On the Number of Independent Invariants for m Unit Vectors and n Symmetric Second Order Tensors

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Abstract—**Invariants play an important role in continuum mechanics. Knowing the number of independent invariants is crucial in modelling and in a rigorous construction of a constitutive equation for a particular material, where it is determined by doing tests that hold all, except one, of the independent invariants constant so that the dependence in the one invariant can be identified. Hence, the aim of this paper is to prove that the number of independent invariants for a set of n symmetric tensors and m unit vectors is at most $2m+6n-3$. The prove requires the construction of spectral invariants. All classical invariants can be explicitly expressed in terms spectral invariants. We show that the number of spectral invariants in an irreducible functional basis is reduced to $2m+6n-3$; a significant reduction to that obtained in the literature if the value of m or n is large. Relations between classical invariants in a classical-minimal integrity basis are given.**

Index Terms—**Continuum mechanics, Independent invariants, Integrity basis, Tensors and vectors, Relations**

I. Introduction

Tensors and vectors play an important role in continuum mechanics [1]. The construction of rotational invariants of sets containing vectors and tensors in continuum mechanics, especially for anisotropic materials [2], [3], [4], [5], has been active for around 70 years. The classical invariants developed by Spencer [1], published in 1971, have been extensively used in the literature. Spencer [1] stated that " It *frequently* happens that polynomial relations exist between invariants which do not permit any one invariant to be expressed as a polynomial in the remainder. Such relations are called syzygies". This suggests that some of the invariants in a given classical-minimal integrity basis may not be independent (see also [6]). However, due to the difficulty in constructing relations (syzygies) among classical invariants, Spencer [1] did not specifically mention the number of existing syzygies for a given classical-minimal integrity basis, and in view of this, the number of independent invariants was, often, not correctly stated in the literature and it is occasionally assumed in the literature (see, for example [7], [8]) that all the invariants in a classical-minimal integrity (or in an irreducible functional [9]) basis are independent. To prove the number of independent classical invariants in a classical-minimal integrity basis is not straightforward. However, recently, in the case of an m-preferred direction anisotropic solid, Shariff [10] proved that the number of independent invariants is at most $2m+3$ (see also [11], [12], [13], [14]) and in the case of $n-symmetric$ tensors, Shariff [15] (see also [16]) proved that at most $6n-3$ classical invariants are independent. In this communication, using spectral invariants [2], [3], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], we extend these results to prove that the number of independent invariants for m vectors and n symmetric tensors is $2m+6n-3$. We also give relations (not necessarily syzygies) between classical invariants in the corresponding classical-minimal integrity basis. Knowing the number of independent invariants is crucial in modelling [4], [5] and in a rigorous construction of a constitutive equation of a particular material, where it is determined by doing tests that hold all, except one, of the independent invariants constant so that the dependence in the one invariant can be identified [17], [29], [30].

It is important to note that the invariants used to characterize a scalar function, required for a constitutive equation, cannot have an arbitrary form. They must satisfy the P-property as described in [21]; we elaborate this property in Appendix A.

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II. Preliminaries

In this paper, the summation convention is not used and all subscripts i,j and k take the values $1, 2, 3$, unless stated otherwise. Preliminary concepts of functional and integrity bases, and syzygies are given in Appendix B.

Consider the set

$$S(m,n) = \{\mathbf{v}^{(r)}, \mathbf{A}^{(s)} \mid r = 0, 1 \ldots, m\,, s = 1, 2 \ldots, n\}\,, \tag{1}$$

where m and n are non-negative integers, $\mathbf{A}^{(s)}$ are symmetric tensors defined on a three-dimensional Euclidean space and $\mathbf{v}^{(r)}$ are linearly independent unit vectors. Using a fixed Cartesian basis $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}$, we have,

$$\mathbf{v}^{(r)} = \sum_{i=1}^{3} \bar{v}_i^{(r)} \boldsymbol{g}_i\,, \quad \mathbf{A}^{(s)} = \sum_{i,j=1}^{3} \bar{A}_{ij}^{(s)} \boldsymbol{g}_i \otimes \boldsymbol{g}_j\,, \tag{2}$$

where $\otimes$ denotes the dyadic product and since $\mathbf{A}^{(s)}$ is symmetric, $\bar{A}_{ij}^{(s)} = \bar{A}_{ji}^{(s)}$. Since $\mathbf{v}^{(r)}$ are unit vectors, we have,

$$\sum_{i=1}^{3} \bar{v}_i^{(r)2} = 1\,, \quad r = 1, 2 \ldots, m\,. \tag{3}$$

In view of (2) and (3) we can say that at most there are $2m+6n$ independent component variables in (1).

Since the classical invariants in a classical-minimal integrity basis are traces of tensors and dot products of vectors [1] and they are explicit functions of the $2m+6n$ components, hence the number of independent invariants in a classical-minimal integrity basis for the set $S(m,n)$ cannot be greater than $2m+6n$. It is important to emphasize that the components $\bar{v}^{(r)}$ and $\bar{A}^{(s)}_{ij}$ are *not* invariants and hence they cannot be explicitly expressed as a function of the classical invariants. However if we use the spectral basis $\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}$ where its elements are eigenvectors of $\mathbf{A}^{(1)}$, we have

$$\begin{aligned}\mathbf{A}^{(1)} &= \sum_{i=1}^{3}\lambda_i\boldsymbol{u}_i\otimes\boldsymbol{u}_i\,, \quad \mathbf{v}^{(r)}=\sum_{i=1}^{3}v_i^{(r)}\boldsymbol{u}_i\,,\\ \mathbf{A}^{(s)} &= \sum_{i,j=1}^{3}A_{ij}^{(s)}\boldsymbol{u}_i\otimes\boldsymbol{u}_j\,,\\ r &= 1,2,\ldots,m\,, s=2,3,\ldots,n\,. \end{aligned}\tag{4}$$

In this case, the spectral components $v_i^{(r)}$, $A_{ij}^{(s)}$ are rotational *invariants* with respect to the rotation tensor $\boldsymbol{Q}$, since

$$v_i^{(r)}=\mathbf{v}^{(r)}\cdot\boldsymbol{u}_i=\boldsymbol{Q}\mathbf{v}^{(r)}\cdot\boldsymbol{Q}\boldsymbol{u}_i\,,$$

$$A_{ij}^{(s)}=\boldsymbol{u}_i\cdot\mathbf{A}^{(s)}\boldsymbol{u}_j=\boldsymbol{Q}\boldsymbol{u}_i\cdot\boldsymbol{Q}\mathbf{A}^{(s)}\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{u}_j\,. \tag{5}$$

