A Combined Laplace Transform and Boundary Element Method for Unsteady Laplace Problems of Several Classes of Anisotropic Functionally Graded Materials

Moh. Ivan Azis*, Imam Solekhudin, Muh. Hajarul Aswad, Suharman Hamzah, Abd. Rasyid Jalil

***Abstract*—In this paper a combined Laplace transform and boundary element method is used to find numerical solutions to unsteady problems of anisotropic functionally graded materials governed by a Laplace type equation. A mathematical transformation is used to transform the variable coefficients equation to a constant coefficients equation from which a boundary-only integral equation is obtained. In addition, the analysis also results in several classes of inhomogeneity functions for the functionally graded materials. Some examples are considered to show the validity of the analysis and accuracy of the numerical solutions. A verification for the effect of the anisotropy and inhomogeneity of the material on the solutions is also demonstrated.**

***Index Terms*—Anisotropic functionally graded materials, variable coefficients equation, Laplace equation, Laplace transform; boundary element method**

I. Introduction

We will consider initial boundary value problems governed by a Laplace type equation with variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\kappa_{ij}(\mathbf{x})\frac{\partial\mu(\mathbf{x},t)}{\partial x_j}\right]=\alpha(\mathbf{x})\frac{\partial\mu(\mathbf{x},t)}{\partial t} \quad (1)$$

The coefficients $[\kappa_{ij}]$ $(i,j=1,2)$ is a real symmetric positive definite matrix. Also, in (1) the summation convention for repeated indices holds. Equation (1) is usually used to model, among others, plane heat conduction problems, for which κ_{ij} may represent the conductivity coefficients, α may depict the rate of change and μ is the temperature. Since the coefficients $\kappa_{ij}(\mathbf{x}), \alpha(\mathbf{x})$ are spatially continuous functions, then the material under consideration has properties which vary spatially according to a specific continuous function. Such a material is called a functionally graded material (FGM). Specifically, since the coefficients $\kappa_{11}, \kappa_{12}, \kappa_{22}$ may differ then the material is called as an anisotropic material. Therefore equation (1) is relevant for anisotropic FGMs.

During the last decade FGMs have become an important topic, and numerous studies on FGMs for a variety of applications have been reported (see for example Bounouara et al. [1] and Karami et al. [2]). On the other hand, in some applications anisotropy of the material of interest needs to be taken into account. Among other studies that considered material anisotropy have been done by Limberkar et al. [3] in material science application, Daghash et al. [4] in chemical engineering application and Yusuf [5] in optics application.

Recently a number of authors had been working on the Laplace equation to find its solutions. However the works mainly focus on problems of isotropic homogeneous materials. For example, Guo et al [6] considered transient heat conduction problems of isotropic and homogeneous media and solved them using a combined Laplace transform and multiple reciprocity boundary face method. In [7] Fu et al. examined a boundary knot method used to find numerical solutions to problems of homogeneous isotropic media governed by a three-dimensional transient heat conduction with a source term. Yang et al. [8] investigated steady nonlinear heat conduction problems of homogeneous isotropic materials and solved them using a radial integration boundary element method. In [9] solutions of a Laplace type equation in unbounded domains are discussed.

For such kind of materials, the boundary element method (BEM) and other methods had been successfully used to find the numerical solutions of problems associated to them. But this is not the case for inhomogeneous materials, due to the unavailability of fundamental solutions for equations of variable coefficients which govern problems of inhomogeneous media. Some progress of solving problems for inhomogeneous media using various techniques has been done. Timpitak and Pochai [10] investigated finite difference solutions of unsteady diffusion-convection problems for heterogeneous media. Noda et al. [11] studied the analytical solutions to a transient heat conduction equation of variable coefficients with a source term for a functionally graded orthotropic strip (FGOS). In this study, the inhomogeneity of the FGOS is simplified to be functionally graded in the x variable only. In [12] Azis and Clements worked on finding numerical solutions to nonlinear transient heat conduction problems for anisotropic quadratically graded materials using a boundary domain element method. The quadratically varying coefficient in the governing equation considered by Azis and Clements [12] can certainly be represented as a sum of constant and variable coefficients. Some later studies on the class of constant-plus-variable coefficients equations had been done a number of authors. Samec and Škerget [13] considered a non-steady diffusive–

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*Corresponding author. M.I. Azis is a lecturer at the Department of Mathematics, Faculty of Mathematics and Natural Sciences, Hasanuddin University, Makassar, INDONESIA. E-mail: ivan@unhas.ac.id

I. Solekhudin is a lecturer at the Department of Mathematics, Universitas Gadjah Mada, Jogjakarta, INDONESIA. E-mail: imams@ugm.ac.id

M.H. Aswad is a lecturer at the Department of Mathematics, Institut Agama Islam Negeri, Palopo, INDONESIA. E-mail: aswad82@yahoo.co.id

S. Hamzah is a lecturer at the Department of Civil Engineering, Hasanuddin University, Makassar, INDONESIA. E-mail: suharmanhz@unhas.ac.id

A.R. Jalil is a lecturer at the Department of Marine Science, Hasanuddin University, INDONESIA. E-mail: abdrasyidjalil@gmail.com

convective transport equation with variable velocity which is represented as a sum of constant and variable terms. Ravnik and Škerget in [14] studied steady state diffusion-convection problems with inhomogeneous isotropic diffusivity, variable velocity and incompressible fluid using a domain boundary integral equation method (DBIEM). In this work both the diffusivity and the velocity take a constant-plus-variable form. Ravnik and Škerget in [15] considered an unsteady state diffusion-convection problems with sources, inhomogeneous isotropic conductivity, variable velocity and incompressible fluid using a DBIEM. In this study both the diffusivity and the velocity are again taken to be of constant-plus-variable form. AL-Bayati and Wrobel [16], [17] focused on convection–diffusion–reaction equation of incompressible flow with constant diffusivity and variable velocity taking the form of constant-plus-variable terms. Ravnik and Tibuat [18] also considered an unsteady diffusion-convection equation with variable diffusivity and velocity. The diffusivity is of the constant-plus-variable form. By taking the variable coefficients as a sum of constant and variable coefficients, the derived integral equation will then involve both boundary and domain integrals. The constant coefficient term will contribute boundary integrals as the fundamental solutions are available, and the variable coefficient term will give domain integrals.

