Numerical Solution of Coupled Burgers' Equation Using Finite Difference and Sinc Collocation Method

Linjun Wang, Honglei Li, Yiping Meng

***Abstract*—This paper is concerned with numerical solution of coupled Burgers' equation by the combination of finite difference and sinc collocation method. Firstly, we derive the semi-discrete scheme by approximating the first order derivative of time with θ-weighted scheme. Different schemes can be obtained by selecting different values of θ. After that, a fully discrete scheme is constructed through the use of sinc collocation and finite difference method to approximate the first and second order derivatives of space. The stability of the fully discrete scheme is analyzed by representing the proposed scheme in matrix form. For Burgers' equation, the similar results could be obtained. At last, some numerical examples are presented to illustrate the efficiency and superiority of present method for solving Burgers' and coupled Burgers' equation.**

***Index Terms*—Burgers' equation, Coupled Burgers' equation, Finite difference method, Sinc collocation method**

I. Introduction

MANY physical phenomena, such as hydrodynamics [1, 2], nonlinear acoustics [3], gas dynamics [4] and traffic flow dynamics [5] can be described by Burgers' equation. It was first introduced by Bateman [6] in 1915 and later treated by Burgers [7]. Burgers' equation can also be considered as a simplified form of Navier-Stokes equation. There are both non-linear convection and diffusion terms in Burgers' equation, which is considered as the most primitive tool to describe convection and diffusion problems. Burgers' equation with the initial and boundary conditions has the following form [8]:

$$\begin{aligned} &u_t + uu_x = \varepsilon u_{xx},\ 0<x<L,\ 0<t\le T,\\ &u(x,0)=\varphi(x),\ 0<x<L,\\ &u(0,t)=0,\ u(L,t)=0,\ 0<t\le T, \end{aligned} \tag{1}$$

where $\varepsilon > 0$ is the coefficient of kinematic viscosity. Coupled Burgers equation plays an important role in physics [9, 10].

In this paper, we study coupled Burgers' equation with initial and boundary conditions [11, 12]:

$$\begin{aligned} &u_t + \delta u_{xx} + \eta u u_x + \alpha(uv)_x = 0,\ 0<x<L,\ 0<t\le T,\\ &v_t + \mu v_{xx} + \xi v v_x + \beta(uv)_x = 0,\ 0<x<L,\ 0<t\le T,\\ &u(x,0)=f(x), v(x,0)=g(x),\ 0<x<L,\\ &u(0,t)=0,\ u(L,t)=0,\\ &v(0,t)=0,\ v(L,t)=0,\ 0<t\le T, \end{aligned} \tag{2}$$

where f, g are known functions, δ, μ, η and ξ are real constants, α and β are arbitrary constants depending on the system parameters.

So far, many powerful techniques have been applied to obtain numerical solutions of Burgers' and coupled Burgers' equation. Deng and Pan [13] made use of a fourth-order singly diagonally implicit Runge-Kutta method for solving Burgers' equation. Sari et al. [14] obtained the numerical solution through sixth-order compact finite difference method. In addition, Kutluay [15] solved coupled Burgers' equation by Galerkin quadratic B-spline finite element method. Srivastava et al. [16] applied a fully implicit finite-difference method to study coupled Burgers' equation. Apart from these methods, there are other methods to solve Burgers' and coupled Burgers' equation numerically, such as Chebyshev wavelet method [17], interval finite-difference method [18], high-order exponential time differening method [19].

Stenger [20] originally introduced Sinc collocation method. In the process of using sinc collocation method, we need to obtain Whittaker cardinal expansion based on sinc function. As mentioned in [21], there are many advantages to investigate numerical solutions by using approximations based on sinc collocation method. It is widely used for solving integral equation [22] and differential equation [23]. Though finite difference method is an easy-to-implement method, it is not flexible enough and cannot be applied directly to solve some complex problems. To overcome its deficiency, many mixed methods have been proposed to solve complex problems by combining other methods with finite difference method [24–26].

Motivated by the idea from [25, 27], we will suggest a mixed method to simulate numerical solution for coupled Burgers' equation by using finite difference and sinc collocation method. Specifically, the semi-discrete scheme is presented by approximating temporal derivative with θ-weighted scheme. After that, we derive a fully discrete scheme of coupled Burgers' equation by the use of sinc collocation and finite difference method to approximate the first and second order derivatives of space. It is noted that the mixed method is also valid for solving Burgers' equation.

Manuscript received October 9, 2020; revised February 7, 2021. This work was supported by National Natural Science Foundation of China (No.11601192, 11901249).

Linjun Wang is a Professor of School of Mathematical Science, Jiangsu University, Zhenjiang (phone:+86051188780161, email: wanglinjun@ujs.edu.cn). Linjun Wang is the corresponding author.

