Modulated Amplitude Waves for Two-color Vector-soliton in Nematic Liquid Crystals in the Local Response Regime

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Abstract—**This paper studies the interactions of two light beams in the nematic liquid crystal in the local response regime, which are referred to as two-color nematicons. For the light of same wavelength, we derive the traveling wave solutions of two nematicon equations under some suitable conditions. Then we find plane-wave solutions of two-color nematicon equations and examine their stability. Using a standing-wave ansatz, we obtain a system of amplitude equations for the spatially modulated states corresponding to the two-color nematicon equations in the non-resonant case. By perturbation theory and the method of averaging, we determine the equilibrium point of this averaged system, which leads to the existence of spatially periodic solutions of the system.**

Index Terms—**Nematicon, liquid crystal, plane-wave solution, average equation.**

I. Introduction

In recent years, the propagation of two-dimensional nonlinear light beams (spatial solitary waves) in nematic liquid crystals [1]–[4] and colloidal suspensions [5]–[7] have attracted considerable attentions. The reason is that scholars have proved under some conditions stable solitary waves can be generated in these media. Assanto and his collaborators first proposed the nematicons in nonlinear light beam existence experiments. They proved that stable spatial solitons may be generated in nematic liquid crystals [8], [9]. Since the equations of the nematic body are the same as those of the thermoelastic waveguide, Kuznetsov and Rubinzick concluded that the nematicon is stable [10]. Usually guided wave propagation in nematic liquid crystals is considered in the nonlocal limit, but in the experiment, the local interaction between light and the nematic will be caused by adjusting the temperature and the external electrostatic field. Therefore, we can study both the local limit and the nonlocal limit [11]. Since the equation of the nematic particles can saturate to the nonlinear Schrödinger (NLS) equation under local conditions, the nematic particles are stable [12]–[14].

Manuscript received September 30, 2020; revised March 08, 2021. This work was supported in part by the National Natural Science Foundation of China (No. 11771105), Guangxi Natural Science Foundation (Nos. 2017GXNSFFA198012, 2017GXNSFBA198130), Guangxi Distinguished Expert Project and Innovation Project of Guangxi Graduate Education (No. YCSW2020157).

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The primary consideration on nematicons researched the propagation of a single soliton. Reimbert et al. proved that in the limit of low light intensity and local material response, the full governing equations can be reduced to a higher-order nonlinear Schrödinger equation [13]. For the plane case of nematicons, the exact traveling wave solutions were obtained [15]–[17]. Recently, many authors studied the interaction of two nematicons with same wave length(color) [18] and two different wave lengths (colors) [19], [20]. Later, using a suitable method, Skuse and Smyth derived the interaction of two-color nematicons, in a nematic liquid crystal in the nonlocal and local limit. In the local limit, they using a suitable trial function derived approximate equations which control the evolution of two color nematicons [21]. In the nonlocal case, they deduced the approximate modulation equations which control the evolution of the beams, and then considered the influence of the diffractive radiation by the beams in this equations [22]. Horikis studied the stable evolution of two nonlinear interacting waves in the nonlocal nematic crystals [23]. At the same time, Horikis and Frantzeskakis also found vector solitons which can be utilized to model beam propagation in nematic liquid crystals, in the nonlinear Schrödinger system of equations [24].

In this paper, we investigate the interactions between the two color nematicons are further investigated mathematically. Firstly, to study the interactions between two color nematicons based on light of the same wavelength, we derive the traveling wave solutions of the two color nematicon equations and depict the phase diagram. In recent years, traveling wave solutions have been widely studied in the fields of physics, mathematics, economics, and so on. Therefore, the study of traveling wave solutions is particularly important. We can refer to [25]–[30] for some related development. Second, inspired by Porter et al. [31], we investigated the interactions between two color nematicons based on light of different wavelengths. In particular, we study the plane wave solutions of the two color nematicon equations and analyze its stability under certain parameter conditions. Finally, we use a standing-wave ansatz to derive a system of amplitude equations for spatially modulated states. Using the averaging method [32] we determine equilibrium points of this system and derive the existence of spatial periodic solutions of the system, which correspond to the modulated amplitude wave solutions of the system when spatial non-resonance occurs. For more results, see [33], [34]. To the best of our knowledge, some phenomena arising from spatial non-resonance states in the two color nematicon models in the local limit have not been considered before.

We consider the interaction between two polarized, coherent light beams of two different wavelengths, in the local regime, propagating through a cell filled with nematic liquid crystal. The light propagates in the z direction orthogonal to the (x, y) plane. The electric field envelopes of the two light beams are defined as $u(x, y, z)$, $v(x, y, z)$. In a static pretilt electric field case, the nondimensional equations controlling two color nematicons propagation are

$$i\frac{\partial u}{\partial z} + \frac{1}{2}D_u\nabla^2 u + A_u u \sin 2\theta = 0, \quad (1)$$

$$i\frac{\partial v}{\partial z} + \frac{1}{2}D_v\nabla^2 v + A_v v \sin 2\theta = 0, \quad (2)$$

$$\nu\nabla^2\theta - q\sin 2\theta = -2A_u|u|^2\cos 2\theta - 2A_v|v|^2\cos 2\theta, \quad (3)$$

where the Laplacian ∇^2 is in the (x, y). The $q \in \mathbb{R}$ denotes the strength of the static electric field and ν measures the elasticity of the nematic. Usually, by changing the operating temperature and the strength of the pretilt field q, we obtain the values range of ν from small(the local regime) to large(the nonlocal regime) [11]. In this work, we consider the local regime with $\nu \to 0$. The parameters D_u, $D_v \in \mathbb{R}$ are the diffraction coefficients for the two wavelengths and A_u, $A_v \in \mathbb{R}$ are the coupling coefficients between the electric fields of the light and the nematic director for the two wavelengths [35]. The θ is the perturbation of the optical director angle due to its static value of the beams. Eq. (3) reduces to

$$\tan 2\theta = \frac{2}{q}\left(A_u|u|^2 + A_v|v|^2\right), \quad (4)$$

as $\nu \to 0$. Substituting (4) into Eqs. (1) and (2), we get

$$i\frac{\partial u}{\partial z} + \frac{1}{2}D_u\nabla^2 u + \frac{2A_u\left(A_u|u|^2 + A_v|v|^2\right)u}{\sqrt{q^2 + 4\left(A_u|u|^2 + A_v|v|^2\right)^2}} = 0, \quad (5)$$

$$i\frac{\partial v}{\partial z} + \frac{1}{2}D_v\nabla^2 v + \frac{2A_v\left(A_u|u|^2 + A_v|v|^2\right)v}{\sqrt{q^2 + 4\left(A_u|u|^2 + A_v|v|^2\right)^2}} = 0. \quad (6)$$

From the above equations, we find the propagation of the two color nematicons can be written as a system of vector saturating nonlinear Schrödinger equations in the local regime.

The paper is organized as following. In Section 2, we derive the traveling wave solutions and draw phase portraits for the two nematicon equations of the same wave color length. In Section 3, we study the plane-wave solutions of two-color nematicon equations and analyze their stability. In Section 4, we introduce the modulated amplitude waves and receive equilibrium points of the average equation to obtain the existence of the space periodic solution of two-color nematicon equations under the non-resonant condition.