Reduction to constant coefficients equation is another technique that can be used to transform a variable coefficients equation to a constant coefficients equation. Therefore the technique will preserve the boundary-only integral equation. Recently Azis and co-workers had been working on steady state problems of anisotropic inhomogeneous media for several types of governing equations, for examples [19]–[24] for Helmholtz equation, [25]–[27] for the modified Helmholtz equation, [28] for elasticity problems, [29]–[33] for the diffusion convection equation, [34]–[37] for the Laplace type equation, [38]–[44] for the diffusion convection reaction equation. Some other classes of inhomogeneity functions for FGMs that differ from the class of constant-plus-variable coefficients are reported from these papers.

This paper is intended to extend the recently published works in [34]–[37] for steady anisotropic Laplace type equation with spatially variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\kappa_{ij}(\mathbf{x})\frac{\partial \mu(\mathbf{x},t)}{\partial x_j}\right]=0$$

to unsteady anisotropic Laplace type equation with spatially variable coefficients of the form (1).

This study is an attempt to solve numerically initial boundary value problems for several types of anisotropic FGMs governed by equation (1) using a boundary-only element method. The analysis of this paper is purely mathematical; the main aim being to construct effective a BEM for (1).

A brief outline of the paper is as follows. Section II defines the initial boundary value problem to be solved. In Section III a boundary integral equation is derived. In Section IV several problems (a test problem in Section IV-A and a problem without analytical solutions in Section IV-B) are solved to primarily show the validity of the analysis used in deriving the boundary integral equation in Section III. Finally, Section V concludes this paper with some remarks.

II. The initial-boundary value problem

Referred to a Cartesian frame Ox_1x_2 solutions $\mu(\mathbf{x},t)$ and its derivatives to (1) are sought which are valid for time interval $t \geq 0$ and in a region Ω in R^2 with boundary $\partial\Omega$ which consists of a finite number of piecewise smooth closed curves. On $\partial\Omega_1$ the dependent variable $\mu(\mathbf{x},t)$ ($\mathbf{x}=(x_1,x_2)$) is specified and on $\partial\Omega_2$

$$P(\mathbf{x},t)=\kappa_{ij}(\mathbf{x})\frac{\partial \mu(\mathbf{x},t)}{\partial x_i}n_j \tag{2}$$

is specified where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ and $\mathbf{n} = (n_1, n_2)$ denotes the outward pointing normal to $\partial\Omega$. The initial condition is taken to be

$$\mu(\mathbf{x},0)=0 \tag{3}$$

The method of solution will be to transform the variable coefficient equation (1) to a constant coefficient equation, and then taking a Laplace transform of the constant coefficient equation, and to obtain a boundary integral equation in the Laplace transform variable s. The boundary integral equation is then solved using a standard boundary element method (BEM). An inverse Laplace transform is taken to get the solution c and its derivatives for all $(\mathbf{x},t)$ in the domain. The inverse Laplace transform is implemented numerically using the Stehfest formula.

The analysis is specially relevant to an anisotropic medium but it equally applies to isotropic media. For isotropy, the coefficients in (1) take the form $\kappa_{11}=\kappa_{22}$ and $\kappa_{12}=0$ and use of these equations in the following analysis immediately yields the corresponding results for an isotropic medium. The analysis also applies for homogeneous materials which occur when the coefficients κ_{ij} and α are constant.

III. The boundary integral equation

The coefficients κ_{ij}, α are required to take the form

$$\kappa_{ij}(\mathbf{x}) = \overline{\kappa}_{ij} g(\mathbf{x}) \tag{4}$$

$$\alpha(\mathbf{x}) = \overline{\alpha} g(\mathbf{x}) \tag{5}$$

where the $\overline{\kappa}_{ij}, \overline{\alpha}$ are constants and g is a differentiable function of $\mathbf{x}$. Use of (4)-(5) in (1) yields

$$\overline{\kappa}_{ij}\frac{\partial}{\partial x_i}\left(g\frac{\partial \mu}{\partial x_j}\right)=\overline{\alpha} g\frac{\partial \mu}{\partial t} \tag{6}$$

Let

$$\mu(\mathbf{x},t)=g^{-1/2}(\mathbf{x})\,\psi(\mathbf{x},t) \tag{7}$$

therefore substitution of (4) and (7) into (2) gives

$$P(\mathbf{x},t)=-P_g(\mathbf{x})\,\psi(\mathbf{x},t)+g^{1/2}(\mathbf{x})\,P_\psi(\mathbf{x},t) \tag{8}$$

where

$$P_g(\mathbf{x})=\overline{\kappa}_{ij}\frac{\partial g^{1/2}}{\partial x_j}n_i \qquad P_\psi(\mathbf{x})=\overline{\kappa}_{ij}\frac{\partial \psi}{\partial x_j}n_i$$

Also, (6) may be written in the form

$$\overline{\kappa}_{ij}\frac{\partial}{\partial x_i}\left[g\frac{\partial\left(g^{-1/2}\psi\right)}{\partial x_j}\right]=\overline{\alpha} g\frac{\partial\left(g^{-1/2}\psi\right)}{\partial t}$$

which can be simplified

$$\overline{\kappa}_{ij}\frac{\partial}{\partial x_i}\left(g^{1/2}\frac{\partial\psi}{\partial x_j}+g\psi\frac{\partial g^{-1/2}}{\partial x_j}\right)=\overline{\alpha}g^{1/2}\frac{\partial\psi}{\partial t}$$

Use of the identity

$$\frac{\partial g^{-1/2}}{\partial x_i}=-g^{-1}\frac{\partial g^{1/2}}{\partial x_i}$$

implies

$$\overline{\kappa}_{ij}\frac{\partial}{\partial x_i}\left(g^{1/2}\frac{\partial\psi}{\partial x_j}-\psi\frac{\partial g^{1/2}}{\partial x_j}\right)=\overline{\alpha}g^{1/2}\frac{\partial\psi}{\partial t}$$

Rearranging and neglecting some zero terms gives

$$g^{1/2}\overline{\kappa}_{ij}\frac{\partial^2\psi}{\partial x_i\partial x_j}-\psi\overline{\kappa}_{ij}\frac{\partial^2 g^{1/2}}{\partial x_i\partial x_j}=\overline{\alpha}g^{1/2}\frac{\partial\psi}{\partial t}$$

It follows that if g is such that

$$\overline{\kappa}_{ij}\frac{\partial^2 g^{1/2}}{\partial x_i\partial x_j}-\lambda g^{1/2}=0 \tag{9}$$

where λ is a constant, then the transformation (7) carries the variable coefficients equation (6) to the constant coefficients equation

$$\overline{\kappa}_{ij}\frac{\partial^2\psi}{\partial x_i\partial x_j}-\lambda\psi=\overline{\alpha}\frac{\partial\psi}{\partial t} \tag{10}$$

Taking the Laplace transform of (7), (8), (10) and applying the initial condition (3) we obtain

$$\psi^*(\mathbf{x},s)=g^{1/2}(\mathbf{x})\,\mu^*(\mathbf{x},s) \tag{11}$$

$$P_{\psi^*}(\mathbf{x},s)=\left[P^*(\mathbf{x},s)+P_g(\mathbf{x})\,\psi^*(\mathbf{x},s)\right]g^{-1/2}(\mathbf{x}) \tag{12}$$

$$\overline{\kappa}_{ij}\frac{\partial^2\psi^*}{\partial x_i\partial x_j}-(\lambda+s\overline{\alpha})\,\psi^*=0 \tag{13}$$

where s is the variable of the Laplace-transformed domain.

