Numerical Simulation for Unsteady Helmholtz Problems of Anisotropic Functionally Graded Materials

Moh. Ivan Azis*, Ikha Magdalena, Widowati, Amiruddin Haddade

Abstract—**The unsteady anisotropic Helmholtz type equation of spatially varying coefficients is considered in this study. The study is to find numerical solutions to initial boundary value problems governed by the equation. Such problems are relevant for anisotropic functionally graded materials. A mathematical analysis is used to transform the variable coefficient equation to a constant coefficient equation which is then Laplace-transformed (LT) to get a time-independent equation. The latest equation is then written as a boundary-only integral equation of a time-free fundamental solution. A boundary element method (BEM) which is derived from the integral equation and combined with the Stehfest formula for numerical Laplace transform inversion is then employed to find the numerical solutions. Some problems are considered. The combined LT-BEM is easy to implement. The results show that the numerical solutions obtained are accurate.**

Index Terms—**anisotropic functionally graded materials, variable coefficients, parabolic equation, Laplace transform, boundary element method.**

I. Introduction

We will consider initial boundary value problems governed by a parabolic equation with variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\kappa_{ij}(\mathbf{x})\frac{\partial \mu(\mathbf{x},t)}{\partial x_j}\right]+\beta^2(\mathbf{x})\,\mu(\mathbf{x},t)=\alpha(\mathbf{x})\frac{\partial \mu(\mathbf{x},t)}{\partial t} \tag{1}$$

The coefficients $[\kappa_{ij}]$ $(i,j=1,2)$ is a real symmetric positive definite matrix. Also, in (1) the summation convention for repeated indices holds. Therefore equation (1) may be written explicitly as

$$\frac{\partial}{\partial x_1}\left(\kappa_{11}\frac{\partial \mu}{\partial x_1}\right)+\frac{\partial}{\partial x_1}\left(\kappa_{12}\frac{\partial \mu}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(\kappa_{12}\frac{\partial \mu}{\partial x_1}\right)$$
$$+\frac{\partial}{\partial x_2}\left(\kappa_{22}\frac{\partial \mu}{\partial x_2}\right)+\beta^2\mu=\alpha\frac{\partial c}{\partial t}$$

Equation (1) is usually used to model acoustic problems. In equation (1), the coefficient κ_{ij} may represent the conductivity or the diffusivity, β describes the wave number, and α may depict the rate of change. Since the coefficients $\kappa_{ij}(\mathbf{x}), \beta(\mathbf{x}), \alpha(\mathbf{x})$ are spatially continuous functions, then the material under consideration has properties which vary spatially according to a specific continuous function. Such a material is called a functionally graded material (FGM). Specifically, since the coefficients $\kappa_{11}, \kappa_{12}, \kappa_{22}$ may differ then the material is called as an anisotropic material. Therefore equation (1) is relevant for anisotropic FGMs.

During the last decade FGMs have become an important topic, and numerous studies on FGMs for a variety of applications have been reported (see for example Bounouara et al. [1] and Karami et al. [2]). On the other hand, in some applications anisotropy of the material of interest needs to be taken into account. Among other studies that considered material anisotropy have been done by Limberkar et al. [3] in material science application, Daghash et al. [4] in chemical engineering application, and Yusuf [5] in optics application.

Recently a number of authors had been working on the Helmholtz equation to find its solutions. However the works focus mainly on problems of isotropic homogeneous materials. For such kind of materials, the boundary element method (BEM) had been successfully used to find numerical solutions of applications associated to the homogeneous materials. But this is not the case for inhomogeneous materials.

In [6] Ma et al. considered a use of the Galerkin boundary element method for exterior problems of 2-D Helmholtz equation with arbitrary wave number. In this paper the authors assumed the internal and the external domain are *homogeneous* media. Loeffler et al. in [7] investigated numerical solutions for Helmholtz problems. The Helmholtz equation is treated like a Laplace equation with non-zero right-hand side of the equation. The resulting boundary-domain integral equation is then solved using a direct radial basis function interpolation. In this study, the medium is supposed to be a *homogeneous* material. In [8] Barucq et al. also considered *homogeneous* media. The study is focused on the numerical aspect of the BEM used for solving the Helmholtz problems. Similarly, Wu and Alkhalifah in [9] also concerned on the numerical aspect of a finite-difference method for solving the Helmholtz equation of *homogeneous* media. Li et al. [10] obtained numerical solutions of the Helmholtz equation of *homogeneous* media using the method of fundamental solutions with Bessel functions, in replacement of Hankel functions, as the fundamental solutions. Some studies for inhomogeneous media have been done, but they are limited to the class of inhomogeneities which take the form of *constant-plus-variable* functions of inhomogeneity.

Recently Azis and co-workers had been working on steady state problems of *anisotropic inhomogeneous* media for several types of governing equations, for examples [11]–[16] for Helmholtz equation, [17]–[19] for the modified Helmholtz

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equation, [20] for elasticity problems, [21]–[25] for the diffusion convection equation, [26]–[29] for the Laplace type equation, [30]–[36] for the diffusion convection reaction equation. These works considered the case of other classes of inhomogeneities which are different from the class of the *constant-plus-variable* inhomogeneity.

This paper is intended to extend the recently published works in [11]–[16] for steady anisotropic Helmholtz type equation with space variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\kappa_{ij}\left(\mathbf{x}\right)\frac{\partial\mu\left(\mathbf{x},t\right)}{\partial x_j}\right]+\beta^2\left(\mathbf{x}\right)\mu\left(\mathbf{x},t\right)=0$$

to unsteady anisotropic parabolic equation with space variable coefficients of the form (1).

This study is an attempt to solve numerically initial boundary value problems for several other classes of anisotropic FGMs governed by equation (1) using a combined LT-BEM. The analysis of this paper is purely mathematical; the main aim being to construct effective a BEM for the three typical equations which fall within the type (1).