II. Traveling wave solutions

To research the traveling wave solution and phase portraits of two color nematicon equations, we shall introduce traveling wave transformations

$$\begin{aligned} u &= R_1(\xi)exp\left(-i\left(\lambda_1 z + \theta(\xi)\right)\right), \\ v &= R_2(\xi)exp\left(-i\left(\lambda_2 z + \theta(\xi)\right)\right), \end{aligned} \quad (7)$$

where $\xi = ax + by$, $R_j (j = 1, 2)$ and θ are functions of ξ, and a, b, $\lambda_j (j = 1, 2) \in \mathbb{R}$. Inserting Eq. (7) into Eqs. (5) and (6) yields

$$\frac{(a^2+b^2)}{2}D_u\left(R_1^{''} - R_1(\theta^{'})^2 - i\left(2R_1^{'}\theta^{'} + R_1\theta^{''}\right)\right) + \lambda_1 R_1 + \frac{2A_u\left(A_u R_1^2 + A_v R_2^2\right)R_1}{\sqrt{q^2 + 4\left(A_u R_1^2 + A_v R_2^2\right)^2}} = 0, \quad (8)$$

$$\frac{(a^2+b^2)}{2}D_v\left(R_2^{''} - R_2(\theta^{'})^2 - i\left(2R_2^{'}\theta^{'} + R_2\theta^{''}\right)\right) + \lambda_2 R_2 + \frac{2A_v\left(A_u R_1^2 + A_v R_2^2\right)R_2}{\sqrt{q^2 + 4\left(A_u R_1^2 + A_v R_2^2\right)^2}} = 0. \quad (9)$$

Suppose that $\lambda_1 \equiv \lambda_2 \equiv \lambda$ and two beams have the same color (wavelength), that is $A_u \equiv A_v \equiv A$, $D_u \equiv D_v \equiv D$, then Eqs. (8) and (9) have solution $R_1(\xi) = R_2(\xi)$. Therefore, Eqs. (8) and (9) can be reduced to a single equation

$$\frac{(a^2+b^2)}{2}D\left(R^{''} - R(\theta^{'})^2 - i\left(2R^{'}\theta^{'} + R\theta^{''}\right)\right) + \lambda R + \frac{4A^2R^3}{\sqrt{q^2 + 16A^2R^4}} = 0. \quad (10)$$

Equating the real and the imaginary components of Eq. (10), we obtain

$$\frac{(a^2+b^2)}{2}D\left(R^{''} - R(\theta^{'})^2\right) + \lambda R + \frac{4A^2R^3}{\sqrt{q^2 + 16A^2R^4}} = 0, \quad (11)$$

$$-\frac{(a^2+b^2)}{2}D\left(2R^{'}\theta^{'} + R\theta^{''}\right) = 0. \quad (12)$$

Eq. (12) implies that $R^2(\xi)\theta^{'}(\xi) = k$, where k is arbitrary integration constant. Next we consider the value of k in two cases, where $R(\xi) \neq 0$.

A. The Case of $k = 0$

When $k = 0$, we obtain $\theta^{'}(\xi) = 0$ and $\theta(\xi) = m$, where m is an arbitrary constant. Assuming $u^* = R(\xi)exp\left(-i(\lambda z + \theta(\xi))\right)$ is the solution of Eq. (10). Then $u^* exp\left(-im_0\right)$ is also the solution of Eq. (10). Therefore, according to the invariability of the solution, we assume $\theta(\xi) = m = 0$. Now Eq. (7) becomes

$$u = v = R(\xi)exp\left(-i\lambda z\right).$$

The form of Eq. (10) can be rewritten as

$$\frac{1}{2}D(a^2+b^2)R^{''} + \lambda R + \frac{4A^2R^3}{\sqrt{q^2 + 16A^2R^4}} = 0. \quad (13)$$

Defining $R^{'} = \varphi$, then Eq. (13) is equivalent to the following Hamiltonian system

$$\begin{aligned} \frac{dR}{d\xi} &= \varphi, \\ \frac{d\varphi}{d\xi} &= -\frac{8A^2}{D(a^2+b^2)}\frac{R^3}{\sqrt{q^2+16A^2R^4}} - \frac{2\lambda}{D(a^2+b^2)}R \end{aligned} \quad (14)$$

with the first integral

$$H(R,\varphi)=\frac{1}{2}\varphi^2+\frac{\lambda}{D(a^2+b^2)}R^2+\frac{\sqrt{q^2+16A^2R^4}}{4D(a^2+b^2)}=h,$$

where h is the Hamiltonian constant. Next, we consider the phase portraits of system (14). We need to obtain equilibrium points of Eq. (13) and determine the type of equilibrium points. According to the theory of planar dynamical systems in integrable systems, if the Jacobian determinant $J<0$, then the equilibrium point is the saddle point. If $J>0$ and the trace of Jacobian matrix $trN(R_i,0)=0$, then the equilibrium point is the center.

If $\lambda>0$, the system (14) has only one equilibrium point $E(0,0)$. The Jacobian determinant of the system (14) at this equilibrium point is

$$J(0,0)=\frac{2\lambda}{D(a^2+b^2)}.$$

It follows from the above conclusions that the equilibrium point is center, when $D>0$ and the equilibrium point is a saddle point, when $D<0$.

If $\lambda<0$, the system (14) has three equilibrium points

$$\begin{aligned}&E_1(0,0),\\&E_2\left(R^+,0\right)=\left(\sqrt[4]{\frac{\lambda^2q^2}{16A^2\left(A^2-\lambda^2\right)}},0\right),\\&E_3\left(R^-,0\right)=\left(-\sqrt[4]{\frac{\lambda^2q^2}{16A^2\left(A^2-\lambda^2\right)}},0\right).\end{aligned}$$

The Jacobian determinants of the system (14) at these equilibrium points are

$$J(0,0)=\frac{2\lambda}{D(a^2+b^2)}$$

and

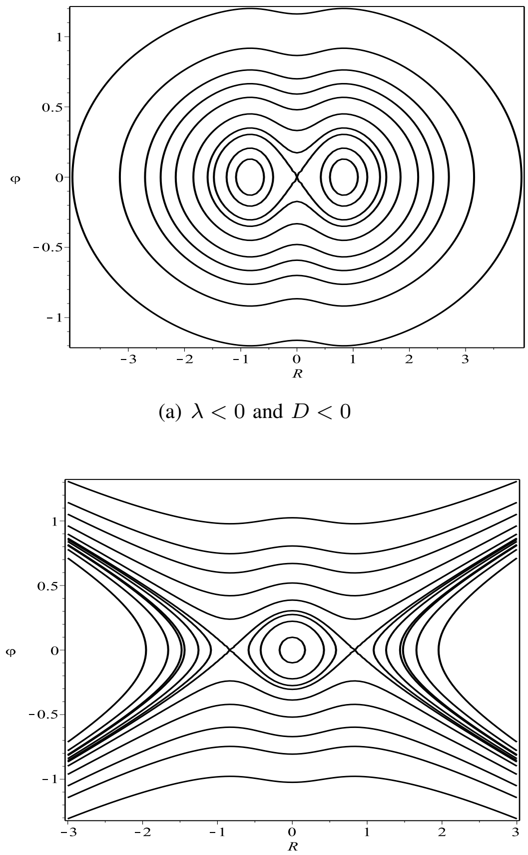
$$\begin{aligned}J(R^\pm,0)=\det N(R^\pm,0)=&\frac{2\lambda}{D(a^2+b^2)}\\&+\frac{8A^2(R^\pm)^2\left(3-32(R^\pm)^4A^2(q^2+16A^2(R^\pm)^4)^{-1}\right)}{D(a^2+b^2)\sqrt{q^2+16A^2(R^\pm)^4}}\\=&-\frac{4\lambda(\lambda^2+A^2)}{D(a^2+b^2)}.\end{aligned}$$