A boundary integral equation for the solution of (13) is given in the form

$$\begin{aligned}\eta(\mathbf{x}_0)\,\psi^*(\mathbf{x}_0,s) &= \int_{\partial\Omega}\left[\Gamma(\mathbf{x},\mathbf{x}_0)\,\psi^*(\mathbf{x},s)-\right.\\ &= \left.\Phi(\mathbf{x},\mathbf{x}_0)\,P_{\psi^*}(\mathbf{x},s)\right]dS(\mathbf{x})\end{aligned} \tag{14}$$

where $\mathbf{x}_0=(a,b)$, $\eta=0$ if $(a,b)\notin\Omega\cup\partial\Omega$, $\eta=1$ if $(a,b)\in\Omega$, $\eta=\frac{1}{2}$ if $(a,b)\in\partial\Omega$ and $\partial\Omega$ has a continuously turning tangent at (a,b). The so called fundamental solution Φ in (14) is any solution of the equation

$$\overline{\kappa}_{ij}\frac{\partial^2\Phi}{\partial x_i\partial x_j}-(\lambda+s\overline{\alpha})\,\Phi=\delta(\mathbf{x}-\mathbf{x}_0) \tag{15}$$

and the Γ is given by

$$\Gamma(\mathbf{x},\mathbf{x}_0)=\overline{\kappa}_{ij}\frac{\partial\Phi(\mathbf{x},\mathbf{x}_0)}{\partial x_j}n_i$$

where δ is the Dirac delta function. For two-dimensional problems, three types of fundamental solutions Φ and Γ that can be obtained from (15), namely the fundamental solutions for Laplace equation $(\lambda+s\overline{\alpha}=0)$, for Helmholtz equation $(\lambda+s\overline{\alpha}<0)$ and for modified Helmholtz equation $(\lambda+s\overline{\alpha}>0)$, are given respectively by

$$\Phi(\mathbf{x},\mathbf{x}_0)=\begin{cases}\frac{K}{2\pi}\ln R & \text{if } \lambda+s\overline{\alpha}=0\\ \frac{\imath K}{4}H_0^{(2)}(\omega R) & \text{if } \lambda+s\overline{\alpha}<0\\ \frac{-K}{2\pi}K_0(\omega R) & \text{if } \lambda+s\overline{\alpha}>0\end{cases} \tag{16}$$

$$\Gamma(\mathbf{x},\mathbf{x}_0)=\begin{cases}\frac{K}{2\pi}\frac{1}{R}\overline{\kappa}_{ij}\frac{\partial R}{\partial x_j}n_i & \text{if } \lambda+s\overline{\alpha}=0\\ \frac{-\imath K\omega}{4}H_1^{(2)}(\omega R)\,\overline{\kappa}_{ij}\frac{\partial R}{\partial x_j}n_i & \text{if } \lambda+s\overline{\alpha}<0\\ \frac{K\omega}{2\pi}K_1(\omega R)\,\overline{\kappa}_{ij}\frac{\partial R}{\partial x_j}n_i & \text{if } \lambda+s\overline{\alpha}>0\end{cases}$$

where

$$\begin{aligned}K &= \ddot{\tau}/D\\ \omega &= \sqrt{|\lambda+s\overline{\alpha}|/D}\\ D &= \left[\overline{\kappa}_{11}+2\overline{\kappa}_{12}\dot{\tau}+\overline{\kappa}_{22}\left(\dot{\tau}^2+\ddot{\tau}^2\right)\right]/2\\ R &= \sqrt{(\dot{x}_1-\dot{a})^2+(\dot{x}_2-\dot{b})^2}\\ \dot{x}_1 &= x_1+\dot{\tau}x_2\\ \dot{a} &= a+\dot{\tau}b\\ \dot{x}_2 &= \ddot{\tau}x_2\\ \dot{b} &= \ddot{\tau}b\end{aligned}$$

where $\dot{\tau}$ and $\ddot{\tau}$ are respectively the real and the positive imaginary parts of the complex root τ of the quadratic

$$\overline{\kappa}_{11}+2\overline{\kappa}_{12}\tau+\overline{\kappa}_{22}\tau^2=0$$

and $H_0^{(2)}$, $H_1^{(2)}$ denote the Hankel function of second kind and order zero and order one respectively. K_0, K_1 denote the modified Bessel function of order zero and order one respectively, $\imath$ represents the square root of minus one. The derivatives $\partial R/\partial x_j$ needed for the calculation of the Γ in (16) are given by

$$\begin{aligned}\frac{\partial R}{\partial x_1} &= \frac{1}{R}(\dot{x}_1-\dot{a})\\ \frac{\partial R}{\partial x_2} &= \dot{\tau}\left[\frac{1}{R}(\dot{x}_1-\dot{a})\right]+\ddot{\tau}\left[\frac{1}{R}\left(\dot{x}_2-\dot{b}\right)\right]\end{aligned}$$

Use of (11) and (12) in (14) yields

$$\eta g^{1/2}\mu^*=\int_{\partial\Omega}\left[\left(g^{1/2}\Gamma-P_g\Phi\right)\mu^*-\left(g^{-1/2}\Phi\right)P^*\right]dS \tag{17}$$

This equation provides a boundary integral equation for determining μ^* and its derivatives at all points of Ω.