A brief outline of the paper is as follows. Section II defines the initial boundary value problem to be solved. In Section III a boundary-only integral equation is derived. In Section IV several problems (test problems in Section IV-A and problems without analytical solutions in Section IV-B) are solved to primarily show the validity of the analysis used in deriving the boundary-only integral equation in Section III, and to specifically examine the accuracy of the present approach. Finally, Section V concludes this paper with some remarks.

II. The initial-boundary value problem

Referred to a Cartesian frame Ox_1x_2 solutions $\mu\left(\mathbf{x},t\right)$ and its derivatives to (1) are sought which are valid for time interval $t\geq 0$ and in a region Ω in R^2 with boundary $\partial\Omega$ which consists of a finite number of piecewise smooth closed curves. On $\partial\Omega_1$ the dependent variable $\mu\left(\mathbf{x},t\right)$ ($\mathbf{x}=(x_1,x_2)$) is specified and on $\partial\Omega_2$

$$P\left(\mathbf{x},t\right)=\kappa_{ij}\left(\mathbf{x}\right)\frac{\partial\mu\left(\mathbf{x},t\right)}{\partial x_i}n_j \tag{2}$$

is specified where $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2$ and $\mathbf{n}=(n_1,n_2)$ denotes the outward pointing normal to $\partial\Omega$. The initial condition is taken to be

$$\mu\left(\mathbf{x},0\right)=0 \tag{3}$$

The method of solution will be to transform the variable coefficient equation (1) to a constant coefficient equation, and then taking a Laplace transform of the constant coefficient equation, and to obtain a boundary integral equation in the Laplace transform variable s. The boundary integral equation is then solved using a standard boundary element method (BEM). An inverse Laplace transform is taken to get the solution c and its derivatives for all $(\mathbf{x},t)$ in the domain. The inverse Laplace transform is implemented numerically using the Stehfest formula.

The analysis is specially relevant to an anisotropic medium but it equally applies to isotropic media. For isotropy, the coefficients in (1) take the form $\kappa_{11}=\kappa_{22}$ and $\kappa_{12}=0$ and use of these equations in the following analysis immediately yields the corresponding results for an isotropic medium. The analysis also applies for homogeneous materials which occur when the coefficients κ_{ij},β^2 and α are constant.

III. The boundary integral equation

The coefficients $\kappa_{ij},\beta^2,\alpha$ are required to take the form

$$\kappa_{ij}\left(\mathbf{x}\right)=\overline{\kappa}_{ij}g(\mathbf{x}) \tag{4}$$

$$\beta^2\left(\mathbf{x}\right)=\overline{\beta}^2g(\mathbf{x}) \tag{5}$$

$$\alpha\left(\mathbf{x}\right)=\overline{\alpha}g(\mathbf{x}) \tag{6}$$

where the $\overline{\kappa}_{ij},\overline{\beta}^2,\overline{\alpha}$ are constants and g is a differentiable function of $\mathbf{x}$. Use of (4)-(6) in (1) yields

$$\overline{\kappa}_{ij}\frac{\partial}{\partial x_i}\left(g\frac{\partial\mu}{\partial x_j}\right)+\overline{\beta}^2g\mu=\overline{\alpha}g\frac{\partial\mu}{\partial t} \tag{7}$$

Let

$$\mu\left(\mathbf{x},t\right)=g^{-1/2}\left(\mathbf{x}\right)\psi\left(\mathbf{x},t\right) \tag{8}$$

therefore substitution of (4) and (8) into (2) gives

$$P\left(\mathbf{x},t\right)=-P_g\left(\mathbf{x}\right)\psi\left(\mathbf{x},t\right)+g^{1/2}\left(\mathbf{x}\right)P_\psi\left(\mathbf{x},t\right) \tag{9}$$

where

$$P_g\left(\mathbf{x}\right)=\overline{\kappa}_{ij}\frac{\partial g^{1/2}}{\partial x_j}n_i \qquad P_\psi\left(\mathbf{x}\right)=\overline{\kappa}_{ij}\frac{\partial\psi}{\partial x_j}n_i$$

Also, (7) may be written in the form

$$\overline{\kappa}_{ij}\frac{\partial}{\partial x_i}\left[g\frac{\partial\left(g^{-1/2}\psi\right)}{\partial x_j}\right]+\overline{\beta}^2g^{1/2}\psi=\overline{\alpha}g\frac{\partial\left(g^{-1/2}\psi\right)}{\partial t}$$

which can be simplified

$$\overline{\kappa}_{ij}\frac{\partial}{\partial x_i}\left(g^{1/2}\frac{\partial\psi}{\partial x_j}+g\psi\frac{\partial g^{-1/2}}{\partial x_j}\right)+\overline{\beta}^2g^{1/2}\psi=\overline{\alpha}g^{1/2}\frac{\partial\psi}{\partial t}$$

Use of the identity

$$\frac{\partial g^{-1/2}}{\partial x_i}=-g^{-1}\frac{\partial g^{1/2}}{\partial x_i}$$

implies

$$\overline{\kappa}_{ij}\frac{\partial}{\partial x_i}\left(g^{1/2}\frac{\partial\psi}{\partial x_j}-\psi\frac{\partial g^{1/2}}{\partial x_j}\right)+\overline{\beta}^2g^{1/2}\psi=\overline{\alpha}g^{1/2}\frac{\partial\psi}{\partial t}$$

Rearranging and neglecting some zero terms gives

$$g^{1/2}\overline{\kappa}_{ij}\frac{\partial^2\psi}{\partial x_i\partial x_j}-\psi\overline{\kappa}_{ij}\frac{\partial^2g^{1/2}}{\partial x_i\partial x_j}+\overline{\beta}^2g^{1/2}\psi=\overline{\alpha}g^{1/2}\frac{\partial\psi}{\partial t}$$