Hence, it is possible that the spectral components can be expressed explicitly in terms of the classical invariants. Since $\mathbf{v}^{(r)}$ are unit vectors,

$$\sum_{i=1}^{3}v_i^{(r)2}=1\,,\quad r=1,2\ldots,m\,. \tag{6}$$

It is clear that the number of independent spectral invariants cannot be greater $2m+6n-3$ and hence the number of independent classical invariants for the $S(m,n)$ set is at most $2m+6n-3<2m+6n$. Note that $v_i^{(r)}$ can take a positive or negative value. If we consider the positive and negative values of $v_i^{(r)}$ as distinct single-valued functions, in view of (6), then we can conclude that the number of invariants in the irreducible functional basis is also $2m+6n-3$, since all invariants can be explicitly expressed in terms of $2m+6n-3$ spectral invariants. We strongly emphasize that, if the value of m or n is large, the number of invariants in an irreducible functional or a minimal integrity basis obtained in the literature (see, for example, references [1], [31]), is significantly higher than $2m+6n-3$.

In the Section III, based on the work of Shariff [2], [10], [15] and Aguiar and Rocha [16] , we show relations between classical invariants using our spectral invariants. We assume, for simplicity, $\mathbf{A}^{(1)}$ is invertible and its eigenvalues λ_i are distinct. In the case when the eigenvalues λ_i and some of the eigenvalues of $\mathbf{A}^{(s)}$ are not distinct, the number of independent invariants is far less than $2m+6n-3$, as exemplified in the Appendix C.

In the case of $S(0,n)$ it is shown in references [15], [16] that the number of independent invariants is $6n-3$ and, in these references, classical invariant relations are given.

III. Classical invariant relations

In this section, we first obtain relations for certain values of m and n and then derive relations for general m and n. The construction of these relations require the invariants

$$\begin{aligned} I_1 &= \operatorname{tr}\mathbf{A}^{(1)}=\sum_{i=1}^{3}\lambda_i\,,\\ I_2 &= \frac{1}{2}\left((\operatorname{tr}\mathbf{A}^{(1)})^2-\operatorname{tr}\mathbf{A}^{(1)2}\right)=\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3\,,\\ I_3 &= \det(\mathbf{A}^{(1)})=\lambda_1\lambda_2\lambda_3\,. \end{aligned}\tag{7}$$

It is commonly known that the above relations are independent and since the eigenvalues λ_i are independent, there are no relations between the classical invariants and hence the three classical invariants are independent. The eigenvalues λ_i can be explicitly expressed in terms of the classical invariants [32] , i.e.,

$$\begin{aligned}\lambda_i &= \frac{1}{3}\left\{I_1+2\sqrt{I_1^2-3I_2}\cos\frac{1}{3}[\theta+2\pi(i-1)]\right\}\,,\\ i &= 1,2,3\,, \end{aligned}\tag{8}$$

where

$$\theta=\arccos\left[\frac{2(I_1^3-9I_1I_2+27I_3}{2[I_1^2-3I_2]^{\frac{3}{2}}}\right]\,, \tag{9}$$

taking note that since the eigenvalues λ_i are distinct, we have, $I_1^2-3I_2\neq 0$.

A. $m=1,n=1$. *Only* 5 *classical invariants are independent*

The set $S(1,1)$ is generally associated with transversely isotropic elastic materials, where their strain energy functions can be written in the form

$$W(\mathbf{v}^{(1)}\otimes\mathbf{v}^{(1)},\mathbf{A}^{(1)})=\hat{W}(\mathbf{v}^{(1)},\mathbf{A}^{(1)})\,. \tag{10}$$

The 5 invariants in the classical-minimal integrity basis are

$$\begin{aligned} I_1^{(1,1)} &= I_1\,,\quad I_2^{(1,1)}=I_2\,,\quad I_3^{(1,1)}=I_3\,,\\ I_4^{(1,1)} &= \mathbf{v}^{(1)}\cdot\mathbf{A}^{(1)}\mathbf{v}^{(1)}=\sum_{i=1}^{3}v_i^{(1)2}\lambda_i\,,\\ I_5^{(1,1)} &= \mathbf{v}^{(1)}\cdot\mathbf{A}^{(1)2}\mathbf{v}^{(1)}=\sum_{i=1}^{3}v_i^{(1)2}\lambda_i^2\,. \end{aligned}\tag{11}$$

In view of (6), only 5 of the spectral invariants are independent and since there are no relations between the 5 spectral invariants, it is clear that there are no relations between the classical invariants. Hence, we have 5 independent classical invariants.

B. $m=2,n=1$. *At most* 7 *classical invariants are independent*

The set $S(2,1)$ is commonly associated with the strain energy function of an elastic solid with two preferred directions, i.e.

$$W(\mathbf{v}^{(1)}\otimes\mathbf{v}^{(1)},\mathbf{v}^{(2)}\otimes\mathbf{v}^{(2)},\mathbf{A}^{(1)})=\hat{W}(\mathbf{v}^{(1)},\mathbf{v}^{(2)},\mathbf{A}^{(1)})\,. \tag{12}$$

There are ten classical invariants in the classical-minimal integrity basis, i.e.,

$$I_1^{(2,1)} = I_1\,, \quad I_2^{(2,1)} = I_2\,, \quad I_3^{(2,1)} = I_3\,, \tag{13}$$

$$I_4^{(2,1)} = I_4^{(1,1)} = \sum_{i=1}^{3} v_i^{(1)2}\lambda_i\,,$$

$$I_5^{(2,1)} = I_5^{(1,1)} = \sum_{i=1}^{3} v_i^{(1)2}\lambda_i^2\,, \tag{14}$$

$$I_6^{(2,1)} = \mathbf{v}^{(2)} \cdot \mathbf{A}^{(1)}\mathbf{v}^{(2)} = \sum_{i=1}^{3} v_i^{(2)2}\lambda_i\,,$$

$$I_7^{(2,1)} = \mathbf{v}^{(2)} \cdot (\mathbf{A}^{(1)})^2\mathbf{v}^{(2)} = \sum_{i=1}^{3} v_i^{(2)2}\lambda_i^2\,, \tag{15}$$