Knowing the solutions $\mu^*(\mathbf{x},s)$ and its derivatives $\partial\mu^*/\partial x_1$ and $\partial\mu^*/\partial x_2$ which are obtained from (17), the numerical Laplace transform inversion technique using the Stehfest formula is then employed to find the values of $\mu(\mathbf{x},t)$ and its derivatives $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$. The Stehfest formula is

$$\begin{aligned}\mu(\mathbf{x},t) &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N}V_m\mu^*(\mathbf{x},s_m)\\ \frac{\partial\mu(\mathbf{x},t)}{\partial x_1} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N}V_m\frac{\partial\mu^*(\mathbf{x},s_m)}{\partial x_1}\\ \frac{\partial\mu(\mathbf{x},t)}{\partial x_2} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N}V_m\frac{\partial\mu^*(\mathbf{x},s_m)}{\partial x_2}\end{aligned} \tag{18}$$

where

$$\begin{aligned}s_m &= \frac{\ln 2}{t}m\\ V_m &= (-1)^{\frac{N}{2}+m}\times\\ &\sum_{k=\left[\frac{m+1}{2}\right]}^{\min\left(m,\frac{N}{2}\right)}\frac{k^{N/2}(2k)!}{\left(\frac{N}{2}-k\right)!k!(k-1)!(m-k)!(2k-m)!}\end{aligned}$$

The analysis of the section requires that the coefficients κ_{ij}, α are of the form (4) and (5) respectively with g satisfying (9). This condition on g allows for considerable choice in the coefficients. For example, when $\lambda = 0$, g can assume a number of multiparameter forms with the parameters being employed to fit the coefficients to numerical data for the coefficients. Possible multiparameter forms include

$$g(\mathbf{x}) = (c_0 + c_1 x_1 + c_2 x_2)^2$$
$$g(\mathbf{x}) = [\Re\{c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n\}]^2$$

where the $c_k, k = 1, 2, \ldots, n$ are constants, $\Re$ denotes the real part of a complex number and $z = x_1 + \tau x_2$. More generally, the square of the real part of any analytical function of the complex variable z can serve as a possible form for g. For the case when $\lambda \neq 0$ some possible multiparameter forms of g are

$$g(\mathbf{x}) = [A\cos(c_0 + c_1 x_1 + c_2 x_2) + B\sin(c_0 + c_1 x_1 + c_2 x_2)]^2, \overline{\kappa}_{ij} c_i c_j + \lambda = 0$$
$$g(\mathbf{x}) = [A\exp(c_0 + c_1 x_1 + c_2 x_2)]^2, \overline{\kappa}_{ij} c_i c_j - \lambda = 0$$

where A, B, c_i are real constants.

IV. Numerical examples

Some particular problems for FGMs will be solved by employing a BEM for the boundary integral equation (17) to obtain numerical solutions in the frame of Laplace transform. The Stehfest formula (18) is used to get the solutions in the time variable t. The main aim is to show the validity of the analysis for deriving the boundary integral equation (17) and the appropriateness of the BEM and Stehfest formula in solving the problems defined in Section II.

For all problems considered, the gradation function $g(\mathbf{x})$ of the considered FGM is required to satisfy equation (9). We assume each problem belongs to a system which is valid in given spatial and time domains. The characteristics of the system which are represented by the coefficients $\kappa_{ij}(\mathbf{x}), \alpha(\mathbf{x})$ in equation (1) are assumed to be of the form (4) and (5).

The BEM with constant elements is employed to obtain numerical results. And the value of N in (18) for the Stehfest formula is chosen to be $N = 10$. For all problems considered, a unit square (depicted in Figure 1) will be taken as the domain, and the boundary of the domain is divided into 320 elements of the same length, that is 80 elements for each side of the unit square, and the time domain is $0 \leq t \leq 5$. The integral on each element is evaluated numerically using the Bode's quadrature. A FORTRAN code is developed to compute the solutions, and a specific FORTRAN command is imposed to calculate the elapsed CPU time for obtaining the results. A simple script is developed and embedded into the main FORTRAN code to calculate the values of the coefficients $V_m, m = 1, 2, \ldots, N$ for any number N. Table (I) shows the values of V_m for $N = 4, 6, 8, 10$ which are obtained from the script.

TABLE I
VALUES OF V_m OF THE STEHFEST FORMULA FOR $N = 4, 6, 8, 10$

V_m	$N=4$	$N=6$	$N=8$	$N=10$
V_1	-2	1	$-1/3$	$1/12$
V_2	26	-49	$145/3$	$-385/12$
V_3	-48	366	-906	1279
V_4	24	-858	$16394/3$	$-46871/3$
V_5		810	$-43130/3$	$505465/6$
V_6		-270	18730	-236957.5
V_7			$-35840/3$	$1127735/3$
V_8			$8960/3$	$-1020215/3$
V_9				164062.5
V_{10}				-32812.5

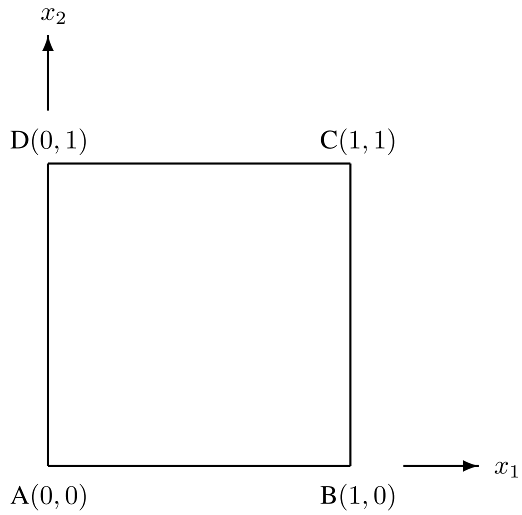

Fig. 1. The domain Ω

A. A test problem

1) Problem 1: In order to see the accuracy of the BEM and the Stehfest formula we will consider a problem of analytical solution. The problem is also aimed to show the steady state solution if exists. Three cases of FGMs will be considered, namely trigonometrically (Case 1), exponentially (Case 2) and quadratically (Case 3) graded materials. The analytical solutions of all cases are assumed to take a separable variables form

$$\mu(\mathbf{x}, t) = g^{-1/2}(\mathbf{x})\, h(\mathbf{x})\, f(t)$$

where the function $h(\mathbf{x})$ satisfies (13). Also, we take a common constant coefficient $\overline{\kappa}_{ij}$ for all cases

$$\overline{\kappa}_{ij} = \begin{bmatrix} 1 & 0.05 \\ 0.05 & 0.45 \end{bmatrix}$$

and a mutual set of boundary conditions (see Figure 1)