It follows that if g is such that

$$\overline{\kappa}_{ij}\frac{\partial^2g^{1/2}}{\partial x_i\partial x_j}-\lambda g^{1/2}=0 \tag{10}$$

where λ is a constant, then the transformation (8) carries the variable coefficients equation (7) to the constant coefficients equation

$$\overline{\kappa}_{ij}\frac{\partial^2\psi}{\partial x_i\partial x_j}+\left(\overline{\beta}^2-\lambda\right)\psi=\overline{\alpha}\frac{\partial\psi}{\partial t} \tag{11}$$

Taking the Laplace transform of (8), (9), (11) and applying the initial condition (3) we obtain

$$\psi^*\left(\mathbf{x},s\right)=g^{1/2}\left(\mathbf{x}\right)\mu^*\left(\mathbf{x},s\right) \tag{12}$$

$$P_{\psi^*}\left(\mathbf{x},s\right)=\left[P^*\left(\mathbf{x},s\right)+P_g\left(\mathbf{x}\right)\psi^*\left(\mathbf{x},s\right)\right]\times g^{-1/2}\left(\mathbf{x}\right) \tag{13}$$

$$\overline{\kappa}_{ij}\frac{\partial^2\psi^*}{\partial x_i\partial x_j}+\left(\overline{\beta}^2-s\overline{\alpha}-\lambda\right)\psi^*=0 \quad (14)$$

where s is the variable of the Laplace-transformed domain.

A boundary integral equation for the solution of (14) is given in the form

$$\eta\left(\mathbf{x}_0\right)\psi^*\left(\mathbf{x}_0,s\right)=\int_{\partial\Omega}\left[\Gamma\left(\mathbf{x},\mathbf{x}_0\right)\psi^*\left(\mathbf{x},s\right)-\right.$$
$$\left.\Phi\left(\mathbf{x},\mathbf{x}_0\right)P_{\psi^*}\left(\mathbf{x},s\right)\right]dS\left(\mathbf{x}\right) \quad (15)$$

where $\mathbf{x}_0=(a,b)$, $\eta=0$ if $(a,b)\notin\Omega\cup\partial\Omega$, $\eta=1$ if $(a,b)\in\Omega$, $\eta=\frac{1}{2}$ if $(a,b)\in\partial\Omega$ and $\partial\Omega$ has a continuously turning tangent at (a,b). The so called fundamental solution Φ in (15) is any solution of the equation

$$\overline{\kappa}_{ij}\frac{\partial^2\Phi}{\partial x_i\partial x_j}+\left(\overline{\beta}^2-s\overline{\alpha}-\lambda\right)\Phi=\delta\left(\mathbf{x}-\mathbf{x}_0\right)$$

and the Γ is given by

$$\Gamma\left(\mathbf{x},\mathbf{x}_0\right)=\overline{\kappa}_{ij}\frac{\partial\Phi\left(\mathbf{x},\mathbf{x}_0\right)}{\partial x_j}n_i$$

where δ is the Dirac delta function. For two-dimensional problems Φ and Γ are given by

$$\begin{aligned}
\Phi\left(\mathbf{x},\mathbf{x}_0\right) &= \begin{cases}\frac{K}{2\pi}\ln R & \text{if } \overline{\beta}^2-s\overline{\alpha}-\lambda=0\\ \frac{\imath K}{4}H_0^{(2)}\left(\omega R\right) & \text{if } \overline{\beta}^2-s\overline{\alpha}-\lambda>0\\ \frac{-K}{2\pi}K_0\left(\omega R\right) & \text{if } \overline{\beta}^2-s\overline{\alpha}-\lambda<0\end{cases}\\
\Gamma\left(\mathbf{x},\mathbf{x}_0\right) &= \begin{cases}\frac{K}{2\pi}\frac{1}{R}\overline{\kappa}_{ij}\frac{\partial R}{\partial x_j}n_i\\ \frac{-\imath K\omega}{4}H_1^{(2)}\left(\omega R\right)\overline{\kappa}_{ij}\frac{\partial R}{\partial x_j}n_i\\ \frac{K\omega}{2\pi}K_1\left(\omega R\right)\overline{\kappa}_{ij}\frac{\partial R}{\partial x_j}n_i\end{cases}\\
&\quad\begin{cases}\text{if } \overline{\beta}^2-s\overline{\alpha}-\lambda=0\\ \text{if } \overline{\beta}^2-s\overline{\alpha}-\lambda>0\\ \text{if } \overline{\beta}^2-s\overline{\alpha}-\lambda<0\end{cases}
\end{aligned} \quad (16)$$

where

$$\begin{aligned}
K &= \ddot{\tau}/D\\
\omega &= \sqrt{|\overline{\beta}^2-s\overline{\alpha}-\lambda|/D}\\
D &= \left[\overline{\kappa}_{11}+2\overline{\kappa}_{12}\dot{\tau}+\overline{\kappa}_{22}\left(\dot{\tau}^2+\ddot{\tau}^2\right)\right]/2\\
R &= \sqrt{(\dot{x}_1-\dot{a})^2+(\dot{x}_2-\dot{b})^2}\\
\dot{x}_1 &= x_1+\dot{\tau}x_2\\
\dot{a} &= a+\dot{\tau}b\\
\dot{x}_2 &= \ddot{\tau}x_2\\
\dot{b} &= \ddot{\tau}b
\end{aligned}$$

where $\dot{\tau}$ and $\ddot{\tau}$ are respectively the real and the positive imaginary parts of the complex root τ of the quadratic

$$\overline{\kappa}_{11}+2\overline{\kappa}_{12}\tau+\overline{\kappa}_{22}\tau^2=0$$

and $H_0^{(2)}$, $H_1^{(2)}$ denote the Hankel function of second kind and order zero and order one respectively. K_0, K_1 denote the modified Bessel function of order zero and order one respectively, $\imath$ represents the square root of minus one. The derivatives $\partial R/\partial x_j$ needed for the calculation of the Γ in (16) are given by

$$\begin{aligned}
\frac{\partial R}{\partial x_1} &= \frac{1}{R}\left(\dot{x}_1-\dot{a}\right)\\
\frac{\partial R}{\partial x_2} &= \dot{\tau}\left[\frac{1}{R}\left(\dot{x}_1-\dot{a}\right)\right]+\ddot{\tau}\left[\frac{1}{R}\left(\dot{x}_2-\dot{b}\right)\right]
\end{aligned}$$

Use of (12) and (13) in (15) yields

$$\eta g^{1/2}\mu^*=\int_{\partial\Omega}\left[\left(g^{1/2}\Gamma-P_g\Phi\right)\mu^*-\left(g^{-1/2}\Phi\right)P^*\right]dS \quad (17)$$

This equation provides a boundary integral equation for determining μ^* and its derivatives at all points of Ω.