$$I_8^{(2,1)} = (\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)})^2 = (\sum_{i=1}^{3} v_i^{(1)}v_i^{(2)})^2\,,$$

$$\begin{aligned} I_9^{(2,1)} &= (\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)})\mathbf{v}^{(1)} \cdot \mathbf{A}^{(1)}\mathbf{v}^{(2)} \\ &= (\sum_{i=1}^{3} v_i^{(1)}v_i^{(2)})\sum_{i=1}^{3} v_i^{(1)}v_i^{(2)}\lambda_i \\ I_{10}^{(2,1)} &= (\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)})\mathbf{v}^{(1)} \cdot (\mathbf{A}^{(1)})^2\mathbf{v}^{(2)} \\ &= (\sum_{i=1}^{3} v_i^{(1)}v_i^{(2)})\sum_{i=1}^{3} v_i^{(1)}v_i^{(2)}\lambda_i^2\,. \end{aligned} \tag{16}$$

Shariff and Bustamante [11] have given 3 classical invariant relations and hence only 7 of the above invariants are independent. However, below we give alternative invariant relations to strengthen our claim that only 7 of the classical invariants are independent.

From (8) and (13), λ_i is expressed in terms of $I_i^{(2,1)}$. From (6), with $r = 1, 2$, (14) and (15) we have 6 linear equations in $v_i^{(1)2}$ and $v_i^{(2)2}$. On solving the two 3 linear equations (see Appendix D) in turn we can express $v_i^{(1)2}$ and $v_i^{(2)2}$ explicitly in terms of $I_\alpha^{(2,1)}\,, \alpha = 1, 2 \ldots, 7$. Taking into consideration the sign of $v^{(1)}$ and $v^{(2)}$ and the fact that $I_8^{(2,1)}, I_9^{(2,1)}, I_{10}^{(2,1)}$ can be explicitly expressed in terms of $\lambda_i, v^{(1)}, v^{(2)}$, it is clear that, in view of (16), they can be expressed explicitly in terms of $I_\alpha^{(1,2)}\,, \alpha = 1, 2 \ldots, 7$. Hence only 7 classical invariants are independent.

C. $m = 1, n = 2$. *At most* 11 *invariants are independent*

An example of a $S(1, 2)$ constitutive function of the form

$$W(\mathbf{v}^{(1)} \otimes \mathbf{v}^{(1)}, \mathbf{A}^{(1)}, \mathbf{A}^{(2)}) = \hat{W}(\mathbf{v}^{(1)}, \mathbf{A}^{(1)}, \mathbf{A}^{(2)}) \tag{17}$$

can be found in Shariff et. al. [3]. There are 18 classical invariants in the classical-minimal integrity basis for the function (17), i.e.,

$$I_1^{(1,2)} = I_1\,, \quad I_2^{(1,2)} = I_2\,, \quad I_3^{(1,2)} = I_3\,, \tag{18}$$

$$\begin{aligned} I_4^{(1,2)} &= \mathbf{v}^{(1)} \cdot \mathbf{A}^{(1)}\mathbf{v}^{(1)} = \sum_{i=1}^{3} v_i^{(1)2}\lambda_i\,, \\ I_5^{(1,2)} &= \mathbf{v}^{(1)} \cdot \mathbf{A}^{(1)2}\mathbf{v}^{(1)} = \sum_{i=1}^{3} v_i^{(1)2}\lambda_i^2\,, \end{aligned} \tag{19}$$

$$\begin{aligned} I_6^{(1,2)} &= \mathrm{tr}\,\mathbf{A}^{(2)} = \sum_{i=1}^{3} A_{ii}^{(2)}\,, \\ I_7^{(1,2)} &\mathrm{tr}\,(\mathbf{A}^{(2)}\mathbf{A}^{(1)}) = \sum_{i=1}^{3} \lambda_i A_{ii}^{(2)}\,, \\ I_8^{(1,2)} &= \mathrm{tr}\,(\mathbf{A}^{(2)}\mathbf{A}^{(1)2}) = \sum_{i=1}^{3} \lambda_i^2 A_{ii}^{(2)}\,, \end{aligned} \tag{20}$$

$$\begin{aligned} I_9^{(1,2)} &= \mathrm{tr}\,(\mathbf{A}^{(2)2}) = A_{11}^{(2)2} + A_{22}^{(2)2} + A_{33}^{(2)2} \\ &+2(A_{12}^{(2)2} + A_{13}^{(2)2} + A_{23}^{(2)2})\,, \\ I_{10}^{(1,2)} &= \mathrm{tr}\,(\mathbf{A}^{(2)2}\mathbf{A}^{(1)}) = \lambda_1(A_{11}^{(2)2} + A_{12}^{(2)2} + A_{13}^{(2)2}) \\ &+\lambda_2(A_{21}^{(2)2} + A_{22}^{(2)2} + A_{23}^{(2)2}) \\ &+\lambda_3(A_{31}^{(2)2} + A_{32}^{(2)2} + A_{33}^{(2)2})\,, \\ I_{11}^{(1,2)} &= \mathrm{tr}\,(\mathbf{A}^{(2)2}\mathbf{A}^{(1)2}) = \lambda_1^2(A_{11}^{(2)2} + A_{12}^{(2)2} + A_{13}^{(2)2}) \\ &+\lambda_2^2(A_{21}^{(2)2} + A_{22}^{(2)2} + A_{23}^{(2)2}) \\ &+\lambda_3^2(A_{31}^{(2)2} + A_{32}^{(2)2} + A_{33}^{(2)2})\,, \end{aligned} \tag{21}$$

$$\begin{aligned} I_{12}^{(1,2)} &= \mathbf{v}^{(1)} \cdot \mathbf{A}^{(2)}\mathbf{v}^{(1)}\,, \quad I_{13}^{(1,2)} = \mathbf{v}^{(1)} \cdot \mathbf{A}^{(2)2}\mathbf{v}^{(1)}\,, \\ I_{14}^{(1,2)} &= \mathrm{tr}\,\mathbf{A}^{(2)3}\,, \quad I_{15}^{(1,2)} = \mathbf{v}^{(1)} \cdot (\mathbf{A}^{(1)}\mathbf{A}^{(2)}\mathbf{v}^{(1)})\,, \\ I_{16}^{(1,2)} &= \mathbf{v}^{(1)} \cdot (\mathbf{A}^{(1)}\mathbf{A}^{(2)2}\mathbf{v}^{(1)})\,, \\ I_{17}^{(1,2)} &= \mathbf{v}^{(1)} \cdot (\mathbf{A}^{(2)}\mathbf{A}^{(1)2}\mathbf{v}^{(1)})\,, \\ I_{18}^{(1,2)} &= \mathbf{v}^{(1)} \cdot (\mathbf{A}^{(1)2}\mathbf{A}^{(2)2}\mathbf{v}^{(1)})\,. \end{aligned} \tag{22}$$

We note that the invariants $I_\alpha^{(1,2)}\,, \alpha = 12, 13, \ldots, 18$ can be explicitly expressed in terms of $\lambda_i, v_i^{(1)}$ and $A_{ij}^{(2)}$ but, for brevity, we omit such explicit expressions.