μ is given on side AB, BC, CD
P is given on side AD

Case 1: trigonometrically graded material: We assume the inhomogeneity function $g(\mathbf{x})$ is a trigonometric function

$$g(\mathbf{x}) = [\cos(1 - 0.55x_1 - 0.25x_2)]^2$$

so that the medium under consideration is a trigonometrically graded material. The time variation function is (see Figure 2)

$$f(t) = 1 - \exp(-1.35t)$$

For $g(\mathbf{x})$ to satisfy (9)

$$\lambda = -0.344375$$

We take

$$h(\mathbf{x}) = 1 - 0.85x_1 - 0.15x_2$$

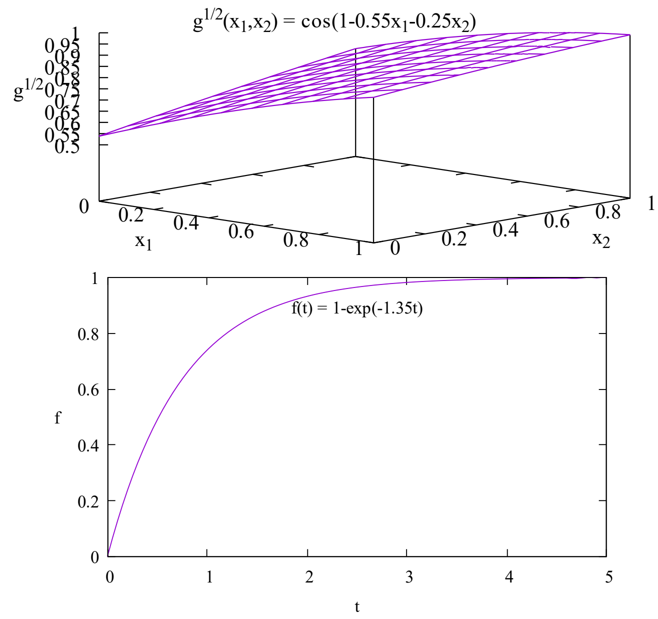

Fig. 2. Functions $g^{1/2}(\mathbf{x})$ and $f(t)$ for Case 1 of Problem 1

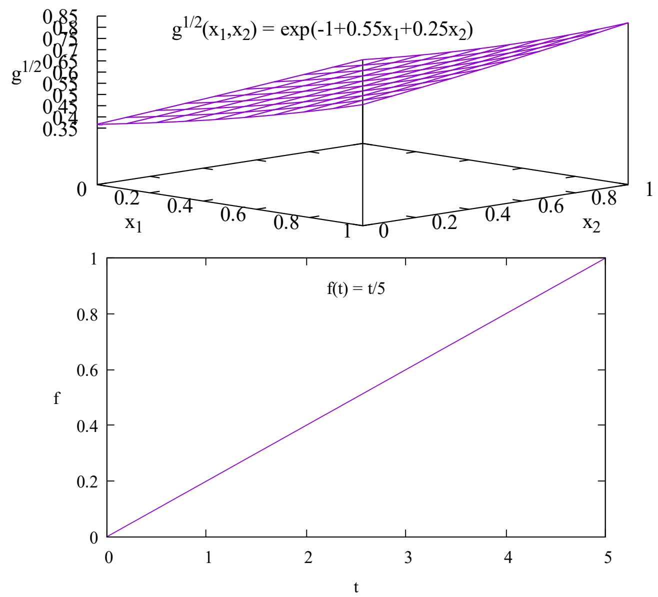

Fig. 3. Functions $g^{1/2}(\mathbf{x})$ and $f(t)$ for Case 2 of Problem 1

so that in order for $h(\mathbf{x})$ to satisfy (13) with $\lambda + s\overline{\alpha} = 0$ (as to use the Laplace fundamental solution in (16))

$$\overline{\alpha} = 0.344375/s$$

Case 2: exponentially graded material: The FGM is supposed to be an exponentially graded material with a gradation function $g(\mathbf{x})$ of the form

$$g(\mathbf{x}) = \left[\exp\left(-1 + 0.55x_1 + 0.25x_2\right)\right]^2$$

so that from (9)

$$\lambda = 0.344375$$

The time variation function is

$$f(t) = t/5$$

Functions $g^{1/2}(\mathbf{x})$ and $f(t)$ are depicted in Figure 3.

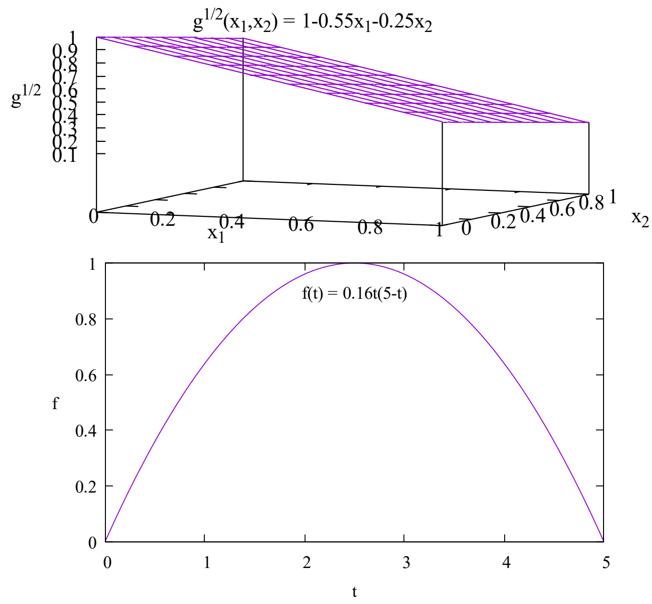

Fig. 4. Functions $g^{1/2}(\mathbf{x})$ and $f(t)$ for Case 3 of Problem 1

We take

$$h(\mathbf{x}) = \sin\left(1 - 0.85x_1 - 0.15x_2\right)$$

so that in order for $h(\mathbf{x})$ to satisfy (13) with $\lambda + s\overline{\alpha} = -0.745375 < 0$ (as to use the Helmholtz fundamental solution in (16))

$$\overline{\alpha} = -1.08975/s$$

Case 3: quadratically graded material: We assume that the material is quadratically graded, with a function of gradation

$$g(\mathbf{x}) = \left(1 - 0.55x_1 - 0.25x_2\right)^2$$

so that from (9)

$$\lambda = 0$$

The time variation function is (see Figure 4)

$$f(t) = 0.16t\left(5 - t\right)$$

We take

$$h(\mathbf{x}) = \exp\left(-1 + 0.85x_1 + 0.15x_2\right)$$

so that in order for $h(\mathbf{x})$ to satisfy (13) with $\lambda + s\overline{\alpha} = 0.745375 > 0$ (as to use the modified Helmholtz fundamental solution in (16))

$$\overline{\alpha} = 0.745375/s$$

The results for the three cases of Problem 1 are shown in Table (II) and Figure 5. Table (II) shows the accuracy of the numerical solutions μ and the derivatives $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$ solutions at $(x_1, x_2) = (0.5, 0.5)$ in the domain. For all cases the errors mainly occur in the fourth decimal place for the $\mu, \partial\mu/\partial x_1, \partial\mu/\partial x_2$ solutions.