Knowing the solutions $\mu^*\left(\mathbf{x},s\right)$ and its derivatives $\partial\mu^*/\partial x_1$ and $\partial\mu^*/\partial x_2$ which are obtained from (17), the numerical Laplace transform inversion technique using the Stehfest formula is then employed to find the values of $\mu\left(\mathbf{x},t\right)$ and its derivatives $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$. The Stehfest formula is

$$\begin{aligned}
\mu\left(\mathbf{x},t\right) &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N}V_m\mu^*\left(\mathbf{x},s_m\right)\\
\frac{\partial\mu\left(\mathbf{x},t\right)}{\partial x_1} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N}V_m\frac{\partial\mu^*\left(\mathbf{x},s_m\right)}{\partial x_1}\\
\frac{\partial\mu\left(\mathbf{x},t\right)}{\partial x_2} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N}V_m\frac{\partial\mu^*\left(\mathbf{x},s_m\right)}{\partial x_2}
\end{aligned} \quad (18)$$

where

$$\begin{aligned}
s_m &= \frac{\ln 2}{t}m\\
V_m &= (-1)^{\frac{N}{2}+m}\\
&\quad\sum_{k=\left[\frac{m+1}{2}\right]}^{\min\left(m,\frac{N}{2}\right)}\frac{k^{N/2}\left(2k\right)!}{\left(\frac{N}{2}-k\right)!k!\left(k-1\right)!\left(m-k\right)!\left(2k-m\right)!}
\end{aligned}$$

Possible multi-parameter solution $g^{1/2}\left(\mathbf{x}\right)$ to (10)

$$\begin{aligned}
g^{1/2}\left(\mathbf{x}\right) &= \begin{cases}\cos\left(c_0+c_ix_i\right)+\sin\left(c_0+c_ix_i\right)\\ \exp\left(c_0+c_ix_i\right)\\ c_0+c_ix_i\end{cases}\\
&\quad\begin{cases}\overline{\kappa}_{ij}c_ic_j+\lambda=0,\lambda\neq 0\\ \overline{\kappa}_{ij}c_ic_j-\lambda=0,\lambda\neq 0\\ \lambda=0\end{cases}
\end{aligned} \quad (19)$$

Specifically, the quadratic inhomogeneity function $g\left(\mathbf{x}\right)=\left(c_0+c_ix_i\right)^2$ in (19) can be written in the form of a sum of a constant and a variable terms as $g\left(\mathbf{x}\right)=c_0^2+\left(2c_0c_ix_i+c_i^2x_i^2\right)$ so that the coefficients $\kappa_{ij}\left(\mathbf{x}\right),\beta^2\left(\mathbf{x}\right),\alpha\left(\mathbf{x}\right)$ fall within the class of constant-plus-variable coefficients. However, the trigonometric inhomogeneity functions can not be written in a simple form of a sum of a constant and a variable terms.

IV. Numerical examples

Some particular problems will be solved numerically by employing the integral equation (17). The main aim is to show the validity of the analysis for deriving the boundary integral equation (17) and the appropriateness of the BEM and Stehfest formula in solving the problems defined in Section II through the derived boundary integral equation (17).

For all problems considered, the function $g(\mathbf{x})$ is required to satisfy equation (10). We assume each problem belongs to a system which is valid in given spatial and time domains. The characteristics of the system which are represented by the coefficients $\kappa_{ij}\left(\mathbf{x}\right),\beta^2\left(\mathbf{x}\right),\alpha\left(\mathbf{x}\right)$ in equation (1) are assumed to be of the form (4), (5) and (6).

TABLE I
VALUES OF V_m OF THE STEHFEST FORMULA FOR $N = 4, 6, 8, 10$

V_m	$N=4$	$N=6$	$N=8$	$N=10$
V_1	-2	1	$-1/3$	$1/12$
V_2	26	-49	$145/3$	$-385/12$
V_3	-48	366	-906	1279
V_4	24	-858	$16394/3$	$-46871/3$
V_5		810	$-43130/3$	$505465/6$
V_6		-270	18730	-236957.5
V_7			$-35840/3$	$1127735/3$
V_8			$8960/3$	$-1020215/3$
V_9				164062.5
V_{10}				-32812.5

Standard BEM with constant elements is employed to obtain numerical results. And the value of N in (18) for the Stehfest formula is chosen to be $N = 10$. For all problems considered, a unit square (depicted in Figure 1) will be taken as the domain, and the boundary of the domain is divided into 320 elements of the same length, that is 80 elements for each side of the unit square, and the time domain is $0 \leq t \leq 5$. The integral on each element is evaluated numerically using the Bode's quadrature. A FORTRAN code is developed to compute the solutions, and a specific FORTRAN command is imposed to calculate the elapsed CPU time for obtaining the results. A simple script is also developed and embedded into the main FORTRAN code to calculate the values of the coefficients $V_m, m = 1, 2, \ldots, N$ for any number N. Table (I) shows the values of V_m for $N = 4, 6, 8, 10$ which are obtained from the script.