From (8) and (18), λ_i is expressed in terms of $I_i^{(1,2)}$. From (6), with $r = 1$, and (19), we have 3 linear equations in $v_i^{(1)2}$. On solving these linear equations we can express $v_i^{(1)2}$ explicitly in terms of $I_\alpha^{(1,2)}\,, \alpha = 1, 2 \ldots, 5$. The invariants $A_{ii}^{(2)}$ can be expressed in terms of $I_\alpha^{(1,2)}\,, \alpha = 1, 2, 3, 6, 7, 8$ by solving the 3 linear equations in (20) for $A_{ii}^{(2)}$. In Eqn. (21), the invariants $A_{12}^{(2)2}, A_{13}^{(2)2}, A_{23}^{(2)2}$ appear linearly. Hence we can solve the 3 linear equations so that these invariants (see Appendix D) can be expressed in terms of $I_\alpha^{(1,2)}\,, \alpha = 1, 2, 3, 6, 7 \ldots, 11$. Since the classical invariants $I_\alpha^{(1,2)}\,, \alpha = 12, 13, \ldots, 17$ can be explicitly expressed in terms of $\lambda_i, v_i^{(1)}$, $A_{ij}^{(2)}$, and taking the appropriate sign for $v_i^{(1)}$ and $A_{ij}^{(2)}$, they can be expressed in terms of $I_\alpha^{(1,2)}\,, \alpha = 1, 2, \ldots, 11$, indicating that only 11 classical invariants are independent.

D. $m = 2, n = 2$. *At most* 13 *invariants are independent*

An example of a $S(2, 2)$ constitutive function of the form

$$\begin{aligned} &W(\mathbf{v}^{(1)} \otimes \mathbf{v}^{(1)}, \mathbf{v}^{(2)} \otimes \mathbf{v}^{(2)}, \mathbf{A}^{(1)}, \mathbf{A}^{(2)}) \\ &= \hat{W}(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{A}^{(1)}, \mathbf{A}^{(2)}) \end{aligned} \tag{23}$$

can be found in [27]. There are 37 classical invariants for the function (23) in the classical-minimal integrity basis and they are:

$$I_1^{(2,2)} = I_1\,, \quad I_2^{(2,2)} = I_2\,, \quad I_3^{(2,2)} = I_3\,, \tag{24}$$

$$\begin{aligned}
I_4^{(2,2)} &= \mathbf{v}^{(1)} \cdot \mathbf{A}^{(1)}\mathbf{v}^{(1)} = \sum_{i=1}^{3} v_i^{(1)2}\lambda_i\,,\\
I_5^{(2,2)} &= \mathbf{v}^{(1)} \cdot \mathbf{A}^{(1)2}\mathbf{v}^{(1)} = \sum_{i=1}^{3} v_i^{(1)2}\lambda_i^2\,,\\
I_6^{(2,2)} &= \mathbf{v}^{(2)} \cdot \mathbf{A}^{(1)}\mathbf{v}^{(2)} = \sum_{i=1}^{3} v_i^{(2)2}\lambda_i\,,\\
I_7^{(2,2)} &= \mathbf{v}^{(2)} \cdot \mathbf{A}^{(1)2}\mathbf{v}^{(2)} = \sum_{i=1}^{3} v_i^{(2)2}\lambda_i^2\,,
\end{aligned} \tag{25}$$

$$\begin{aligned}
I_8^{(2,2)} &= \operatorname{tr}\mathbf{A}^{(2)} = \sum_{i=1}^{3} A_{ii}^{(2)}\,,\\
I_9^{(2,2)} &\operatorname{tr}(\mathbf{A}^{(2)}\mathbf{A}^{(1)}) = \sum_{i=1}^{3} \lambda_i A_{ii}^{(2)}\,,\\
I_{10}^{(2,2)} &= \operatorname{tr}(\mathbf{A}^{(2)}\mathbf{A}^{(1)2}) = \sum_{i=1}^{3} \lambda_i^2 A_{ii}^{(2)}\,,
\end{aligned} \tag{26}$$

$$\begin{aligned}
&I_{11}^{(2,2)} = \operatorname{tr}(\mathbf{A}^{(2)2}) = A_{11}^{(2)2} + A_{22}^{(2)2} + A_{33}^{(2)2}\\
&+2(A_{12}^{(2)2} + A_{13}^{(2)2} + A_{23}^{(2)2})\,,\\
&I_{12}^{(2,2)} = \operatorname{tr}(\mathbf{A}^{(2)2}\mathbf{A}^{(1)}) = \lambda_1(A_{11}^{(2)2} + A_{12}^{(2)2} + A_{13}^{(2)2})\\
&+\lambda_2(A_{21}^{(2)2} + A_{22}^{(2)2} + A_{23}^{(2)2}) +\\
&\lambda_3(A_{31}^{(2)2} + A_{32}^{(2)2} + A_{33}^{(2)2})\,,\\
&I_{13}^{(2,2)} = \operatorname{tr}(\mathbf{A}^{(2)2}\mathbf{A}^{(1)2}) = \lambda_1^2(A_{11}^{(2)2} + A_{12}^{(2)2} + A_{13}^{(2)2})\\
&+\lambda_2^2(A_{21}^{(2)2} + A_{22}^{(2)2} + A_{23}^{(2)2}) +\\
&\lambda_3^2(A_{31}^{(2)2} + A_{32}^{(2)2} + A_{33}^{(2)2})\,,
\end{aligned} \tag{27}$$