Figure 5 shows a variation of the μ solution values at interior points $(x_1, x_2) = (0.2, 0.2), (0.8, 0.8)$ as the time increases from $t = 0.0005$ to $t = 5$. As expected, the variation follows the way the associated function $f(t)$ changes. Specifically for the Case 2 of associated function $f(t) = 1 - \exp(-1.25t)$ the μ solutions tends to approach

TABLE II
THE ACCURACY OF THE NUMERICAL SOLUTIONS AT POSITION $(x_1, x_2) = (0.5, 0.5)$ FOR PROBLEM 1

t	Numerical			Errors		
	μ	$\frac{\partial\mu}{\partial x_1}$	$\frac{\partial\mu}{\partial x_2}$	μ	$\frac{\partial\mu}{\partial x_1}$	$\frac{\partial\mu}{\partial x_2}$
Case 1						
0.0005	0.0004	-0.0008	-0.0002	0.0000	0.0000	0.0000
0.5	0.2973	-0.6176	-0.1401	0.0001	0.0002	0.0001
1.0	0.4486	-0.9320	-0.2114	0.0001	0.0002	0.0001
1.5	0.5255	-1.0916	-0.2476	0.0004	0.0002	0.0001
2.0	0.5646	-1.1730	-0.2661	0.0005	0.0004	0.0001
2.5	0.5847	-1.2145	-0.2755	0.0004	0.0003	0.0000
3.0	0.5950	-1.2360	-0.2804	0.0003	0.0001	0.0000
3.5	0.6004	-1.2471	-0.2829	0.0001	0.0005	0.0001
4.0	0.6031	-1.2529	-0.2841	0.0000	0.0007	0.0000
4.5	0.6045	-1.2557	-0.2850	0.0001	0.0008	0.0003
5.0	0.6052	-1.2573	-0.2854	0.0001	0.0009	0.0004
Case 2						
0.0005	0.0001	-0.0002	-0.0000	0.0000	0.0000	0.0000
0.5	0.0873	-0.1840	-0.0458	0.0000	0.0000	0.0000
1.0	0.1747	-0.3680	-0.0917	0.0001	0.0001	0.0000
1.5	0.2620	-0.5520	-0.1375	0.0001	0.0001	0.0000
2.0	0.3493	-0.7360	-0.1833	0.0001	0.0002	0.0000
2.5	0.4366	-0.9201	-0.2292	0.0001	0.0002	0.0000
3.0	0.5240	-1.1041	-0.2750	0.0002	0.0003	0.0000
3.5	0.6113	-1.2881	-0.3208	0.0002	0.0003	0.0000
4.0	0.6986	-1.4721	-0.3666	0.0002	0.0004	0.0000
4.5	0.7860	-1.6561	-0.4125	0.0002	0.0004	0.0001
5.0	0.8733	-1.8401	-0.4583	0.0003	0.0004	0.0000
Case 3						
0.0005	0.0004	0.0007	0.0002	0.0000	0.0000	0.0000
0.5	0.3641	0.6431	0.2063	0.0001	0.0002	0.0001
1.0	0.6472	1.1433	0.3668	0.0003	0.0003	0.0002
1.5	0.8495	1.5006	0.4814	0.0004	0.0004	0.0003
2.0	0.9709	1.7151	0.5502	0.0005	0.0006	0.0003
2.5	1.0114	1.7865	0.5732	0.0005	0.0006	0.0004
3.0	0.9710	1.7150	0.5505	0.0005	0.0005	0.0005
3.5	0.8495	1.5009	0.4812	0.0003	0.0008	0.0001
4.0	0.6474	1.1436	0.3671	0.0005	0.0007	0.0004
4.5	0.3644	0.6436	0.2065	0.0005	0.0007	0.0003
5.0	0.0004	0.0005	0.0002	0.0004	0.0005	0.0002

TABLE III
THE ELAPSED CPU TIME (IN SECONDS) FOR PROBLEM 1

Case 1	Case 2	Case 3
472	685.984375	338.65625

a steady state solution. This is also expected, as the function $f(t) = 1 - \exp(-1.25t)$ converges to 1 as t gets bigger.

The elapsed CPU time for the computation of the numerical solutions at 19×19 spatial positions and 11 time steps from $t = 0.0005$ to $t = 5$ is shown in Table III.

B. An example without analytical solution

1) Problem 2:: The aim is to show the effect of inhomogeneity and anisotropy of the considered medium to the solution μ. The medium is supposed to be an anisotropic or isotropic, and inhomogeneous (functionally graded) or homogeneous material. For all combinations of the material's anisotropy and inhomogeneity (isotropic homogeneous, isotropic inhomogeneous, anisotropic homogeneous, anisotropic inhomogeneous) we choose