When $D>0$, the equilibrium points $E_1(0,0)$ is a saddle point, $E_2(R^+,0)$ and $E_3(R^-,0)$ are center. When $D<0$, the equilibrium point $E_1(0,0)$ is center, $E_2(R^+,0)$ and $E_3(R^-,0)$ are saddle points. Fig. 1 shows the phase portraits of system (14), when $\lambda<0$.

B. The Case of $k\neq 0$

When $k\neq 0$, we obtain $\theta^{'}(\xi)=k/R^2(\xi)$. Therefore, the form of Eq. (10) can be rewritten as

$$\begin{aligned}\frac{(a^2+b^2)}{2}D\left(R^{''}-\frac{k^2}{R^3}\right)+\lambda R&\\+\frac{4A^2R^3}{\sqrt{q^2+16A^2R^4}}=0.&\end{aligned}\tag{15}$$



(a) $\lambda<0$ and $D<0$

(b) $\lambda<0$ and $D>0$

Fig. 1: The phase portraits of (14)

Similarly, Eq. (15) is equivalent to the following Hamiltonian system

$$\begin{aligned}\frac{dR}{d\xi}=&\varphi,\\\frac{d\varphi}{d\xi}=&\frac{k^2}{R^3}-\frac{2\lambda}{D(a^2+b^2)}R\\&-\frac{8A^2}{D(a^2+b^2)}\frac{R^3}{\sqrt{q^2+16A^2R^4}}\end{aligned}\tag{16}$$

with the first integral

$$\begin{aligned}H(R,\varphi)=&\frac{1}{2}\varphi^2+\frac{k^2}{2R^2}+\frac{\lambda}{D(a^2+b^2)}R^2\\&+\frac{\sqrt{q^2+16A^2R^4}}{4D(a^2+b^2)}\\=&h.\end{aligned}$$

The Jacobian determinant of system (16) is

$$\begin{aligned}J(R,0)=&\det N(R,0)\\=&\frac{3k^2}{R^4}+\frac{2\lambda}{D(a^2+b^2)}+\frac{8A}{D(a^2+b^2)}\\&\left(\frac{6R^5}{\sqrt{16A^2R^4+q^2}}-\frac{32A^2R^9}{(16A^2R^4+q^2)^{3/2}}\right).\end{aligned}$$

Since it is difficult to judge the type of equilibrium points from the above formula, we analyze it numerically. Given the parameters $\lambda=-1$, $q=2$, $D=0.8$, $A=0.9$, $a^2+b^2=10$, $k^2=1$, we obtain two equilibrium points

$E_1(R_1,0) = (-2.49288,0)$ and $E_2(R_2,0) = (2.49288,0)$ of the system (16). And both of these two equilibrium points are center. The phase portraits of system (16) are shown in Fig. 2. The relevant discussion on phase portraits of such system is provided in [36].

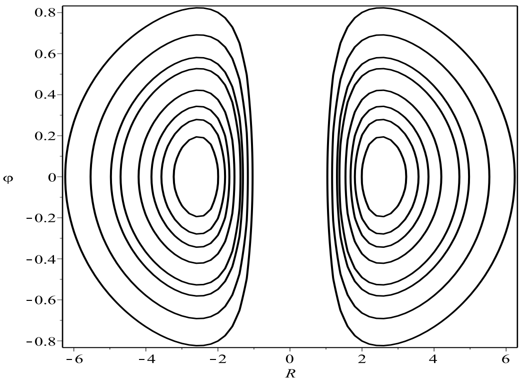


Fig. 2: The phase portraits of (16)

III. Plane-wave solutions

In this section, we derive plane-wave solutions of two-color nematicon equations with different wavelengths and analyze their stability. Eqs. (5) and (6) have the plane-wave solutions of the form

$$\begin{aligned} u &= R_1 exp\left(-i(\lambda z + cx + dy)\right), \\ v &= R_2 exp\left(-i(\lambda z + cx + dy)\right), \end{aligned} \tag{17}$$

where R_j $(j=1,2)$ are constant functions and c, d, $\lambda \in \mathbb{R}$. Inserting Eq. (17) into Eqs. (5) and (6) yields

$$\begin{aligned} &-\frac{d^2+c^2}{2}D_u R_1 + \lambda R_1 \\ &+\frac{2A_u\left(A_u R_1^2 + A_v R_2^2\right)R_1}{\sqrt{q^2+4\left(A_u R_1^2 + A_v R_2^2\right)^2}} = 0, \end{aligned} \tag{18}$$

$$\begin{aligned} &-\frac{d^2+c^2}{2}D_v R_2 + \lambda R_2 \\ &+\frac{2A_v\left(A_u R_1^2 + A_v R_2^2\right)R_2}{\sqrt{q^2+4\left(A_u R_1^2 + A_v R_2^2\right)^2}} = 0. \end{aligned} \tag{19}$$

In addition to the zero solution, Eqs. (18) and (19) have four solutions

$$\begin{aligned} &(R_1,R_2) \\ &= \left(\pm\sqrt{\frac{q}{A_u}}\frac{1}{\sqrt[4]{4A_u^2-(c^2+d^2)D_u(D_u+4\lambda)+4\lambda^2}}, 0\right), \\ &(R_1,R_2) \\ &= \left(0, \pm\sqrt{\frac{q}{A_v}}\frac{1}{\sqrt[4]{4A_v^2-(c^2+d^2)D_v(D_v+4\lambda)+4\lambda^2}}\right). \end{aligned}$$

The necessary existence conditions for these solutions are

$$\begin{aligned} &qA_u > 0, \\ &qA_v > 0, \\ &A_u^2 > \frac{D_u}{4}(D_u+4\lambda)\left(c^2+d^2\right) - \lambda^2, \\ &A_v^2 > \frac{D_v}{4}(D_v+4\lambda)\left(c^2+d^2\right) - \lambda^2. \end{aligned}$$

Eqs. (18) and (19) also have nonzero solutions (R_1, R_2), which satisfy the elliptic equation

$$A_u R_1^2 + A_v R_2^2 = \frac{q\left(D_u(c^2+d^2)-2\lambda\right)}{2\sqrt{4A_u^2-\left(D_u(c^2+d^2)-2\lambda\right)^2}}, \tag{20}$$

where $A_v\left(D_u(c^2+d^2)-2\lambda\right) = A_u\left(D_v(c^2+d^2)-2\lambda\right)$,

$$\left|\frac{D_u(c^2+d^2)-2\lambda}{2A_u}\right| < 1,$$

and $q\left(D_u(c^2+d^2)-2\lambda\right)$, A_u, A_v have the same sign. In this case, each point (R_1, R_2) of Eq. (20) corresponds to a set of plane-wave solutions Eqs. (5) and (6).