$$\begin{aligned}
&I_{14}^{(2,2)} = \mathbf{v}^{(1)} \cdot \mathbf{A}^{(2)}\mathbf{v}^{(1)}\,, \quad I_{15}^{(2,2)} = \mathbf{v}^{(1)} \cdot \mathbf{A}^{(2)2}\mathbf{v}^{(1)}\,,\\
&I_{16}^{(2,2)} = \operatorname{tr}\mathbf{A}^{(2)3}\,, \quad I_{17}^{(2,2)} = \mathbf{v}^{(1)} \cdot (\mathbf{A}^{(1)}\mathbf{A}^{(2)}\mathbf{v}^{(1)})\,,\\
&I_{18}^{(2,2)} = \mathbf{v}^{(1)} \cdot (\mathbf{A}^{(1)}\mathbf{A}^{(2)2}\mathbf{v}^{(1)})\,,\\
&I_{19}^{(2,2)} = \mathbf{v}^{(1)} \cdot (\mathbf{A}^{(2)}\mathbf{A}^{(1)2}\mathbf{v}^{(1)})\,,\\
&I_{20}^{(2,2)} = \mathbf{v}^{(1)} \cdot (\mathbf{A}^{(1)2}\mathbf{A}^{(2)2}\mathbf{v}^{(1)})\,,\\
&I_{21}^{(2,2)} = \mathbf{v}^{(2)} \cdot \mathbf{A}^{(2)}\mathbf{v}^{(2)}\,,\\
&I_{22}^{(2,2)} = \mathbf{v}^{(2)} \cdot \mathbf{A}^{(2)2}\mathbf{v}^{(2)}\,,\\
&I_{23}^{(2,2)} = \mathbf{v}^{(2)} \cdot (\mathbf{A}^{(1)}\mathbf{A}^{(2)}\mathbf{v}^{(2)})\,,\\
&I_{24}^{(2,2)} = \mathbf{v}^{(2)} \cdot (\mathbf{A}^{(1)}\mathbf{A}^{(2)2}\mathbf{v}^{(2)})\,,\\
&I_{25}^{(2,2)} = \mathbf{v}^{(2)} \cdot (\mathbf{A}^{(2)}\mathbf{A}^{(1)2}\mathbf{v}^{(2)})\,,\\
&I_{26}^{(2,2)} = \mathbf{v}^{(2)} \cdot (\mathbf{A}^{(1)2}\mathbf{A}^{(2)2}\mathbf{v}^{(2)})\,,\\
&I_{27}^{(2,2)} = (\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)})(\mathbf{v}^{(1)} \cdot \mathbf{A}^{(1)}\mathbf{v}^{(2)})\,,\\
&I_{28}^{(2,2)} = (\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)})(\mathbf{v}^{(1)} \cdot \mathbf{A}^{(2)}\mathbf{v}^{(2)})\,,\\
&I_{29}^{(2,2)} = (\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)})(\mathbf{v}^{(1)} \cdot \mathbf{A}^{(1)}\mathbf{A}^{(2)}\mathbf{v}^{(2)})\,,\\
&I_{30}^{(2,2)} = (\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)})(\mathbf{v}^{(1)} \cdot \mathbf{A}^{(2)}\mathbf{A}^{(1)}\mathbf{v}^{(2)})\,,\\
&I_{31}^{(2,2)} = (\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)})(\mathbf{v}^{(1)} \cdot \mathbf{A}^{(1)2}\mathbf{A}^{(2)}\mathbf{v}^{(2)})\,,\\
&I_{32}^{(2,2)} = (\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)})(\mathbf{v}^{(1)} \cdot \mathbf{A}^{(2)}\mathbf{A}^{(1)2}\mathbf{v}^{(2)})\,,\\
&I_{33}^{(2,2)} = (\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)})(\mathbf{v}^{(1)} \cdot \mathbf{A}^{(1)}\mathbf{A}^{(2)2}\mathbf{v}^{(2)})\,,\\
&I_{34}^{(2,2)} = (\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)})(\mathbf{v}^{(1)} \cdot \mathbf{A}^{(2)2}\mathbf{A}^{(1)}\mathbf{v}^{(2)})\,,\\
&I_{35}^{(2,2)} = (\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)})(\mathbf{v}^{(1)} \cdot \mathbf{A}^{(2)2}\mathbf{A}^{(1)2}\mathbf{v}^{(2)})\,,\\
&I_{36}^{(2,2)} = (\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)})(\mathbf{v}^{(1)} \cdot \mathbf{A}^{(1)2}\mathbf{A}^{(2)2}\mathbf{v}^{(2)})\,,\\
&I_{37}^{(2,2)} = (\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)})^2\,.
\end{aligned} \tag{28}$$

From (8) and (24), λ_i is expressed in terms of $I_i^{(2,2)}$. From (6), with $r = 1, 2$, and (25) we have 6 linear equations in $v_i^{(r)2}$. On solving 3 linear equations for each r, we can express $v_i^{(1)2}$ and $v_i^{(2)2}$ explicitly in terms of $I_\alpha^{(2,2)}\,, \alpha = 1, 2 \ldots, 7$. The invariants $A_{ii}^{(2)}$ appear linearly in (26). On solving the 3 linear equations in (26), we can express $A_{ii}^{(2)}$ explicitly in terms of $I_\alpha^{(2,2)}\,, \alpha = 1, 2, 3, 8, 9, 10$. The invariants $A_{ij}^{(2)2}\,, i \neq j$ appear linearly in the 3 equations given by (27). On solving these equations we can express explicitly for $A_{ij}^{(2)2}$ in terms of $I_\alpha^{(2,2)}\,, \alpha = 1, 2, 3, 8, \ldots, 13$. The remaining classical invariants $I_\alpha^{(2,2)}\,, \alpha = 14, 15, \ldots, 37$ in (28) can be expressed explicitly in terms of $\lambda_i\,, v_i^{(r)}\, A_{ij}^{(2)}$. Using the appropriate sign for $v^{(r)}$ and $A_{ij}^{(2)}\,, i \neq j$, we can express the remaining classical invariants explicitly in terms of the 13 independent invariants $I_\alpha^{(2,2)}\,, \alpha = 1, 2 \ldots, 13$.