$$\overline{\alpha} = 1$$

and a common set of boundary conditions that

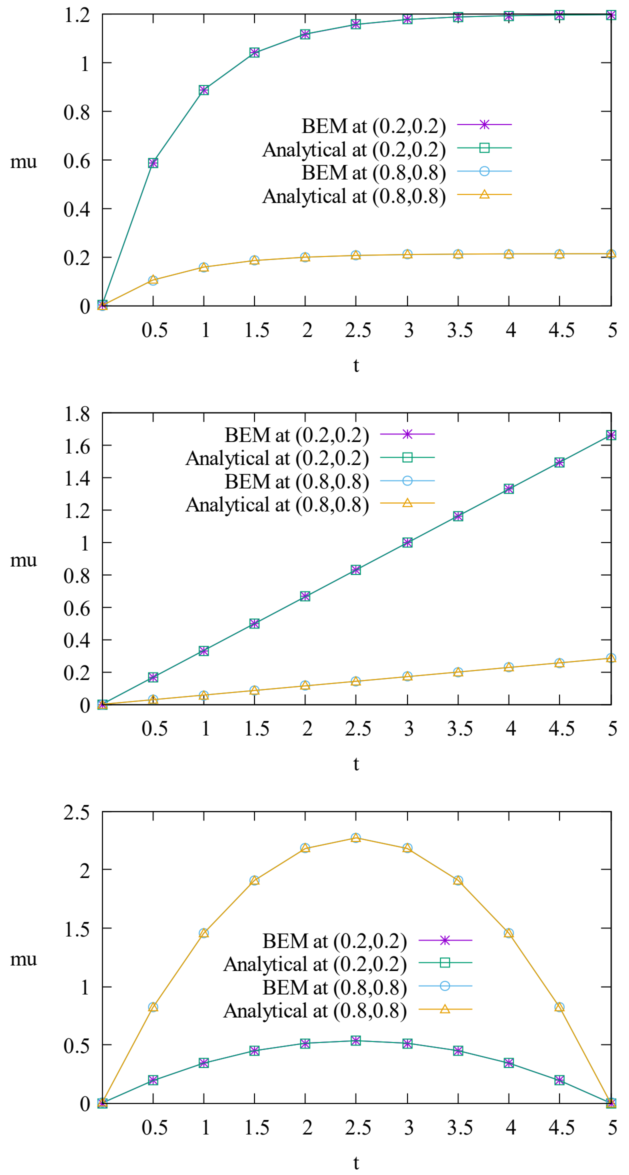

Fig. 5. Solutions μ at interior points $(x_1, x_2) = (0.2, 0.2), (0.8, 0.8)$ for the Case 1 (top), Case 2 (center) and Case 3 (bottom) of Problem 1

$$\begin{aligned} P &= f(t) \text{ on side AB} \\ P &= 0 \text{ on side BC} \\ \mu &= 0 \text{ on side CD} \\ P &= 0 \text{ on side AD} \end{aligned}$$

where the function $f(t)$ is defined as one of the following two forms

$$\begin{aligned} f(t) = f_1(t) &= 1 \\ f(t) = f_2(t) &= 1 - \exp(-1.35t) \end{aligned}$$

If the material is anisotropic then the constant coefficient $\overline{\kappa}_{ij}$ is

$$\overline{\kappa}_{ij} = \begin{bmatrix} 1 & 0.05 \\ 0.05 & 0.45 \end{bmatrix}$$

and

$$\overline{\kappa}_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

when it is isotropic. Regarding its inhomogeneity, three cases of FGMs will be considered, namely trigonometrically, exponentially and quadratically graded materials. We take

$g^{1/2}(\mathbf{x}) = 1$ (that is $\lambda = 0$) for the case of homogeneous material.

Case 1: trigonometrically graded material: The gradation function is

$$g^{1/2}(\mathbf{x}) = \cos(1 - 0.55x_1 - 0.25x_2)$$

So that if the material is anisotropic then $\lambda = -0.344375$, and $\lambda = -0.365$ when it is isotropic.

Case 2: exponentially graded material: We assume

$$g^{1/2}(\mathbf{x}) = \exp(-1 + 0.55x_1 + 0.25x_2)$$

So that $\lambda = 0.344375$ if the material is anisotropic and $\lambda = 0.365$ when it is isotropic.

Case 3: quadratically graded material: We take

$$g^{1/2}(\mathbf{x}) = 1 - 0.55x_1 - 0.25x_2$$

So that $\lambda = 0$ for all combinations of the material's anisotropy and inhomogeneity.

It should be noted that when the considered material is isotropic homogeneous then the problem is symmetric about the axis $x_1 = 0.5$. This symmetry condition will be used to verify the numerical solutions.

The results for Problem 2 are shown in Figures 6, 7, 8 and 9.

When the material under consideration is homogeneous the problems for all Cases 1, 2, 3 are identical. The results are shown in Figure 6. Specifically, when the material is isotropic homogeneous the solutions μ at point $(0.1, 0.5)$ will coincide with the solutions at point $(0.9, 0.5)$. This is expected as for isotropic homogeneous material the problem is symmetric about the axis $x_1 = 0.5$. Otherwise, if the material is anisotropic then the values of μ at points $(0.1, 0.5)$ and $(0.9, 0.5)$ differ. This indicates that anisotropy of the material gives effects on the μ values.

Figures 6 – 9 also indicate that anisotropy and inhomogeneity of material give effect on the values of solution μ. This suggests that it is important to take the anisotropy and inhomogeneity into account in any applications.

Moreover, in all Figures 6 – 9 it is observed that at a point $(0.1, 0.5)$ or $(0.9, 0.5)$ the solutions μ of problems with boundary condition (on side AB) $f(t) = f_1(t) = 1$ and $f(t) = f_2(t) = 1 - \exp(-1.35t)$ converge to a steady state solution as the time increases from $t = 0.25$ to $t = 5$. This is expected as for big value of t the limit of the function $f(t) = f_2(t) = 1 - \exp(-1.35t)$ is equal to $f(t) = f_1(t) = 1$.

V. Conclusion

A combined Laplace transform and standard BEM has been used to find numerical solutions to initial boundary value problems for anisotropic functionally graded materials which are governed by the parabolic equation (1). The method is easy to implement as it uses a pure boundary integral equation (17). It also involves a time variable free fundamental solution therefore it gives more accurate solutions. It does not involve round-off error propagation as it

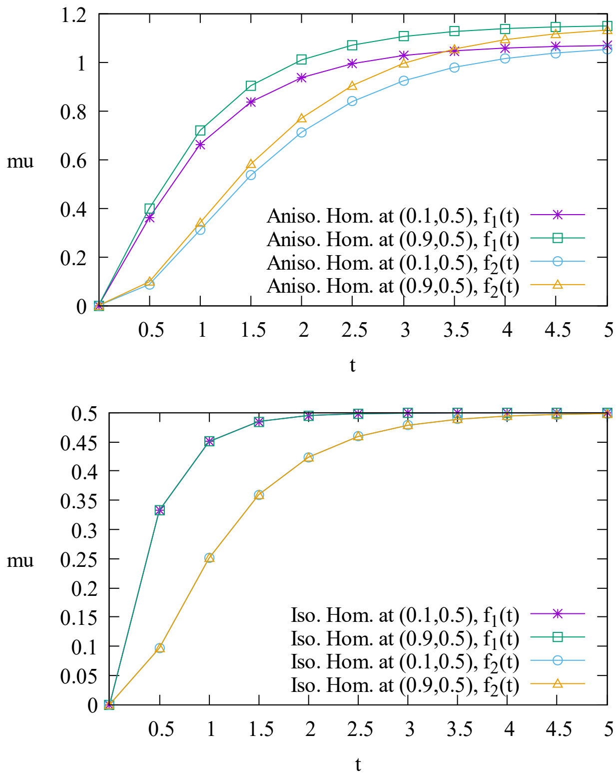

Fig. 6. Solutions μ at $(x_1, x_2) = (0.1, 0.5), (0.9, 0.5)$ for Problem 2 when the material is homogeneous with $g(\mathbf{x}) = 1$