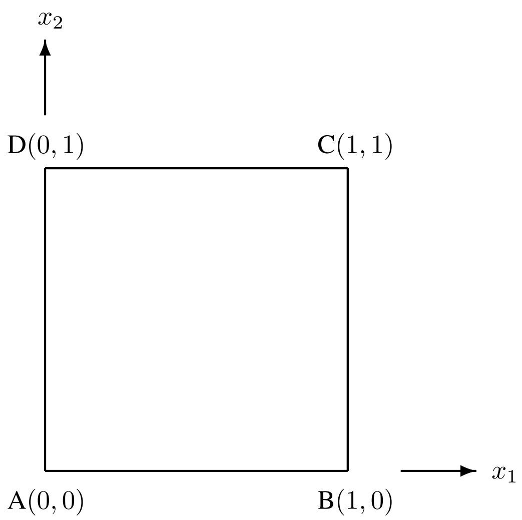

Fig. 1. The domain Ω

A. Problem 1: Examples with analytical solutions

In order to see the accuracy of the BEM and the Stehfest formula we will consider some problems with analytical solutions. The examples are also aimed to show the steady state solution. The analytical solutions are assumed to take a separable variables form

$$\mu(\mathbf{x}, t) = g^{-1/2}(\mathbf{x})\, h(\mathbf{x})\, f(t)$$

where the functions $h(\mathbf{x}), g^{1/2}(\mathbf{x}), f(t)$ take the forms

$$\begin{aligned} h(\mathbf{x}) &= 1 - 0.15x_1 - 0.85x_2 \\ h(\mathbf{x}) &= \cos(-1 + 0.15x_1 + 0.85x_2) \\ h(\mathbf{x}) &= \exp(-1 + 0.15x_1 + 0.85x_2) \end{aligned}$$

$$\begin{aligned} g^{1/2}(\mathbf{x}) &= \cos(-0.85 + 0.35x_1 + 0.4x_2) \\ g^{1/2}(\mathbf{x}) &= \exp(-0.85 + 0.35x_1 + 0.4x_2) \\ g^{1/2}(\mathbf{x}) &= 0.85 - 0.35x_1 - 0.4x_2 \end{aligned}$$

$$\begin{aligned} f(t) &= 1 - \exp(-1.25t) \\ f(t) &= t/5 \\ f(t) &= 0.16t\,(5 - t) \end{aligned}$$

Also, we take a mutual constant coefficient $\overline{\kappa}_{ij}$ for the problems

$$\overline{\kappa}_{ij} = \begin{bmatrix} 1 & 0.15 \\ 0.15 & 0.75 \end{bmatrix} \quad \overline{\beta}^2 = 0.1$$

and a mutual set of boundary conditions (see Figure 1)

μ is given on side AB, BC, CD
P is given on side AD

1) Case 1 (trigonometrically graded material): We assume the inhomogeneity function $g(\mathbf{x})$ is a trigonometric function

$$g(\mathbf{x}) = [\cos(-0.85 + 0.35x_1 + 0.4x_2)]^2$$

so that the medium under consideration is a trigonometrically graded material. The time variation function is

$$f(t) = 0.16t\,(5 - t)$$

For $g(\mathbf{x})$ to satisfy (19)

$$\lambda = -0.2845$$

We take

$$h(\mathbf{x}) = 1 - 0.15x_1 - 0.85x_2$$

so that in order for $h(\mathbf{x})$ to satisfy (14) with $\overline{\beta}^2 - s\overline{\alpha} - \lambda = 0$ (as to use the Laplace fundamental solution in (16)) then

$$\overline{\alpha} = 0.3845/s$$

2) Case 2 (exponentially graded material): Now we consider an exponentially graded material with inhomogeneity function $g(\mathbf{x})$ of the form

$$g(\mathbf{x}) = [\exp(-0.85 + 0.35x_1 + 0.4x_2)]^2$$

so that from (19)

$$\lambda = 0.2845$$

The time variation function is

$$f(t) = 1 - \exp(-1.25t)$$

We assume

$$h(\mathbf{x}) = \cos(-1 + 0.15x_1 + 0.85x_2)$$

so that in order for $h(\mathbf{x})$ to satisfy (14) with $\overline{\beta}^2 - s\overline{\alpha} - \lambda = 0.602625 > 0$ (as to use the Helmholtz fundamental solution in (16))

$$\overline{\alpha} = -0.787125/s$$

3) Case 3 (quadratically graded material): We assume that the material is quadratically graded, with function of gradation

$$g(\mathbf{x}) = (0.85 - 0.35x_1 - 0.4x_2)^2$$

so that from (19)

$$\lambda = 0$$

The time variation function is

$$f(t) = t/5$$

We suppose

$$h(\mathbf{x}) = \exp(-1 + 0.15x_1 + 0.85x_2)$$

so that in order for $h(\mathbf{x})$ to satisfy (14) with $\overline{\beta}^2 - s\overline{\alpha} - \lambda = -0.602625 < 0$ (as to use the modified Helmholtz fundamental solution in (16))

$$\overline{\alpha} = 0.702625/s$$

The results for Problem 1 are shown in Table II and Figure 2. Table II shows the accuracy of the numerical solutions μ and the derivatives $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$ solutions in the domain. For the Case 1, 2 and 3 the errors mainly occur in the fourth decimal place for the $\mu, \partial\mu/\partial x_1, \partial\mu/\partial x_2$ solutions. Figure 2 shows a variation of the μ solution values at some interior points as the time increases from $t = 0.0005$ to $t = 5$. As expected, the variation follows the way the associated function $f(t)$ changes. Specifically for the Case 2 of associated function $f(t) = 1 - \exp(-1.25t)$ the μ solution will tend to approach a steady state solution. This is also expected, as the function $f(t) = 1 - \exp(-1.25t)$ will converge to 1 as t gets bigger. The elapsed CPU time for the computation of the numerical solutions at 19×19 spatial positions and 11 time steps from $t = 0.0005$ to $t = 5$ is 451.171875 seconds for the Case 1, 642.40625 seconds for the Case 2, and 319.078125 seconds for the Case 3.