To examine the stability of the plane-wave solutions, we consider perturbed solutions of the form

$$u = \hat{u}(x,y,z)[1+\epsilon_1(x,y,z)], \tag{21}$$

$$v = \hat{v}(x,y,z)[1+\epsilon_2(x,y,z)], \tag{22}$$

where

$$\begin{aligned} \hat{u} &= R_1 exp\left(-i(\lambda z + cx + dy)\right), \\ \hat{v} &= R_2 exp\left(-i(\lambda z + cx + dy)\right), \end{aligned}$$

and $|\epsilon_j^2| \ll 1$, $\;j=1,2$. When R_1, $R_2 \neq 0$, we substitute Eqs. (21) and (22) into Eqs. (5) and (6) and reserve the first order term in ϵ_j $(j=1,2)$. Then we obtain

$$\begin{aligned} \epsilon_{1z} =& 2A_u^2 i\left(\frac{R_1^2}{\sqrt{4\left(R_1^2A_u+R_2^2A_v\right)^2+q^2}}\right. \\ &\left.-\frac{4R_1^2\left(R_1^2A_u+R_2^2A_v\right)^2}{\sqrt{\left(4\left(R_1^2A_u+R_2^2A_v\right)^2+q^2\right)^3}}\right)\epsilon_1 \\ &+i\frac{2q^2R_1^2A_u^2}{\sqrt{\left(4\left(R_1^2A_u+R_2^2A_v\right)^2+q^2\right)^3}}\epsilon_1^* \\ &+i\frac{2q^2R_2^2A_uA_v}{\sqrt{\left(4\left(R_1^2A_u+R_2^2A_v\right)^2+q^2\right)^3}}\left(\epsilon_2+\epsilon_2^*\right) \\ &+D_u\left(c\epsilon_{1x}+d\epsilon_{1y}\right)+\frac{iD_u}{2}\left(\epsilon_{1xx}+\epsilon_{1yy}\right), \end{aligned} \tag{23}$$

$$\begin{aligned} \epsilon_{2z} =& 2A_v^2 i\left(\frac{R_2^2}{\sqrt{4\left(R_1^2A_u+R_2^2A_v\right)^2+q^2}}\right. \\ &\left.-\frac{4R_2^2\left(R_1^2A_u+R_2^2A_v\right)^2}{\sqrt{\left(4\left(R_1^2A_u+R_2^2A_v\right)^2+q^2\right)^3}}\right)\epsilon_2 \\ &+i\frac{2q^2R_2^2A_v^2}{\sqrt{\left(4\left(R_1^2A_u+R_2^2A_v\right)^2+q^2\right)^3}}\epsilon_2^* \\ &+i\frac{2q^2R_1^2A_uA_v}{\sqrt{\left(4\left(R_1^2A_u+R_2^2A_v\right)^2+q^2\right)^3}}\left(\epsilon_1+\epsilon_1^*\right) \\ &+D_v\left(c\epsilon_{2x}+d\epsilon_{2y}\right)+\frac{iD_v}{2}\left(\epsilon_{2xx}+\epsilon_{2yy}\right), \end{aligned} \tag{24}$$

where $*$ denotes the complex conjugate. Assuming that ϵ_j is periodic in (x, y), it can be expanded in Fourier series

$$\epsilon_1(x,y,z) = \sum_{n=-\infty}^{\infty} \overline{\epsilon}_n(z) exp(i\mu_n(x+y)), \tag{25}$$

$$\epsilon_2(x,y,z) = \sum_{n=-\infty}^{\infty} \widetilde{\epsilon}_n(z) exp(i\mu_n(x+y)), \tag{26}$$

where the n-th mode has wavenumber μ_n. It follows from Eqs. (23), (24), (25) and (26), that the systems of ordinary differential equations

$$\frac{d}{dz}\begin{pmatrix} \overline{\epsilon}_n \\ \overline{\epsilon}^*_{-n} \\ \widetilde{\epsilon}_n \\ \widetilde{\epsilon}^*_{-n} \end{pmatrix} = G_n \begin{pmatrix} \overline{\epsilon}_n \\ \overline{\epsilon}^*_{-n} \\ \widetilde{\epsilon}_n \\ \widetilde{\epsilon}^*_{-n} \end{pmatrix}, \quad n = -\infty, \cdots \infty, \; n \neq 0$$

hold, where

$$G_n := i\begin{pmatrix} E & MR_1^2 & NR_2^2 & NR_2^2 \\ -MR_1^2 & P & -NR_2^2 & -NR_2^2 \\ NR_1^2 & NR_1^2 & F & MR_2^2 \\ -NR_1^2 & -NR_1^2 & -MR_2^2 & Q \end{pmatrix},$$

and

$$\begin{aligned} E =& 2A_u^2\left(\frac{R_1^2}{\sqrt{4\left(R_1^2A_u+R_2^2A_v\right)^2+q^2}}\right. \\ & \left. -\frac{4R_1^2\left(R_1^2A_u+R_2^2A_v\right)^2}{\sqrt{\left(4\left(R_1^2A_u+R_2^2A_v\right)^2+q^2\right)^3}}\right) \\ & + D_u\left(c+d\right)\mu_n - D_u\mu_n^2, \\ P =& -2A_u^2\left(\frac{R_1^2}{\sqrt{4\left(R_1^2A_u+R_2^2A_v\right)^2+q^2}}\right. \\ & \left. -\frac{4R_1^2\left(R_1^2A_u+R_2^2A_v\right)^2}{\sqrt{\left(4\left(R_1^2A_u+R_2^2A_v\right)^2+q^2\right)^3}}\right) \\ & + D_u\left(c+d\right)\mu_n + D_u\mu_n^2, \\ F =& 2A_v^2\left(\frac{R_2^2}{\sqrt{4\left(R_1^2A_u+R_2^2A_v\right)^2+q^2}}\right. \\ & \left. -\frac{4R_2^2\left(R_1^2A_u+R_2^2A_v\right)^2}{\sqrt{\left(4\left(R_1^2A_u+R_2^2A_v\right)^2+q^2\right)^3}}\right) \\ & + D_v\left(c+d\right)\mu_n - D_v\mu_n^2, \\ Q =& -2A_v^2\left(\frac{R_2^2}{\sqrt{4\left(R_1^2A_u+R_2^2A_v\right)^2+q^2}}\right. \\ & \left. -\frac{4R_2^2\left(R_1^2A_u+R_2^2A_v\right)^2}{\sqrt{\left(4\left(R_1^2A_u+R_2^2A_v\right)^2+q^2\right)^3}}\right) \\ & + D_v\left(c+d\right)\mu_n + D_v\mu_n^2, \end{aligned}$$

$$\begin{aligned} M =& \frac{2q^2A_u^2}{\sqrt{\left(4\left(R_1^2A_u+R_2^2A_v\right)^2+q^2\right)^3}}, \\ N =& \frac{2q^2A_uA_v}{\sqrt{\left(4\left(R_1^2A_u+R_2^2A_v\right)^2+q^2\right)^3}}. \end{aligned}$$