E. Relations for general m and n. At most $2m + 6n - 3$ invariants are independent

In this section we only construct relations between classical invariants for an isotropic function of the form

$$\begin{aligned}
&W(\mathbf{v}^{(1)} \otimes \mathbf{v}^{(1)}, \mathbf{v}^{(2)} \otimes \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(m)} \otimes \mathbf{v}^{(m)},\\
&\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \ldots, \mathbf{A}^{(n)})\,.
\end{aligned} \tag{29}$$

Our intention is just to show relations, we shall not construct all the classical invariants in the general classical-minimal integrity basis, only the independent $2m + 6n - 3$ invariants, and they are:

$$I_1^{(m,n)} = I_1\,, \quad I_2^{(m,n)} = I_2\,, \quad I_3^{(m,n)} = I_3\,, \tag{30}$$

$$\begin{aligned}
&I_{2r+2}^{(m,n)} = \mathbf{v}^{(r)} \cdot \mathbf{A}^{(1)}\mathbf{v}^{(r)} = \sum_{i=1}^{3} v_i^{(r)2}\lambda_i\,,\\
&I_{2r+3}^{(m,n)} = \mathbf{v}^{(r)} \cdot \mathbf{A}^{(1)2}\mathbf{v}^{(r)} = \sum_{i=1}^{3} v_i^{(r)2}\lambda_i^2\,,\\
&r = 1, 2, \ldots, m\,,
\end{aligned} \tag{31}$$

$$\begin{aligned}
&I_{2m+3+6s-5}^{(m,n)} = \operatorname{tr}\mathbf{A}^{(s+1)} = \sum_{i=1}^{3} A_{ii}^{(s+1)}\,,\\
&I_{2m+3+6s-4}^{(m,n)} = \operatorname{tr}(\mathbf{A}^{(s+1)}\mathbf{A}^{(1)}) = \sum_{i=1}^{3} \lambda_i A_{ii}^{(s+1)}\,,\\
&I_{2m+3+6s-3}^{(m,n)} = \operatorname{tr}(\mathbf{A}^{(s+1)}\mathbf{A}^{(1)2})\\
&= \sum_{i=1}^{3} \lambda_i^2 A_{ii}^{(s+1)}\,,
\end{aligned} \tag{32}$$

$I^{(m,n)}_{2m+3+6s-2} = \text{tr}(\mathbf{A}^{(s+1)2}) = A^{(s+1)2}_{11} + A^{(s+1)2}_{22} +$
$A^{(s+1)2}_{33} + 2(A^{(s+1)2}_{12} + A^{(s+1)2}_{13} + A^{(s+1)2}_{23})\,,$
$I^{(m,n)}_{2m+3+6s-1} = \text{tr}(\mathbf{A}^{(2)2}\mathbf{A}^{(1)}) = \lambda_1(A^{(s+1)2}_{11} +$
$A^{(s+1)2}_{12} + A^{(s+1)2}_{13}) + \lambda_2(A^{(s+1)2}_{21} + A^{(s+1)2}_{22} +$
$A^{(s+1)2}_{23}) + \lambda_3(A^{(s+1)2}_{31} + A^{(s+1)2}_{32} + A^{(s+1)2}_{33})\,,$
$I^{(m,n)}_{2m+3+6s} = \text{tr}(\mathbf{A}^{(2)2}\mathbf{A}^{(1)2}) = \lambda_1^2(A^{(s+1)2}_{11} +$
$A^{(s+1)2}_{12} + A^{(s+1)2}_{13}) + \lambda_2^2(A^{(s+1)2}_{21} + A^{(s+1)2}_{22} +$
$A^{(s+1)2}_{23}) + \lambda_3^2(A^{(s+1)2}_{31} + A^{(s+1)2}_{32} + A^{(s+1)2}_{33})\,,$
$s = 1, 2 \ldots n-1\,, \quad n > 1\,.$ (33)

From (8) and (30), λ_i is expressed in terms of $I^{(m,n)}_i$. From (6), with $r = 1, 2, \ldots, m$, and (31) we have $3m$ linear equations in $v^{(r)2}_i$. On solving the 3 linear equations for each r, we can explicitly express, for all $v^{(r)2}_i$, in terms of $I^{(m,n)}_\alpha\,, \alpha = 1, 2\ldots, 2m+3$. The invariants $A^{(s+1)}_{ii}, s = 1, 2, \ldots, n-1$ appear linearly in (32). On solving the 3 linear equations in (32) for each s, we can express $A^{(s+1)}_{ii}$ explicitly in terms of $I^{(m,n)}_\alpha\,, \alpha = 1, 2, 3$ and $\alpha = 2m+3+6s-5, 2m+3+6s-4, 2m+3+6s-3$. The invariants $A^{(s+1)2}_{ij}\,, i \neq j$ appear linearly in the 3 equations given by (33). On solving these equations for each s, we can explicitly express $A^{(s+1)2}_{ij}$ in terms of $I^{(m,n)}_\alpha\,, \alpha = 1, 2, 3, 2m+4, \ldots, 2m+6n-3$. The remaining classical invariants $I^{(m,n)}_{2m+6n-3+\alpha}\,, \alpha = 1, 2, \ldots$ can be expressed explicitly in terms of $\lambda_i\,, v^{(r)}_i\; A^{(s+1)}_{ij}$. Using the appropriate sign for $v^{(r)}$ and $A^{(s+1)}_{ij}\,, i \neq j$, we can express the remaining classical invariants explicitly in terms of the independent invariants $I^{(m,n)}_\alpha\,, \alpha = 1, 2\ldots, 2m+6n-3$.

Note that for $S(m,1)$, we have $2m+3$ independent invariants, which concurs with the result of Shariff [10]. However, Shariff did not give classical invariant relations in his work [10]; the relations above for $S(m,1)$ supplement the results of [10]. In the case of $S(0,n)$, we obtain $6n-3$ independent invariants; this agrees with the result of Shariff [15], however, the relation forms in [15] are different from the above.