Honglei Li is a graduate student of School of Mathematical Science, Jiangsu University, Zhenjiang (email: 843419945@qq.com).

Yiping Meng is an associate Professor of School of Science, Jiangsu University of Science and Technology, Zhenjiang (email: myp_just@163.com).

The rest of the paper is organized as follows. In Section II, we give the discrete scheme of coupled Burgers' equation. In Section III, we discuss the stability of the discrete scheme based upon the knowledge of matrix analysis. Section IV present some numerical examples. Section V concludes.

II. Discrete scheme of coupled Burgers' equation

In this section, some basic knowledge of sinc function is reviewed.

On the whole real line, the sinc function is defined by [21, 28, 29]

$$sinc(x)=\begin{cases}\dfrac{sin(\pi x)}{\pi x}, & x\neq 0,\\ 0, & x=0,\end{cases} \tag{3}$$

and the j-th translate of the sinc function is defined by

$$S(j,h)(x)=sinc(\frac{x-jh}{h}), \tag{4}$$

where j is an integer and h is a positive number. It is not difficult to obtain the following property

$$S(j,h)(kh)=\delta_{kj}=\begin{cases}1, & k=j,\\ 0, & k\neq j,\end{cases} \tag{5}$$

where k is an integer.

The Whittaker cardinal expansion of u defined on the whole real line is

$$C(u,h)(x)=\sum_{j=-\propto}^{\propto}u(jh)S(j,h)(x), \tag{6}$$

whenever this series converges.

A. The semi-discrete scheme

Take two positive integers M and N. Let $h=L/M$, $\tau=T/N$; $x_i=ih$, $0\leq i\leq M$; $t_k=k\tau$, $0\leq k\leq N$. By approximating the first-order derivative of time with θ-weighted scheme, the semi-discrete scheme of coupled Burgers' equation can be derived

$$\begin{aligned}&\frac{u^{k+1}-u^k}{\tau}+\theta(\delta u_{xx}^{k+1}+\eta u^{k+1}u_x^{k+1}+\alpha u^{k+1}v_x^{k+1}+\alpha v^{k+1}\\&\times u_x^{k+1})+(1-\theta)(\delta u_{xx}^k+\eta u^ku_x^k+\alpha u^kv_x^k+\alpha v^ku_x^k)=0,\\&\frac{v^{k+1}-v^k}{\tau}+\theta(\mu v_{xx}^{k+1}+\xi v^{k+1}v_x^{k+1}+\beta u^{k+1}v_x^{k+1}+\beta v^{k+1}\\&\times u_x^{k+1})+(1-\theta)(\mu v_{xx}^k+\xi v^kv_x^k+\beta u^kv_x^k+\beta v^ku_x^k)=0,\end{aligned} \tag{7}$$

where $x\in(0,L)$, $\theta\in[0,1]$, u^{k+1} denotes $u(x,t_{k+1})$. For the special case of $\theta=0$, semi-discrete scheme (7) can be considered as explicit scheme. When $\theta=1$, scheme (7) can be regarded as implicit scheme and when $\theta=\frac{1}{2}$, scheme (7) is just Crank-Nicholson scheme.

Next, we use Taylor expansion to deal with the nonlinear terms in (7). According to Taylor expansion, we have

$$\begin{aligned}&u(x,t_{k+1})=u(x,t_k)+\tau u_t(x,t_k)+O(\tau^2),\\&u_x(x,t_{k+1})=u_x(x,t_k)+\tau u_{xt}(x,t_k)+O(\tau^2),\\&v(x,t_{k+1})=v(x,t_k)+\tau v_t(x,t_k)+O(\tau^2),\\&v_x(x,t_{k+1})=v_x(x,t_k)+\tau v_{xt}(x,t_k)+O(\tau^2).\end{aligned}$$

Thus, we have

$$\begin{aligned}&u^{k+1}u_x^{k+1}\\&=u^ku_x^k+\tau u_x(x,t_k)u_t(x,t_k)+\tau u(x,t_k)u_{xt}(x,t_k)+O(\tau^2)\\&=u^ku_x^k+\tau u_x^k(\frac{u^{k+1}-u^k}{\tau})+\tau u^k(\frac{u_x^{k+1}-u_x^k}{\tau})+O(\tau^2)\\&=u^{k+1}u_x^k+u^ku_x^{k+1}-u^ku_x^k+O(\tau^2).\end{aligned}$$

The similar results can be obtained

$$\begin{aligned}v^{k+1}u_x^{k+1}&=u_x^{k+1}v^k+v^{k+1}u_x^k-v^ku_x^k+O(\tau^2),\\u^{k+1}v_x^{k+1}&=v_x^{k+1}u^k+u^{k+1}v_x^k-u^kv_x^k+O(\tau^2).\end{aligned}$$