Furthermore, the eigenvalues $\hat{\lambda}_n$ of G_n are given by

$$\hat{\lambda}_n^4 + A\hat{\lambda}_n^3 + B\hat{\lambda}_n^2 + C\hat{\lambda}_n + D = 0, \tag{27}$$

where

$$\begin{aligned} A =& -2i\mu_n(c+d)\left(D_u+D_v\right), \\ B =& -4(c+d)^2\mu_n^2D_uD_v - FQ - EP \\ & + M^2\left(R_1^4+R_2^4\right), \\ C =& 2i\mu_n(c+d)\left(FQD_u + EPD_v\right. \\ & \left. -M^2\left(R_2^4D_u + R_1^4D_v\right)\right), \\ D =& \left(EP - M^2R_1^4\right)\left(FQ - M^2R_2^4\right) \\ & - N^2(E-P)(F-Q)R_1^2R_2^2. \end{aligned}$$

Then the expression of $\hat{\lambda}_n$ is

$$\begin{aligned} \hat{\lambda}_n =& -\frac{A}{4} \mp \frac{1}{2}\sqrt{\frac{A^2}{4} - \frac{2B}{3} + \Delta} \\ & \mp \sqrt{\frac{A^2}{2} - \frac{4B}{3} - \Delta - \frac{-A^3 + 4AB - 8C}{4\sqrt{\frac{A^2}{4} - \frac{2B}{3} + \Delta}}}, \end{aligned}$$

where

$$\begin{aligned} \Delta =& \frac{\sqrt[3]{2}\Delta_1}{3\sqrt[3]{\Delta_2 + \sqrt{-4\Delta_1^3 + \Delta_2^2}}} + \frac{\sqrt[3]{\Delta_2 + \sqrt{-4\Delta_1^3 + \Delta_2^2}}}{3\sqrt[3]{2}}, \\ \Delta_1 =& B^2 - 3AC + 12D, \\ \Delta_2 =& 2B^3 - 9ABC + 27C^2 + 27A^2C - 72BD. \end{aligned}$$

The $\hat{\lambda}_n$ are the perturbation growth rates determining the stability of the n-th mode. If $\hat{\lambda}_n$ has positive real part, then we obtain instable perturbation solutions and the side-band modes $c+\mu_n$, $c-\mu_n$, $d+\mu_n$, $d-\mu_n$ of perturbed solutions to grow exponentially, which are close enough to the fundamental modes c and d. Therefore, the plane-wave solutions are instable, see [31], [37]. Next we present a numerical example. When the parameters $\lambda = -0.7$, $q = 2$, $A_u = 1$, $D_u = 1$, $A_v = 0.9$, $D_v = 0.62$, $c^2 + d^2 = 0.5$, we obtain nonzero solutions (R_1, R_2), which satisfy elliptic equation

$$10R_1^2 + 9R_2^2 = \frac{190}{\sqrt{39}}, \tag{28}$$

where each point of Eq. (28) corresponds to a set of plane-wave solutions. For simplicity, we substitute $R_2^2 = 1$ into Eq. (28) and obtain $R_1^2 = 2.1424349$. Then substituting the above parameters into Eq. (27), we can obtain that the eigenvalue satisfies

$$\begin{aligned} & \hat{\lambda}_n^4 - 1.62\mu_n i\hat{\lambda}_n^3 + \left(-0.62\mu_n^2 - FQ - EP + 0.0051814\right)\hat{\lambda}_n^2 \\ & +\mu_n\left(FQ + 0.62EP - 0.0035647\right)i\hat{\lambda}_n \\ & + \left(EP - 0.0042545\right)\left(FQ - 0.0009269\right) \\ & -0.0000115 + 0.0004365\mu_n^2 - 0.0039892\mu_n^4 = 0, \end{aligned} \tag{29}$$

where

$$\begin{aligned}E =&0.0652251 + 0.5\mu_n - \mu_n^2,\\ P =&-0.0652251 + 0.5\mu_n + \mu_n^2,\\ F =&0.0274 + 0.31\mu_n - 0.62\mu_n^2,\\ Q =&-0.0274 + 0.31\mu_n + 0.62\mu_n^2.\end{aligned}$$

From Eq. (29), the $\hat{\lambda}_n$ has positive real part, so in the above parameters the plane-wave solution is instable.

IV. Modulate amplitude waves

In this section, we assume solutions of Eqs. (5) and (6) that describe coherent structures of the form

$$\begin{aligned}u &= R_1(\xi)exp\left(-i(\lambda_1 z + \theta(\xi))\right),\\ v &= R_2(\xi)exp\left(-i(\lambda_2 z + \theta(\xi))\right),\end{aligned} \tag{30}$$

where $\xi = ax + by$, $R_j(j = 1, 2)$ and θ are functions of ξ, and a, b, $\lambda_j(j = 1, 2) \in \mathbb{R}$. Inserting Eq. (30) into Eqs. (5) and (6), we obtain

$$\begin{aligned}&\frac{(a^2+b^2)}{2}D_u\left(R_1^{''} - R_1(\theta^{'})^2 - i\left(2R_1^{'}\theta^{'} + R_1\theta^{''}\right)\right)\\ &\quad+\lambda_1R_1 + \frac{2A_u\left(A_uR_1^2 + A_vR_2^2\right)R_1}{\sqrt{q^2 + 4\left(A_uR_1^2 + A_vR_2^2\right)^2}} = 0,\end{aligned} \tag{31}$$

$$\begin{aligned}&\frac{(a^2+b^2)}{2}D_v\left(R_2^{''} - R_2(\theta^{'})^2 - i\left(2R_2^{'}\theta^{'} + R_2\theta^{''}\right)\right)\\ &\quad+\lambda_2R_2 + \frac{2A_v\left(A_uR_1^2 + A_vR_2^2\right)R_2}{\sqrt{q^2 + 4\left(A_uR_1^2 + A_vR_2^2\right)^2}} = 0.\end{aligned} \tag{32}$$

Equating real and imaginary parts of Eqs. (31) and (32), we conclude that

$$\begin{aligned}&\frac{(a^2+b^2)}{2}D_u\left(R_1^{''} - R_1(\theta^{'})^2\right) + \lambda_1R_1\\ &\quad+\frac{2A_u\left(A_uR_1^2 + A_vR_2^2\right)R_1}{\sqrt{q^2 + 4\left(A_uR_1^2 + A_vR_2^2\right)^2}} =0,\end{aligned} \tag{33}$$

$$\begin{aligned}&\frac{(a^2+b^2)}{2}D_v\left(R_2^{''} - R_2(\theta^{'})^2\right) + \lambda_2R_2\\ &\quad+\frac{2A_v\left(A_uR_1^2 + A_vR_2^2\right)R_2}{\sqrt{q^2 + 4\left(A_uR_1^2 + A_vR_2^2\right)^2}} =0,\end{aligned} \tag{34}$$

$$-\frac{(a^2+b^2)}{2}D_u\left(2R_1^{'}\theta^{'} + R_1\theta^{''}\right) =0, \tag{35}$$

$$-\frac{(a^2+b^2)}{2}D_v\left(2R_2^{'}\theta^{'} + R_2\theta^{''}\right) =0. \tag{36}$$