Appendix A: P-property

The description of the P-property uses the eigenvalues λ_i and eigenvectors $\boldsymbol{u}_i$ of the symmetric tensor $\mathbf{A}^{(1)}$. A general anisotropic invariant, where its arguments are expressed in terms spectral invariants with respect to the basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ can be written in the form

$$\begin{aligned}\Phi &= W(\lambda_i, \boldsymbol{u}_i \cdot \mathbf{A}^{(s)}\boldsymbol{u}_j, \boldsymbol{u}_i \cdot \mathbf{v}^{(r)}) \\ &= \tilde{W}(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)\,,\end{aligned} \tag{A1}$$

where

$$r = 1, 2, \ldots, m, \,, \quad s = 2, \ldots, m, \tag{A2}$$

and, in Eqn. $(\text{A1})_2$, the appearance of $\mathbf{A}^{(s)}$ and $\mathbf{v}^{(r)}$ is suppressed to facilitate the description of the P-property. In view of $(4)_1$, $\tilde{W}$ must satisfy the symmetrical property

$$\begin{aligned}&\tilde{W}(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) = \tilde{W}(\lambda_2, \lambda_1, \lambda_3, \boldsymbol{u}_2, \boldsymbol{u}_1, \boldsymbol{u}_3) \\ &= \tilde{W}(\lambda_3, \lambda_2, \lambda_1, \boldsymbol{u}_3, \boldsymbol{u}_2, \boldsymbol{u}_1)\,.\end{aligned} \tag{A3}$$

In view of the non-unique values of $\boldsymbol{u}_i$ and $\boldsymbol{u}_j$ when $\lambda_i = \lambda_j$, a function $\tilde{W}$ should be independent of $\boldsymbol{u}_i$ and $\boldsymbol{u}_j$ when $\lambda_i = \lambda_j$, and $\tilde{W}$ should be independent of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ when $\lambda_1 = \lambda_2 = \lambda_3$. Hence, when two or three of the principal stretches have equal values the scalar function Φ must have any of the following forms

$$\Phi = \begin{cases} W_{(a)}(\lambda, \lambda_k, \boldsymbol{u}_k)\,, & \lambda_i = \lambda_j = \lambda\,, i \neq j \neq k \neq i \\ W_{(b)}(\lambda)\,, & \lambda_1 = \lambda_2 = \lambda_3 = \lambda \end{cases}$$

For example, consider

$$\Phi = \boldsymbol{a} \bullet \mathbf{A}^{(1)}\boldsymbol{a} = \sum_{i=1}^{3} \lambda_i (\boldsymbol{a} \bullet \boldsymbol{u}_i)^2\,, \tag{A4}$$

where $\boldsymbol{a}$ is a fixed unit vector and

$$\sum_{i=1}^{3} (\boldsymbol{a} \bullet \boldsymbol{u}_i)^2 = 1\,. \tag{A5}$$

. If

$$\lambda_1 = \lambda_2 = \lambda\,, \tag{A6}$$

we have

$$\Phi = W_{(a)}(\lambda, \lambda_3, \boldsymbol{u}_3) = \lambda + (\lambda_3 - \lambda)(\boldsymbol{a} \bullet \boldsymbol{u}_3)^2 \tag{A7}$$

and in the case of $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$,

$$\Phi = W_{(b)}(\lambda) = \lambda\,. \tag{A8}$$

Hence, the invariant (A4) satisfies the P-property and we note that all the classical invariants described in Spencer [1] satisfy the P-property. In reference [33], the P-property described here is extended to non-symmetric tensors such as the two-point deformation tensor $\boldsymbol{F}$.

Appendix B: Functional and integrity bases

Let $J_1, \ldots, J_r$ be isotropic invariants of the set $S(m,n)$ given in (1).

1) A single-valued function of $J_1, \ldots, J_p$

$$f(S(m,n)) = g(J_1, , \ldots J_q) \tag{B1}$$

is called a *representation* for isotropic scalar-valued functions of $S(m,n)$. If one of the invariants in the set $\{J_1, \ldots, J_q\}$ can be expressed as a single-valued function of the remainders, the invariant is declared to be *functionally reducible*. We call the representation to be *complete*, if any isotropic scalar-valued function of $S(m,n)$ is expressible in the form (B1). A set of invariants in a complete representation is called a functional basis. A functional basis is declared to be *irreducible*, if none of its proper subsets is a functional basis.

2) If the function $f(S(m,n))$ is constrained to polynomial functions, then we are dealing with integrity bases. A *reducible* polynomial invariant is an invariant that can be expressed as a *polynomial* in other invariants; otherwise, it is an *irreducible* polynomial invariant. An integrity basis is a polynomial invariant set which has the property that any polynomial scalar function is expressible as a polynomial in members of the given set. A minimal integrity basis is an integrity basis, where none of its proper subset is an integrity basis. *Syzygies* are polynomial relations between invariants, which do not permit any one invariant to be expressed as a polynomial in the remainder.

3) A minimal integrity basis is not necessarily an irreducible functional basis. In general, an irreducible functional basis contains fewer elements than a minimal integrity basis.

APPENDIX C

In this Appendix we only give results for the case of $m = 1$ and $n = 2$ and when $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$. Results when two of the eigenvalues λ_i are not-distinct are not given. Our main intention is to show that at most $2m+6n-3$ invariants are independent and that when the eigenvalues $\mathbf{A}^{(s)}$ are not distinct, the number of independent invariants is less than $2m+6n-3$.

When $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, the eigenvectors $\boldsymbol{u}_i$ are arbitrary. We select $\boldsymbol{u}_i$ to coincide with the eigenvectors of $\mathbf{A}^{(2)}$. Hence,

$$\mathbf{A}^{(1)} = \lambda \boldsymbol{I}\,, \quad \mathbf{A}^{(2)} = \sum_{i=1}^{3} \bar{\lambda}_i \boldsymbol{u}_i \otimes \boldsymbol{u}_i\,. \tag{C1}$$