TABLE II
COMPARISON OF THE NUMERICAL (NUM) AND THE ANALYTICAL (ANAL) SOLUTIONS AT $(x_1, x_2) = (0.5, 0.5)$ FOR PROBLEM 1

t	μ		$\frac{\partial\mu}{\partial x_1}$		$\frac{\partial\mu}{\partial x_2}$	
	Num	Anal	Num	Anal	Num	Anal
Case 1						
0.0005	0.0002	0.0002	-0.0001	-0.0001	-0.0004	-0.0004
0.5	0.2024	0.2024	-0.0972	-0.0972	-0.3859	-0.3857
1.0	0.3598	0.3598	-0.1726	-0.1727	-0.6860	-0.6857
1.5	0.4723	0.4723	-0.2267	-0.2267	-0.9005	-0.9000
2.0	0.5398	0.5398	-0.2590	-0.2591	-1.0291	-1.0286
2.5	0.5622	0.5622	-0.2699	-0.2699	-1.0720	-1.0715
3.0	0.5399	0.5398	-0.2592	-0.2591	-1.0292	-1.0286
3.5	0.4724	0.4723	-0.2268	-0.2267	-0.9007	-0.9000
4.0	0.3599	0.3598	-0.1725	-0.1727	-0.6863	-0.6857
4.5	0.2026	0.2024	-0.0974	-0.0972	-0.3862	-0.3857
5.0	0.0002	0.0000	-0.0003	-0.0000	-0.0004	-0.0000
Case 2						
0.0005	0.0009	0.0009	-0.0002	-0.0002	0.0001	0.0001
0.5	0.6558	0.6558	-0.1757	-0.1758	0.0423	0.0422
1.0	1.0069	1.0069	-0.2698	-0.2699	0.0650	0.0648
1.5	1.1944	1.1948	-0.3200	-0.3203	0.0771	0.0769
2.0	1.2946	1.2953	-0.3469	-0.3472	0.0836	0.0834
2.5	1.3484	1.3492	-0.3614	-0.3616	0.0871	0.0868
3.0	1.3776	1.3780	-0.3692	-0.3694	0.0890	0.0887
3.5	1.3934	1.3934	-0.3734	-0.3735	0.0900	0.0897
4.0	1.4020	1.4017	-0.3757	-0.3757	0.0905	0.0902
4.5	1.4066	1.4061	-0.3769	-0.3769	0.0908	0.0905
5.0	1.4091	1.4084	-0.3776	-0.3775	0.0910	0.0906
Case 3						
0.0005	0.0001	0.0001	0.0001	0.0001	0.0002	0.0002
0.5	0.1277	0.1277	0.1133	0.1132	0.2162	0.2161
1.0	0.2555	0.2554	0.2265	0.2265	0.4323	0.4321
1.5	0.3831	0.3831	0.3397	0.3397	0.6484	0.6482
2.0	0.5108	0.5108	0.4530	0.4530	0.8646	0.8643
2.5	0.6387	0.6385	0.5663	0.5662	1.0809	1.0803
3.0	0.7662	0.7661	0.6793	0.6794	1.2969	1.2964
3.5	0.8940	0.8938	0.7927	0.7927	1.5132	1.5125
4.0	1.0217	1.0215	0.9057	0.9059	1.7293	1.7285
4.5	1.1494	1.1492	1.0191	1.0192	1.9454	1.9446
5.0	1.2771	1.2769	1.1323	1.1324	2.1618	2.1607

B. Problem 2: Examples without analytical solutions

The aim is to show the effect of inhomogeneity and anisotropy of the considered medium to the solution μ. The medium is supposed to be an anisotropic or isotropic, and inhomogeneous (functionally graded) or homogeneous material. For all combinations of the material's anisotropy and inhomogeneity (isotropic homogeneous, isotropic inhomogeneous, anisotropic homogeneous, anisotropic inhomogeneous) we choose

$$\overline{\beta}^2 = 0.1 \quad \overline{\alpha} = 1$$

and a common set of boundary conditions that

$$\begin{aligned} &P = f(t) \text{ on side AB} \\ &P = 0 \text{ on side BC} \\ &\mu = 0 \text{ on side CD} \\ &P = 0 \text{ on side AD} \end{aligned}$$

where the function $f(t)$ is defined as one of the following two forms

$$\begin{aligned} f(t) = f_1(t) &= 1 \\ f(t) = f_2(t) &= 1 - \exp(-1.25t) \end{aligned}$$

If the material is anisotropic then the constant coefficient $\overline{\kappa}_{ij}$ is

$$\overline{\kappa}_{ij} = \begin{bmatrix} 1 & 0.15 \\ 0.15 & 0.75 \end{bmatrix}$$

and

$$\overline{\kappa}_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

when it is isotropic. Regarding its inhomogeneity, three cases of the material will be considered, namely a trigonometrically, exponentially and quadratically graded material.

1) Case 1: The medium is supposed to be a trigonometrically graded material with

$$g^{1/2}(\mathbf{x}) = \cos(-0.85 + 0.35x_1 + 0.4x_2)$$

or it is homogeneous with

$$g^{1/2}(\mathbf{x}) = 1$$

So that if the material is anisotropic inhomogeneous then $\lambda = -0.2845$, and $\lambda = -0.2825$ when it is isotropic inhomogeneous, and $\lambda = 0$ when it is homogeneous.