It follows from Eqs. (35) and (36) that

$$\theta^{'}(\xi) = \frac{k_1}{R_1^2(\xi)} = \frac{k_2}{R_2^2(\xi)}$$

with arbitrary constants k_1, k_2. When $k_1,\ k_2 \neq 0$, we have $R_1(\xi) = mR_2(\xi)$, where $m \in \mathbb{R}$. Only when $k_1 = k_2 = 0$, the solutions $R_1(\xi)$ and $R_2(\xi)$ are different. In this case, u and v are different. In other words, under consideration of the solutions of null angular momenta, $R_1(\xi)$ and $R_2(\xi)$ are different. In this situation, Eqs. (33) and (34) have the form

$$\begin{aligned}&\frac{(a^2+b^2)}{2}D_uR_1^{''} + \lambda_1R_1\\ &\quad+\frac{2A_u\left(A_uR_1^2 + A_vR_2^2\right)R_1}{\sqrt{q^2 + 4\left(A_uR_1^2 + A_vR_2^2\right)^2}} = 0,\end{aligned} \tag{37}$$

$$\begin{aligned}&\frac{(a^2+b^2)}{2}D_vR_2^{''} + \lambda_2R_2\\ &\quad+\frac{2A_v\left(A_uR_1^2 + A_vR_2^2\right)R_2}{\sqrt{q^2 + 4\left(A_uR_1^2 + A_vR_2^2\right)^2}} = 0.\end{aligned} \tag{38}$$

Assuming $R_i^{'} = \varphi_i(i = 1, 2)$, then Eqs. (37) and (38) are equivalent to the following Hamiltonian system

$$\begin{aligned}\frac{dR_1}{d\xi} =&\varphi_1,\\ \frac{d\varphi_1}{d\xi} =&-\frac{2\lambda_1R_1}{(a^2+b^2)D_u}\\ &-\frac{4A_u\left(A_uR_1^2 + A_vR_2^2\right)R_1}{(a^2+b^2)D_u\sqrt{q^2 + 4\left(A_uR_1^2 + A_vR_2^2\right)^2}},\\ \frac{dR_2}{d\xi} =&\varphi_2,\\ \frac{d\varphi_2}{d\xi} =&-\frac{2\lambda_2R_2}{(a^2+b^2)D_v}\\ &-\frac{4A_v\left(A_uR_1^2 + A_vR_2^2\right)R_2}{(a^2+b^2)D_v\sqrt{q^2 + 4\left(A_uR_1^2 + A_vR_2^2\right)^2}}.\end{aligned}$$

By simplifying Eqs. (37) and (38), we obtain

$$\begin{aligned}&R_1^{''} + \frac{2\lambda_1}{(a^2+b^2)D_u}R_1\\ &= -\frac{4A_u\left(A_uR_1^2 + A_vR_2^2\right)R_1}{(a^2+b^2)D_u\sqrt{q^2 + 4\left(A_uR_1^2 + A_vR_2^2\right)^2}},\end{aligned} \tag{39}$$

$$\begin{aligned}&R_2^{''} + \frac{2\lambda_2}{(a^2+b^2)D_v}R_2\\ &= -\frac{4A_v\left(A_uR_1^2 + A_vR_2^2\right)R_2}{(a^2+b^2)D_v\sqrt{q^2 + 4\left(A_uR_1^2 + A_vR_2^2\right)^2}}.\end{aligned} \tag{40}$$

When $\lambda_1/D_u = \lambda_2/D_v$, we use the notation

$$\alpha := \frac{2\lambda_1}{(a^2+b^2)D_u} = \frac{2\lambda_2}{(a^2+b^2)D_v}.$$

Then Eqs. (39) and (40) have the form

$$\begin{aligned}&R_1^{''} + \alpha R_1\\ &= -\frac{4A_u\left(A_uR_1^2 + A_vR_2^2\right)R_1}{(a^2+b^2)D_u\sqrt{q^2 + 4\left(A_uR_1^2 + A_vR_2^2\right)^2}},\end{aligned} \tag{41}$$

$$\begin{aligned}&R_2^{''} + \alpha R_2\\ &= -\frac{4A_v\left(A_uR_1^2 + A_vR_2^2\right)R_2}{(a^2+b^2)D_v\sqrt{q^2 + 4\left(A_uR_1^2 + A_vR_2^2\right)^2}}.\end{aligned} \tag{42}$$

In fact, Porter et al. used perturbation method and first order averaging to discuss resonant and non-resonant modulated amplitude waves for binary Bose-Einstein condensates in optical lattices, see [31]. To achieve some analytical understanding of the spatial resonances in two-color vector-soliton in nematic liquid crystals, we use a perturbation method to average Eqs. (41) and (42). Defining $A_u \equiv \sqrt{\epsilon}\widetilde{A}_u$, $A_v \equiv \sqrt{\epsilon}\widetilde{A}_v$, then Eqs. (41) and (42) can be written as

$$\begin{aligned} & R_1^{''} + \alpha R_1 \\ & = -\frac{4\epsilon\widetilde{A}_u\left(\widetilde{A}_u R_1^2 + \widetilde{A}_v R_2^2\right) R_1}{(a^2+b^2)D_u\sqrt{q^2+4\epsilon\left(\widetilde{A}_u R_1^2 + \widetilde{A}_v R_2^2\right)^2}}, \end{aligned} \quad (43)$$

$$\begin{aligned} & R_2^{''} + \alpha R_2 \\ & = -\frac{4\epsilon\widetilde{A}_v\left(\widetilde{A}_u R_1^2 + \widetilde{A}_v R_2^2\right) R_2}{(a^2+b^2)D_v\sqrt{q^2+4\epsilon\left(\widetilde{A}_u R_1^2 + \widetilde{A}_v R_2^2\right)^2}}. \end{aligned} \quad (44)$$

Assuming $\alpha > 0$, we insert the ansatz

$$R_j = A_j(\xi)\cos\left(\sqrt{\alpha}\xi\right) + B_j(\xi)\sin\left(\sqrt{\alpha}\xi\right), \quad (45)$$

$$\begin{aligned} R_j^{'} = & -\sqrt{\alpha}A_j(\xi)\sin\left(\sqrt{\alpha}\xi\right) \\ & + \sqrt{\alpha}B_j(\xi)\cos\left(\sqrt{\alpha}\xi\right), \quad j=1,2 \end{aligned} \quad (46)$$

into Eqs. (43) and (44). By differentiating Eq. (45) and comparing it with Eq. (46), we obtain

$$A_j^{'}(\xi)\cos\left(\sqrt{\alpha}\xi\right) + B_j^{'}(\xi)\sin\left(\sqrt{\alpha}\xi\right) = 0, \quad j=1,2.$$