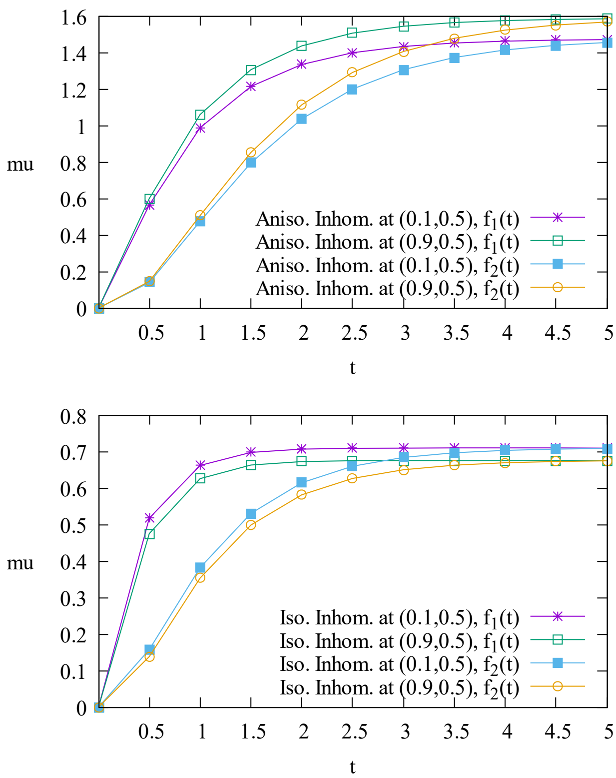

Fig. 7. Solutions μ at $(x_1, x_2) = (0.1, 0.5), (0.9, 0.5)$ for Case 1 of Problem 2

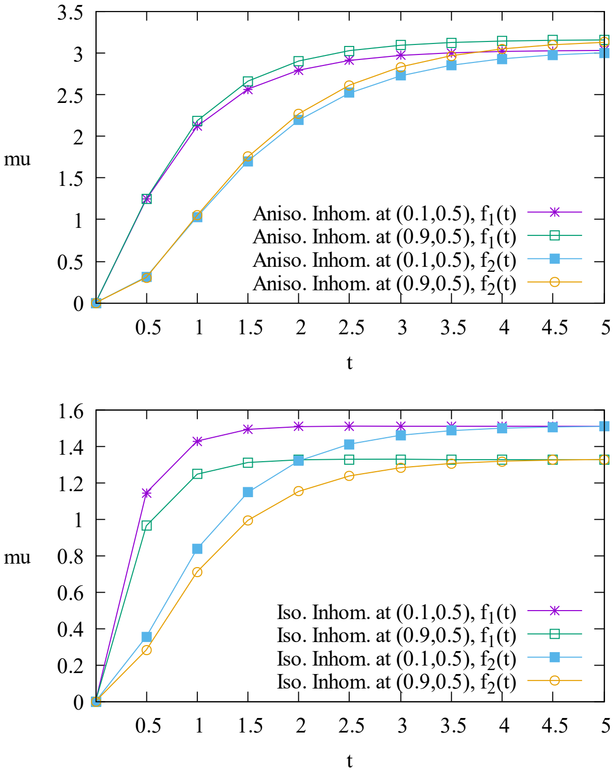

Fig. 8. Solutions μ at $(x_1, x_2) = (0.1, 0.5), (0.9, 0.5)$ for Case 2 of Problem 2

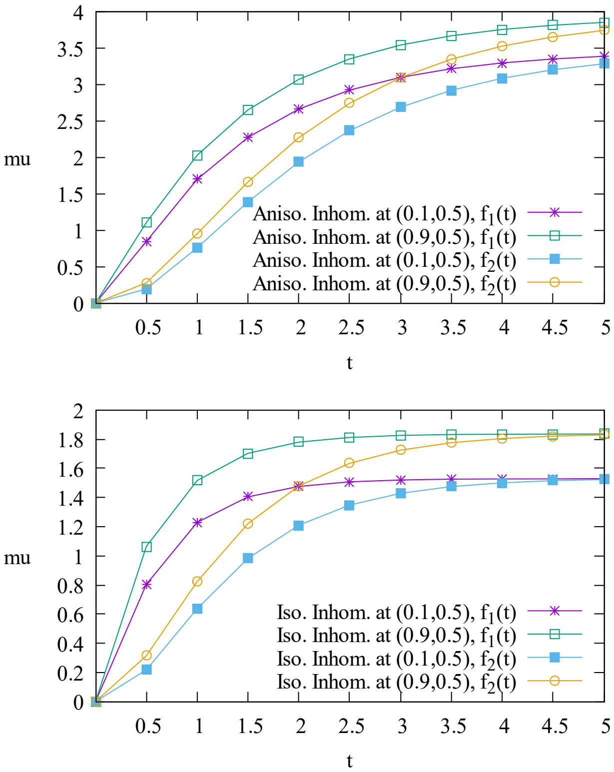

Fig. 9. Solutions μ at $(x_1, x_2) = (0.1, 0.5), (0.9, 0.5)$ for Case 3 of Problem 2

solves the boundary integral equation (17) independently for each specific value of t at which the solution is computed. Unlikely, the methods with time variable fundamental solution may produce less accurate solutions as the fundamental solution sometimes contain time singular points and also solution for the next time step is based on the solution of the previous time step so that the round-off error may propagate.

It has been applied to three classes of anisotropic functionally graded materials, namely quadratically, exponentially and trigonometrically graded materials. The quadratic inhomogeneity can be certainly written as a constant-plus-variable inhomogeneity, but each of the other two types of inhomogeneities (exponential and trigonometric) can not be simply represented as constant-plus-variable inhomogeneity.

In order to use the boundary integral equation (17), the values $\mu(\mathbf{x}, t)$ or $P(\mathbf{x}, t)$ of the boundary conditions as stated in Section II of the original system in time variable t have to be Laplace transformed first. This means that from the beginning when we set up a problem, we actually put a set of approached boundary conditions. Therefore it is really important to find a very accurate technique of numerical Laplace transform inversion. Based on the obtained results, the Stehfest formula is a quite accurate technique for the calculation of the numerical Laplace transform inverse.

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