{aligned} \quad (4.6)$$

By the arbitrariness of t', it gets from (4.5) and (4.6) that

$${}^cD_0^\alpha|p(t)|_{\max} \leq -\gamma|p(t)|_{\max}, \quad t \geq t_1. \quad (4.7)$$

Considering the following equations generated by (4.7):

$${}^cD^\alpha\varphi(t) = -\gamma\varphi(t),\ \varphi(t_1) = |p(t_1)|_{\max}, \quad t > t_1. \quad (4.8)$$

By Lemma II.7, it has

$$\varphi(t) = \varphi(t_1)E_\alpha(-\gamma t^\alpha), \quad t > t_1,$$

which deduces from Lemma II.9 that

$$|p(t)|_{\max} \leq \varphi(t) = \varphi(t_1)E_\alpha(-\gamma t^\alpha), \quad t \geq t_1.$$

From (H_3)-(H_4), the solution of system (1.3) is globally Mittag-Leffler stable. The proof is finished. ■

V. Global Asymptotic Periodic Solution

Theorem V.1. Assume that (H_5) and (H_6) hold, then system (1.3) is ω-asymptotic global periodic, that is,

$$\lim_{t\to\infty} |u_i(t+\omega) - u_i(t)| = 0, \quad i = 1, 2, \cdots, n,$$
$$\lim_{t\to\infty} |v_j(t+\omega) - v_j(t)| = 0, \quad j = 1, 2, \cdots, m.$$

Proof: Let $N_i(t) = u_i(t+\omega) - u_i(t)$, $M_i(t) = v_i(t+\omega) - v_i(t)$, and $N(t) = \sum_{i=1}^{n} N_i(t)$, $M(t) = \sum_{j=1}^{m} M_j(t)$, from the first equation of system (1.3), it gets

$$\begin{aligned}
{}^cD^\alpha N(t) &= {}^cD^\alpha \sum_{i=1}^{n} N_i(t) \\
&= -\sum_{i=1}^{n}[b_i(t+\omega) - b_i(t)]u_i(t+\omega) - \sum_{i=1}^{n} b_i(t)N_i(t) \\
&\quad + \sum_{i=1}^{n}\sum_{k=1}^{n}[a_{ik}(t+\omega) - a_{ik}(t)]\frac{u_i(t)}{u_k(t)} \\
&\quad + \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}(t)\frac{N_i(t)}{u_k(t+\omega)} \\
&\quad - \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}(t)\frac{u_i(t)N_k(t)}{u_k(t)u_k(t+\omega)} + \sum_{i=1}^{n}\sum_{k=1}^{m}\Bigg[\\
&\quad \frac{c_{ik}(t)N_i(t)}{v_k(t+\omega)[1+u_i^2(t+\omega)f_{ik}(t+\omega)]} \\
&\quad - \frac{c_{ik}(t)u_i(t)[u_i(t)+u_i(t+\omega)]f_{ik}(t)N_i(t)}{[1+u_i^2(t+\omega)f_{ik}(t+\omega)]v_k(t)[1+u_i^2(t)f_{ik}(t)]} \\
&\quad + \frac{[c_{ik}(t+\omega) - c_{ik}(t)]u_i(t+\omega)}{v_k(t+\omega)[1+u_i^2(t+\omega)f_{ik}(t+\omega)]} \\
&\quad + \frac{c_{ik}(t)u_i(t)[v_k(t) - v_k(t+\omega)]}{v_k(t+\omega)[1+u_i^2(t+\omega)f_{ik}(t+\omega)]v_k(t)} \\
&\quad + \frac{c_{ik}(t)u_i(t)u_i^2(t+\omega)[f_{ik}(t) - f_{ik}(t+\omega)]}{[1+u_i^2(t+\omega)f_{ik}(t+\omega)]v_k(t)[1+u_i^2(t)f_{ik}(t)]}\Bigg] \\
&\le -\sum_{i=1}^{n} b_i^m N_i(t) + \sum_{k=1}^{n}\frac{a_{ik}^M}{u_k^m - \epsilon}N_i(t) \\
&\quad + \sum_{i=1}^{n}\sum_{k=1}^{m}\frac{c_{ik}^M N_i(t)}{v_k^m - \epsilon} \\
&\quad - 2\sum_{i=1}^{n}\sum_{k=1}^{m}\frac{c_{ik}^m(u_i^m - \epsilon)^2 f_{ik}^m N_i(t)}{[1+(u_i^M+\epsilon)^2 f_{ik}^M]^2(v_k^M+\epsilon)} \\
&\quad + \sum_{i=1}^{n}\sum_{k=1}^{n}\frac{a_{ik}^M(u_i^M+\epsilon)N_k(t)}{(u_k^m - \epsilon)^2} \\
&\quad + \sum_{i=1}^{n}\sum_{k=1}^{m}\frac{c_{ik}(t)u_i(t)[v_k(t) - v_k(t+\omega)]}{v_k(t+\omega)[1+u_i^2(t+\omega)f_{ik}(t+\omega)]v_k(t)} \\
&\quad - [b_i(t+\omega) - b_i(t)]u_i(t+\omega) \\
&\quad + \sum_{i=1}^{n}\sum_{k=1}^{n}[a_{ik}(t+\omega) - a_{ik}(t)]\frac{u_i(t)}{u_k(t)} + \sum_{i=1}^{n}\sum_{k=1}^{m}\Bigg[\\
&\quad \frac{[c_{ik}(t+\omega) - c_{ik}(t)]u_i(t+\omega)}{v_k(t+\omega)[1+u_i^2(t+\omega)f_{ik}(t+\omega)]} \\
&\quad + \frac{c_{ik}(t)u_i(t)u_i^2(t+\omega)[f_{ik}(t) - f_{ik}(t+\omega)]}{[1+u_i^2(t+\omega)f_{ik}(t+\omega)]v_k(t)[1+u_i^2(t)f_{ik}(t)]}\Bigg],
\end{aligned}$$