Inserting the above three equations into Eqs. (43) and (44) and doing Taylor series in ϵ at $\epsilon = 0$ of the Eqs. (43) and (44) generates a set of coupled differential equations for A_j and B_j, whose right side comes from the contributions of different harmonics [31]. Hence we obtain

$$\begin{aligned} A_1^{'} = & \epsilon\frac{4\widetilde{A}_u\left(\widetilde{A}_u R_1^2 + \widetilde{A}_v R_2^2\right) R_1 \sin\left(\sqrt{\alpha}\xi\right)}{(a^2+b^2)D_u q\sqrt{\alpha}} + O(\epsilon^2), \\ B_1^{'} = & -\epsilon\frac{4\widetilde{A}_u\left(\widetilde{A}_u R_1^2 + \widetilde{A}_v R_2^2\right) R_1 \cos\left(\sqrt{\alpha}\xi\right)}{(a^2+b^2)D_u q\sqrt{\alpha}} + O(\epsilon^2), \\ A_2^{'} = & \epsilon\frac{4\widetilde{A}_v\left(\widetilde{A}_u R_1^2 + \widetilde{A}_v R_2^2\right) R_2 \sin\left(\sqrt{\alpha}\xi\right)}{(a^2+b^2)D_v q\sqrt{\alpha}} + O(\epsilon^2), \\ B_2^{'} = & -\epsilon\frac{4\widetilde{A}_v\left(\widetilde{A}_u R_1^2 + \widetilde{A}_v R_2^2\right) R_2 \cos\left(\sqrt{\alpha}\xi\right)}{(a^2+b^2)D_v q\sqrt{\alpha}} + O(\epsilon^2), \end{aligned} \quad (47)$$

where $R_1 \equiv R_1(A_1, B_1, \xi)$, $R_2 \equiv R_2(A_2, B_2, \xi)$. The system (47) has the form

$$\begin{aligned} A_1^{'} = & \epsilon F_{A_1}(A_1, A_2, B_1, B_2, \xi) + O(\epsilon^2), \\ B_1^{'} = & \epsilon F_{B_1}(A_1, A_2, B_1, B_2, \xi) + O(\epsilon^2), \\ A_2^{'} = & \epsilon F_{A_2}(A_1, A_2, B_1, B_2, \xi) + O(\epsilon^2), \\ B_2^{'} = & \epsilon F_{B_2}(A_1, A_2, B_1, B_2, \xi) + O(\epsilon^2), \end{aligned} \quad (48)$$

where

$$\begin{aligned} F_{A_1} = & \frac{G_{A_1}}{(a^2+b^2)D_u q\sqrt{\alpha}}, \quad F_{B_1} = \frac{G_{B_1}}{(a^2+b^2)D_u q\sqrt{\alpha}}, \\ F_{A_2} = & \frac{G_{A_2}}{(a^2+b^2)D_v q\sqrt{\alpha}}, \quad F_{B_2} = \frac{G_{B_2}}{(a^2+b^2)D_v q\sqrt{\alpha}}. \end{aligned}$$

For convenience, we write $\widetilde{A}_u$ and $\widetilde{A}_v$ as A_u and A_v. Since the right side of the equation consists of different harmonics, the functions F_{A_j} and F_{B_j} can be written as a sum of harmonic contributions.

Through calculation, we obtain

$$\begin{aligned} & G_{A_1}(A_1, B_1, A_2, B_2, \xi) \\ & = \left[A_1A_2B_2A_uA_v + \frac{B_1A_uA_v}{2}\left(A_2^2+3B_2^2\right)\right. \\ & \quad \left. + \frac{3B_1A_u^2}{2}\left(A_1^2+B_1^2\right)\right] \\ & \quad + \left[2A_2B_1B_2A_uA_v + A_1A_uA_v\left(A_2^2+B_2^2\right)\right. \\ & \quad \left. + A_1A_u^2\left(A_1^2+3B_1^2\right)\right]\sin\left(2\sqrt{\alpha}x\right) \\ & \quad + \left[-A_vA_2B_1B_2A_u + \frac{A_1A_uA_v}{2}\left(A_2^2-B_2^2\right)\right. \\ & \quad \left. + \frac{A_1A_u^2}{2}\left(A_1^2-3B_1^2\right)\right]\sin\left(4\sqrt{\alpha}\xi\right) \\ & \quad + \left[-2B_2^2B_1A_uA_v - 2B_1^3A_u^2\right]\cos\left(2\sqrt{\alpha}\xi\right) \\ & \quad + \left[-A_vA_1A_2B_2A_u + \frac{B_1A_uA_v}{2}\left(B_2^2-A_2^2\right)\right. \\ & \quad \left. + \frac{B_1A_u^2}{2}\left(B_1^2-3A_1^2\right)\right]\cos\left(4\sqrt{\alpha}\xi\right), \end{aligned}$$

$$\begin{aligned} & G_{B_1}(A_1, B_1, A_2, B_2, \xi) \\ & = -\left[A_2B_1B_2A_uA_v + \frac{A_1A_uA_v}{2}\left(3A_2^2+B_2^2\right)\right. \\ & \quad \left. + \frac{3A_1A_u^2}{2}\left(A_1^2+B_1^2\right)\right] \\ & \quad - \left[2A_1A_2B_2A_uA_v + B_1A_uA_v\left(A_2^2+B_2^2\right)\right. \\ & \quad \left. + B_1A_u^2\left(3A_1^2+B_1^2\right)\right]\sin\left(2\sqrt{\alpha}\xi\right) \\ & \quad - \left[A_1A_2B_2A_uA_v + \frac{B_1A_uA_v}{2}\left(A_2^2-B_2^2\right)\right. \\ & \quad \left. + \frac{B_1A_u^2}{2}\left(3A_1^2-B_1^2\right)\right]\sin\left(4\sqrt{\alpha}\xi\right) \\ & \quad - \left[2A_2^2A_1A_uA_v + 2A_1^3A_u^2\right]\cos\left(2\sqrt{\alpha}\xi\right) \\ & \quad - \left[-A_vA_2B_1B_2A_u + \frac{A_1A_uA_v}{2}\left(A_2^2-B_2^2\right)\right. \\ & \quad \left. + \frac{A_1A_u^2}{2}\left(A_1^2-3B_1^2\right)\right]\cos\left(4\sqrt{\alpha}x\right), \end{aligned}$$

$$\begin{aligned} & G_{A_2}(A_1, B_1, A_2, B_2, \xi) \\ & = \left[A_1A_2B_1A_uA_v + \frac{B_2A_uA_v}{2}\left(A_1^2+3B_1^2\right)\right. \\ & \quad \left. + \frac{3B_2A_v^2}{2}\left(A_2^2+B_2^2\right)\right] \\ & \quad + \left[A_2A_uA_v\left(A_1^2+B_1^2\right) + 2A_1B_1B_2A_uA_v\right. \\ & \quad \left. + A_2A_v^2\left(A_2^2+3B_2^2\right)\right]\sin\left(2\sqrt{\alpha}\xi\right) \\ & \quad + \left[-A_uA_1B_2B_1A_v + \frac{A_2A_uA_v}{2}\left(A_1^2-B_1^2\right)\right. \\ & \quad \left. + \frac{A_2A_v^2}{2}\left(A_2^2-3B_2^2\right)\right]\sin\left(4\sqrt{\alpha}\xi\right) \\ & \quad + \left[-2B_1^2B_2A_uA_v - 2B_2^3A_v^2\right]\cos\left(2\sqrt{\alpha}\xi\right) \\ & \quad + \left[-A_uA_1A_2B_1A_v + \frac{B_2A_uA_v}{2}\left(B_1^2-A_1^2\right)\right. \\ & \quad \left. + \frac{B_2A_v^2}{2}\left(B_2^2-3A_2^2\right)\right]\cos\left(4\sqrt{\alpha}\xi\right), \end{aligned}$$