2) Case 2: The medium is supposed to be a exponentially graded material with

$$g^{1/2}(\mathbf{x}) = \exp(-0.85 + 0.35x_1 + 0.4x_2)$$

or it is homogeneous with

$$g^{1/2}(\mathbf{x}) = 1$$

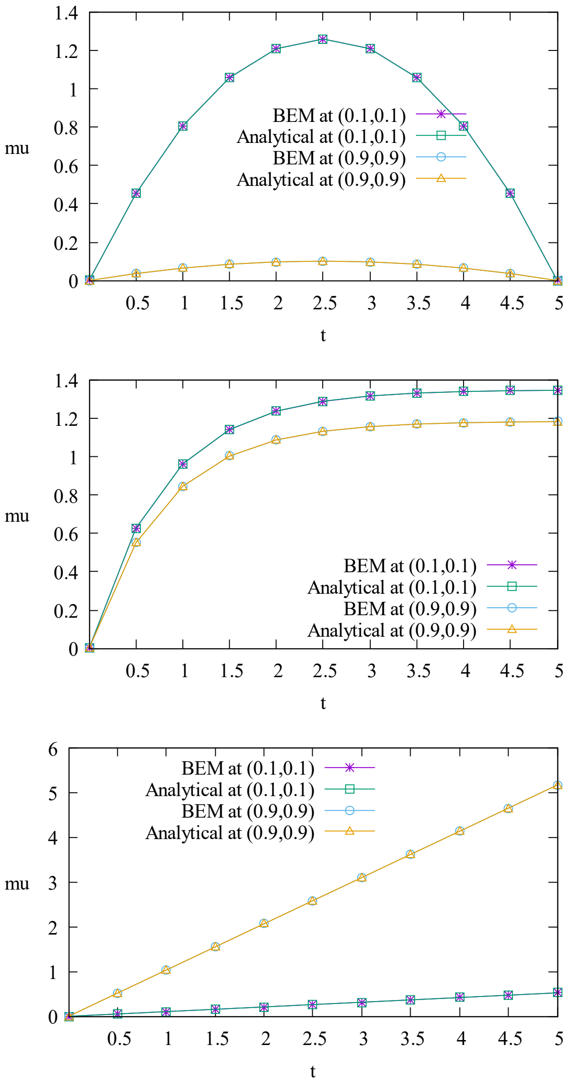

Fig. 2. Solutions μ at some interior points (x_1, x_2) for the Case 1 (top), Case 2 (center) and Case 3 (bottom) of Problem 1

So that $\lambda = 0.2845$ if the material is anisotropic inhomogeneous, and $\lambda = 0.2825$ when it is isotropic inhomogeneous, and $\lambda = 0$ when it is homogeneous.

3) Case 3: The medium is supposed to be a quadratically graded material with

$$g^{1/2}(\mathbf{x}) = 0.85 - 0.35x_1 - 0.4x_2$$

or it is homogeneous with

$$g^{1/2}(\mathbf{x}) = 1$$

So that $\lambda = 0$ for all combinations of the material's anisotropy and inhomogeneity.

The results for Problem 2 are shown in Figures 3, 4, 5 and 6. The figures show solutions μ at points $(x_1, x_2) = (0.3, 0.5), (0.7, 0.5)$ and at time $t = 0.25, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5$ for all four combinations of anisotropy and inhomogeneity, all cases (Case 1: trigonometrically graded material, Case 2: exponentially graded material, Case 3: quadratically graded material), both types of functions $f_1(t) = 1, f_2(t) = 1 - \exp(-1.25t)$.

When the material under consideration is homogeneous the problems for all Cases 1, 2, 3 are identical, and the results are shown in Figure 3. Specifically, when the material

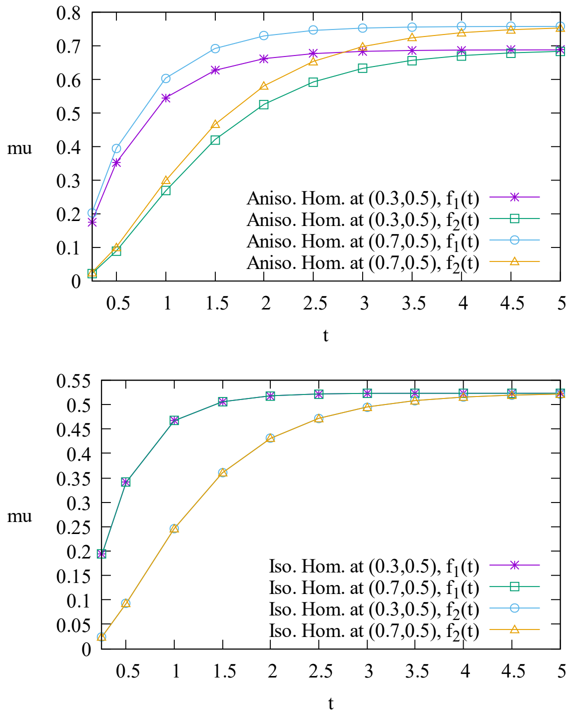

Fig. 3. Solutions μ at $(x_1, x_2) = (0.3, 0.5), (0.7, 0.5)$ for Case 1, 2, 3 of Problem 2

is isotropic homogeneous the solutions μ at point $(0.3, 0.5)$ will coincide with the solutions at point $(0.7, 0.5)$. This is expected as for isotropic homogeneous material the problem is symmetric about the axis $x_1 = 0.5$.

Moreover, as the time increases from $t = 0.25$ to $t = 5$ the solutions μ of problems with functions $f(t) = f_1(t) = 1$ and $f(t) = f_2(t) = 1 - \exp(-1.25t)$ as the boundary condition on side AB tend to approach a same steady state solution. This is expected as for big value of t the limit of the function $f(t) = f_2(t) = 1 - \exp(-1.25t)$ is equal to $f(t) = f_1(t) = 1$.

Figures 3 – 6 also indicate that anisotropy and inhomogeneity of material give effect on the values of solution μ. This suggests that it is important to take the anisotropy and inhomogeneity into account in any practical application.

V. Conclusion

A combined Laplace transform and standard BEM has been used to find numerical solutions to initial boundary value problems for anisotropic functionally graded materials which are governed by the parabolic equation (1). The boundary-only integral equation (17) is employed to find the numerical solutions so that the method is easy to implement. The method involves a time variable free fundamental solution therefore it gives more accurate solutions. It does not involve round-off error propagation as it solves the boundary integral equation (17) independently for each specific value of t at which the solution is computed. Unlikely, the methods with time variable fundamental solution may produce less accurate solutions as the fundamental solution sometimes contain time singular points and also solution for the next time step is based on the solution of the previous time step so that the round-off error may propagate.