thus by condition (H_5)-(H_6), there exist positive constants A_1, B_1, K_1 and C_1, such that

$$ {}^cD^\alpha N(t) \le -A_1 N(t) + B_1 M(t) + C_1\frac{\epsilon}{K_1}, \qquad (5.1)$$

where $i = 1, 2, \cdots, n$. In the same way,

$$\begin{aligned}
{}^cD^\alpha M(t) &= {}^cD^\alpha \sum_{j=1}^{m} M_j(t) \\
&\le \sum_{j=1}^{m} r_j^M M_j(t) + \sum_{j=1}^{m}\sum_{k=1}^{m}\frac{e_{jk}^M}{v_k^m - \epsilon}M_j(t) \\
&\quad - \sum_{j=1}^{m}\sum_{k=1}^{m}\frac{(v_j^m+\epsilon)e_{jk}^m}{(v_k^M+\epsilon)^2}M_k(t) \\
&\quad - \sum_{j=1}^{m}\sum_{k=1}^{n}\frac{d_{kj}^m M_j(t)}{[1+(u_k^M+\epsilon)^2 f_{kj}^M]} \\
&\quad + 2\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{d_{kj}^m(v_j - \epsilon)^2 f_{kj}^m N_k(t)}{[1+(u_k^M+\epsilon)^2 f_{kj}^M]^2} \\
&\quad + \sum_{j=1}^{m}\sum_{k=1}^{m}[e_{jk}(t+\omega) - e_{jk}(t)]\frac{v_j(t+\omega)}{v_k(t+\omega)} \\
&\quad + \sum_{j=1}^{m}\sum_{k=1}^{n}\Bigg[\frac{[d_{kj}(t+\omega) - d_{kj}(t)]v_j(t+\omega)}{[1+u_k^2(t+\omega)f_{kj}(t+\omega)]} \\
&\quad + \frac{d_{kj}(t)v_j(t)u_k^2(t+\omega)[f_{kj}(t) - f_{kj}(t+\omega)]}{[1+u_k^2(t+\omega)f_{kj}(t+\omega)][1+u_k^2(t)f_{kj}(t)]}\Bigg],
\end{aligned}$$

and there exist positive constants A_2, B_2, K_2 and C_2 such that

$$ {}^cD^\alpha M(t) \le -A_2 M(t) + B_2 N(t) + C_2\frac{\epsilon}{K_2}, \qquad (5.2)$$

where $j = 1, 2, \cdots, m$. From (5.1)-(5.2), it yields

$$\begin{aligned}
{}^cD^\alpha(N(t)+M(t)) &\le (B_2 - A_1)N(t) + (B_1 - A_2)M(t) + C\epsilon \\
&\le -A(N(t)+M(t)) + C\epsilon,
\end{aligned}$$

by Lemma II.6, it leads

$$ {}^cD^\alpha|N(t)+M(t)| \le -A|N(t)+M(t)| + C\epsilon. \qquad (5.3)$$

Set $V(t) = |N(t)+M(t)| - \frac{C\epsilon}{A}$, substitute it into (5.3), it leads

$$ {}^cD^\alpha V(t) \le -AV(t), \quad t > t_\epsilon,$$

it is obvious that V is globally asymptotically stable, i.e., there exists $t_\epsilon^* > t_\epsilon$, such that

$$|V(t)| < \frac{\epsilon}{2},$$

that is,

$$|N(t)+M(t)| - \frac{C\epsilon}{A}| < \frac{\epsilon}{2}, \quad t > t_\epsilon^*.$$

So

$$|N(t)+M(t)| < \frac{C\epsilon}{A} + \frac{\epsilon}{2} < \epsilon, \quad t > t_\epsilon^*.$$

This completes the proof. ■

VI. An example

Consider the fractional order system of two-species as follows:

$$\begin{cases} {}^cD_0^{0.5}u(t) = -10u(t) + 1 + \dfrac{0.5u(t)}{v(t)[1+u^2(t)]}, \\ {}^cD_0^{0.5}v(t) = 0.001v(t) - \dfrac{0.5v(t)}{1+u^2(t)} + 0.1, \quad t \ge 0. \end{cases} \qquad (6.1)$$

By simple computing,

$$u^m = 0.1, \quad v^m = 0.2, \quad b^* = 10, \quad d^* = 0.5,$$
$$u^M = 0.35, \quad r^* = 0.45, \quad v^M = 0.2,$$

which implies that (H_1)-(H_2) hold. By Theorem III.1, system (6.1) is permanent.

Furthermore,

$$\lambda_1 \geq 9.75, \quad \lambda_2 = 0, \quad \lambda_3 = 0.44,$$
$$\beta_1 = 0.45, \quad \beta_2 = 0, \quad \beta_3 = 0.07.$$

Then (H_3)-(H_4) are valid. By Theorem IV.1, system (6.1) is globally Mittag-Leffler stable.

VII. Conclusions

By utilizing comparison principles and fractional calculus theory, some new conditions are established to ensure the permanence, global Mittag-Leffler stability and global asymptotic periodic solution of a kind of predator-prey system with fraction-order derivative scheme. The main methods and results can be able to study other models of fractional order in science and engineering.

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