$$\begin{aligned} & G_{B_2}(A_1, B_1, A_2, B_2, \xi) \\ & = -\left[A_1B_1B_2A_uA_v + \frac{A_2A_uA_v}{2}\left(3A_1^2+B_1^2\right)\right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{3A_2A_v^2}{2}\left(A_2^2+B_2^2\right)\Big]\\
&-\left[2A_1A_2B_1A_uA_v+B_2A_uA_v\left(A_1^2+B_1^2\right)\right.\\
&\left.+B_2A_v^2\left(3A_2^2+B_2^2\right)\right]\sin\left(2\sqrt{\alpha}\xi\right)\\
&-\Big[A_1A_2B_1A_uA_v+\frac{B_2A_uA_v}{2}\left(A_1^2-B_1^2\right)\\
&+\frac{B_2A_v^2}{2}\left(3A_2^2-B_2^2\right)\Big]\sin\left(4\sqrt{\alpha}\xi\right)\\
&-\left[2A_1^2A_2A_uA_v+2A_2^3A_v^2\right]\cos\left(2\xi\sqrt{\alpha}\right)\\
&-\Big[-A_1B_2B_1A_uA_v+\frac{A_2A_uA_v}{2}\left(A_1^2-B_1^2\right)\\
&+\frac{A_2A_v^2}{2}\left(A_2^2-3B_2^2\right)\Big]\cos\left(4\sqrt{\alpha}\xi\right).
\end{aligned}
$$

Next we consider the non-resonant case. First, in system (48), we separate the parts of the functions $A_j(\xi)$ and $B_j(\xi)$ which vary slowly in contrast with the fast oscillations of $\cos(\sqrt{\alpha}\xi)$ and $\sin(\sqrt{\alpha}\xi)$ and to receive averaged equations governing their slow evolution. Then we use a method which is equivalent to first order averaging to compute the equilibrium points of the averaged equations. All the equilibrium points, except for the trivial equilibrium point, correspond to spatially periodic traveling wave solutions of the Eqs. (43) and (44). In the non-resonant, the averaged equations governing the slow evolution are

$$
\begin{aligned}
A_1^{'}=&\frac{\epsilon}{(a^2+b^2)D_uq\sqrt{\alpha}}\Big[A_1A_2B_2A_uA_v+\frac{B_1A_uA_v}{2}\\
&\left(A_2^2+3B_2^2\right)+\frac{3B_1A_u^2}{2}\left(A_1^2+B_1^2\right)\Big], \qquad (49)\\
A_2^{'}=&\frac{\epsilon}{(a^2+b^2)D_vq\sqrt{\alpha}}\Big[A_1A_2B_1A_uA_v+\frac{B_2A_uA_v}{2}\\
&\left(A_1^2+3B_1^2\right)+\frac{3B_2A_v^2}{2}\left(A_2^2+B_2^2\right)\Big], \qquad (50)\\
B_1^{'}=&\frac{\epsilon}{(a^2+b^2)D_uq\sqrt{\alpha}}\Big[-A_2B_1B_2A_uA_v-\frac{A_1A_uA_v}{2}\\
&\left(3A_2^2+B_2^2\right)-\frac{3A_1A_u^2}{2}\left(A_1^2+B_1^2\right)\Big], \qquad (51)\\
B_2^{'}=&\frac{\epsilon}{(a^2+b^2)D_vq\sqrt{\alpha}}\Big[-A_1B_1B_2A_uA_v-\frac{A_2A_uA_v}{2}\\
&\left(3A_1^2+B_1^2\right)-\frac{3A_2A_v^2}{2}\left(A_2^2+B_2^2\right)\Big]. \qquad (52)
\end{aligned}
$$

When $A_1^{'}=A_2^{'}=B_1^{'}=B_2^{'}=0$, the non-resonant averaged equations (49), (50), (51) and (52) have three types of equilibria, which are trivial equilibria, double mode equilibria and quadruple mode equilibria. Double modes and quadruple modes indicate that there are two and four nonzero amplitudes A_j, B_j, respectively. Through calculation, we obtain two types of double mode equilibria. They are "A_1A_2" equilibria with $A_1, A_2 \neq 0$ and $B_1 = B_2 = 0$ and "B_1B_2" equilibria with $B_1, B_2 \neq 0$ and $A_1 = A_2 = 0$.

When $A_u/A_v < 0$, double mode equilibria are

$$
(A_1,A_2,B_1,B_2)=\left(A_1,\pm\sqrt{-\frac{A_u}{A_v}}A_1,0,0\right),
$$

$$
(A_1,A_2,B_1,B_2)=\left(0,0,B_1,\pm\sqrt{-\frac{A_u}{A_v}}B_1\right),
$$

respectively "A_1A_2" equilibria and "B_1B_2" equilibria, where A_1, B_1 are arbitrary. In this case, the two components have unequal amplitudes. Therefore, these types of equilibria correspond to the asymmetric periodic traveling wave solutions of Eqs. (43) and (44).

When $A_v = -3A_u$, two sets of quadruple mode equilibria satisfy $A_1 = -A_2,\ B_1 = B_2$ and $A_1 = A_2,\ B_1 = -B_2$, respectively. The form of quadruple mode equilibria are

$$
\begin{aligned}
(A_1,A_2,B_1,B_2)&=\left(-A_2,A_2,\pm A_2,\pm A_2\right),\\
(A_1,A_2,B_1,B_2)&=\left(\pm B_2,\pm B_2,-B_2,B_2\right),
\end{aligned}
$$

with A_2, B_2 are arbitrary.

Except for the trivial equilibrium point, all equilibrium points of Eqs. (49), (50), (51) and (52) correspond to spatially periodic traveling wave solutions of Eqs. (43) and (44). Therefore, each of the above four sets of equilibrium points correspond to the spatially periodic traveling wave solutions of Eqs. (43) and (44). Therefore, two color nematicon equations exist spatially periodic traveling wave solutions.

V. CONCLUSION

In this paper, we discuss the interactions of two polarized, coherent light beams of different wavelengths propagating through a cell filled with nematic liquid crystals. In the case of light of the same wavelengths and different wavelength, traveling wave solutions and plane-wave solutions of two-color nematicon equations are given, respectively. The methods of perturbation and the first order averaging are employed in this paper. By solving the average equation of slowly varying parts in the non-resonant, the existence of the space periodic solutions of two-color nematicon systems are proved. These spatial periodic solutions correspond to the modulated amplitude waves of two-color nematicon equations in the spatially non-resonant state.

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