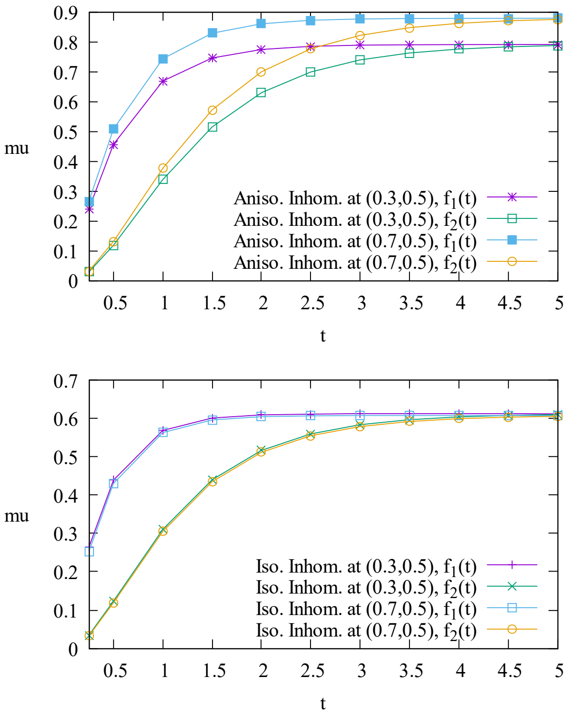

Fig. 4. Solutions μ at $(x_1, x_2) = (0.3, 0.5), (0.7, 0.5)$ for Case 1 (trigonometrically graded material) of Problem 2

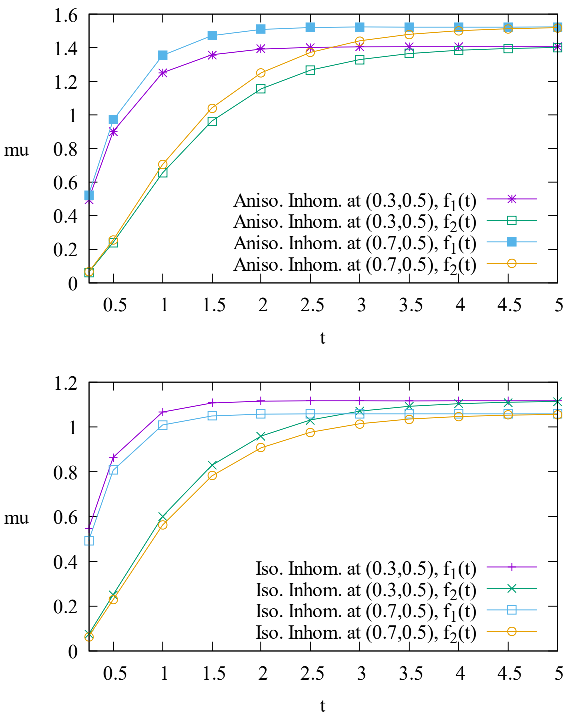

Fig. 5. Solutions μ at $(x_1, x_2) = (0.3, 0.5), (0.7, 0.5)$ for Case 2 (exponentially graded material) of Problem 2

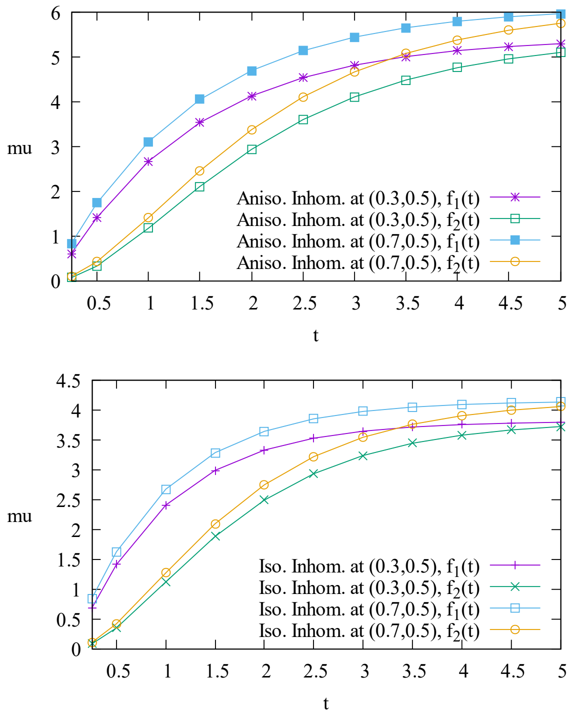

Fig. 6. Solutions μ at $(x_1, x_2) = (0.3, 0.5), (0.7, 0.5)$ for Case 3 (quadratically graded material) of Problem 2

It has been applied to a wide range of functionally graded materials, namely quadratically, exponentially and trigonometrically graded materials. As the coefficients $\kappa_{ij}(\mathbf{x}), \beta^2(\mathbf{x}), \alpha(\mathbf{x})$ do depend on the spatial variable $\mathbf{x}$ only and on the same inhomogeneity or grading function $g(\mathbf{x})$, it is interesting to extend the study in the future to the case when the coefficients depend on different gradation functions varying also with the time variable t.

In order to use the boundary integral equation (17), the values $\mu(\mathbf{x}, t)$ or $P(\mathbf{x}, t)$ of the boundary conditions as stated in Section II of the original system in time variable t have to be Laplace transformed first. This means that from the beginning when we set up a problem, we actually put a set of approached boundary conditions. Therefore it is really important to find a very accurate technique of numerical Laplace transform inversion. Based on the obtained results, the Stehfest formula is a quite accurate technique for the calculation of values of the numerical Laplace transform inverse.

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