From Section III-C, we have,

$$\begin{aligned} &I_1^{(1,2)} = 3\lambda\,, \quad I_2^{(1,2)} = 3\lambda^2\,, \quad I_3^{(1,2)} = \lambda^3\,,\\ &I_4^{(1,2)} = \lambda\,, \quad I_5^{(1,2)} = \lambda^2\,, \end{aligned} \tag{C2}$$

$$\begin{aligned} I_6^{(1,2)} &= \sum_{i=1}^{3} \bar{\lambda}_i\,, \quad I_7^{(1,2)} \mathrm{tr}(\mathbf{A}^{(2)}\mathbf{A}^{(1)}) = \lambda \sum_{i=1}^{3} \bar{\lambda}_i\,,\\ I_8^{(1,2)} &= \lambda^2 \sum_{i=1}^{3} \bar{\lambda}_i\,, \end{aligned} \tag{C3}$$

$$\begin{aligned} I_9^{(1,2)} &= \sum_{i=1}^{3} \bar{\lambda}_i^2\,, \quad I_{10}^{(1,2)} = \lambda \sum_{i=1}^{3} \bar{\lambda}_i^2\,,\\ I_{11}^{(1,2)} &= \lambda^2 \sum_{i=1}^{3} \bar{\lambda}_i^2\,, \end{aligned} \tag{C4}$$

$$\begin{aligned} I_{12}^{(1,2)} &= \sum_{i=1}^{3} v_i^{(1)2} \bar{\lambda}_i\,, \quad I_{13}^{(1,2)} = \sum_{i=1}^{3} v_i^{(1)2} \bar{\lambda}_i^2\,,\\ I_{14}^{(1,2)} &= \sum_{i=1}^{3} \bar{\lambda}_i^3\,,\\ I_{15}^{(1,2)} &= \lambda \sum_{i=1}^{3} v_i^{(1)2} \bar{\lambda}_i\,, \quad I_{16}^{(1,2)} = \lambda \sum_{i=1}^{3} v_i^{(1)2} \bar{\lambda}_i^2\,,\\ I_{17}^{(1,2)} &= \lambda^2 \sum_{i=1}^{3} v_i^{(1)2} \bar{\lambda}_i\,,\\ I_{18}^{(1,2)} &= \lambda^2 \sum_{i=1}^{3} v_i^{(1)2} \bar{\lambda}_i^2\,. \end{aligned} \tag{C5}$$

It is clear from (C2) to (C5) that only 6 of the classical invariants are independent and we can consider the independent invariants to be

$$I_1^{(1,2)}\,, \quad I_6^{(1,2)}\,, \quad I_9^{(1,2)}\,, \quad I_{12}^{(1,2)}\,, \quad I_{13}^{(1,2)}\,, \quad I_{14}^{(1,2)}\,. \tag{C6}$$

We also have 6 independent spectral invariants

$$\lambda\,, \quad A_{ii}^{(2)} = \bar{\lambda}_i\,, \quad v_1^{(1)}\,, \quad v_2^{(1)}\,. \tag{C7}$$

The number of independent invariants is further reduced if, for example, $\bar{\lambda}_1 = \bar{\lambda}_2 = \bar{\lambda}_3 = \bar{\lambda}$. In this case, we have,

$$\begin{aligned} &I_6^{(1,2)} = 3\bar{\lambda}\,, \quad I_7^{(1,2)} = 3\lambda\bar{\lambda}\,, \quad I_8^{(1,2)} = 3\lambda^2\bar{\lambda}\,,\\ &I_9^{(1,2)} = 3\bar{\lambda}^2\,, \quad I_{10}^{(1,2)} = 3\lambda\bar{\lambda}^2\,, \end{aligned}$$

$$\begin{aligned} &I_{11}^{(1,2)} = 3\lambda^2\bar{\lambda}^2\,, \quad I_{12}^{(1,2)} = \bar{\lambda}\,, \quad I_{13}^{(1,2)} = \bar{\lambda}^2\,,\\ &I_{14}^{(1,2)} = \bar{\lambda}^3\,, \quad I_{15}^{(1,2)} = \lambda\bar{\lambda}\,, \end{aligned}$$

$$I_{16}^{(1,2)} = \lambda\bar{\lambda}^2\,, \quad I_{17}^{(1,2)} = \lambda^2\bar{\lambda}\,, \quad I_{18}^{(1,2)} = \lambda^2\bar{\lambda}^2\,. \tag{C8}$$

Hence, it is clear from (C8) that the classical invariants are independent of $v_i^{(1)}$ and only 2 of them are independent. We can consider the invariants

$$I_1^{(1,2)}\,, \quad I_6^{(1,2)} \tag{C9}$$

to be the independent invariants. However, the number of independent spectral invariants is 4 and they are

$$\lambda\,, \quad \bar{\lambda}\,, \quad v_1^{(1)}\,, \quad v_2^{(1)}\,. \tag{C10}$$

It is expected that the number of independent classical invariants in (C9) is less than that of spectral invariants in (C10) due to the fact that the classical invariants satisfy the P-property [21] while the spectral invariants are not constrained to satisfy the P-property.

APPENDIX D

The solutions in the main body require the results

$$\begin{pmatrix} 1 & 1 & 1\\ \lambda_1 & \lambda_2 & \lambda_3\\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} \alpha_1\lambda_2\lambda_3 & -\alpha_1(\lambda_2+\lambda_3) & \alpha_1\\ \alpha_2\lambda_1\lambda_3 & -\alpha_2(\lambda_1+\lambda_3) & \alpha_2\\ \alpha_3\lambda_1\lambda_2 & -\alpha_3(\lambda_1+\lambda_2) & \alpha_3 \end{pmatrix} \tag{D1}$$

and

$$\begin{pmatrix} 2 & 2 & 2\\ \lambda_1+\lambda_2 & \lambda_1+\lambda_3 & \lambda_2+\lambda_3\\ \lambda_1^2+\lambda_2^2 & \lambda_1^2+\lambda_3^2 & \lambda_2^2+\lambda_3^2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -\dfrac{\alpha_3\beta_3}{2} & \alpha_3(\lambda_1+\lambda_2) & -\alpha_3\\ -\dfrac{\alpha_2\beta_2}{2} & \alpha_2(\lambda_1+\lambda_3) & -\alpha_2\\ -\dfrac{\alpha_1\beta_1}{2} & \alpha_1(\lambda_2+\lambda_3) & -\alpha_1 \end{pmatrix}, \tag{D2}$$

where

$$\begin{aligned} \alpha_1 &= \frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\,, \quad \alpha_2 = \frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\,,\\ \alpha_3 &= \frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\,,\\ \beta_1 &= \lambda_1\lambda_2+\lambda_1\lambda_3-\lambda_1^2+\lambda_2\lambda_3\,,\\ \beta_2 &= \lambda_1\lambda_2+\lambda_1\lambda_3-\lambda_2^2+\lambda_2\lambda_3\,,\\ \beta_3 &= \lambda_1\lambda_2+\lambda_1\lambda_3-\lambda_3^2+\lambda_2\lambda_3\,. \end{aligned} \tag{D3}$$

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