Dropping the second order small quantities with respect to τ^2 and substituting them into (7), we get the semi-discrete scheme of coupled Burgers' equation

$$\begin{aligned}&u^{k+1}+\theta\tau\delta u_{xx}^{k+1}+\theta\tau\eta u^{k+1}u_x^k+\theta\tau\eta u^ku_x^{k+1}+\theta\tau\alpha u_x^{k+1}v^k\\&+\theta\tau\alpha v^{k+1}u_x^k+\theta\tau\alpha u^{k+1}v_x^k+\theta\tau\alpha v_x^{k+1}u^k=\\&u^k+(\theta-1)\tau\delta u_{xx}^k+(2\theta-1)\tau\eta u^ku_x^k+(2\theta-1)\tau\alpha u_x^kv^k\\&+(2\theta-1)\tau\alpha v_x^ku^k,\\&v^{k+1}+\theta\tau\mu v_{xx}^{k+1}+\theta\tau\xi v^{k+1}v_x^k+\theta\tau\xi v^kv_x^{k+1}+\theta\tau\beta u_x^{k+1}v^k\\&+\theta\tau\beta v^{k+1}u_x^k+\theta\tau\beta u^{k+1}v_x^k+\theta\tau\beta v_x^{k+1}u^k=\\&v^k+(\theta-1)\tau\mu v_{xx}^k+(2\theta-1)\tau\xi v^kv_x^k+(2\theta-1)\tau\beta u_x^kv^k\\&+(2\theta-1)\tau\beta v_x^ku^k,\ x\in(0,L).\end{aligned} \tag{8}$$

Remark 1. Similarly, Burgers' equation has the semi-discrete scheme read as

$$\begin{aligned}&u^{k+1}+\theta\tau(u^{k+1}u_x^k+u^ku_x^{k+1}-\varepsilon u_{xx}^{k+1})\\&=u^k+(2\theta-1)\tau u^ku_x^k+(1-\theta)\tau\varepsilon u_{xx}^k,\ x\in(0,L).\end{aligned} \tag{9}$$

B. The fully discrete scheme

We use central difference formulas [30] to approximate the first and second order spatial derivatives as follows

$$\begin{aligned}u_x(x_i,t_k)&\approx\frac{u_{i+1}^k-u_{i-1}^k}{2h},\\u_{xx}(x_i,t_k)&\approx\frac{u_{i+1}^k-2u_i^k+u_{i-1}^k}{h^2},\end{aligned} \tag{10}$$

where u_i^k denotes $u(x_i,t_k)$.

On the other hand, by (3)-(5), we get

$$S^{'}(j,h)(x_i)=\begin{cases}\dfrac{(-1)^{i-j}}{(i-j)h}, & i\neq j,\\ 0, & i=j.\end{cases} \tag{11}$$

For convenience, we denote $\mathbf{S}=(\mathbf{S_{ij}})=S^{'}(j,h)(x_i)$. By using the boundary conditions of coupled Burgers' equation (2) together with (6), we have

$$u_x(x_i,t_k)=\sum_{j=1}^{M-1}S_{ij}u_j^k. \tag{12}$$

Just like [25], we rewrite (12) in the following form

$$\mathbf{u}_x=\mathbf{S}\mathbf{u}^k,$$

where

$$\begin{aligned}\mathbf{u}^k&=(u_1^k,u_2^k,\cdots,u_{M-1}^k)^T,\\\mathbf{u}_x&=(u_x(x_1,t_k),u_x(x_2,t_k),\cdots,u_x(x_{M-1},t_k))^T.\end{aligned}$$

Substituting (10), (12) into (8), the fully discrete scheme of coupled Burgers' equation yields

$$
\begin{aligned}
&u_i^{k+1} + \theta\tau\delta\frac{u_{i+1}^{k+1} - 2u_i^{k+1} + u_{i-1}^{k+1}}{h^2} + \theta\tau\eta u_i^{k+1}\sum_{j=1}^{M-1} S_{ij}u_j^k \\
&+ \theta\tau\eta u_i^k\frac{u_{i+1}^{k+1} - u_{i-1}^{k+1}}{2h} + \theta\tau\alpha v_i^k\frac{u_{i+1}^{k+1} - u_{i-1}^{k+1}}{2h} + \theta\tau\alpha v_i^{k+1}\times \\
&\sum_{j=1}^{M-1} S_{ij}u_j^k + \theta\tau\alpha u_i^{k+1}\sum_{j=1}^{M-1} S_{ij}v_j^k + \theta\tau\alpha u_i^k\frac{v_{i+1}^{k+1} - v_{i-1}^{k+1}}{2h} = \\
&u_i^k + (2\theta-1)\tau\eta u_i^k\sum_{j=1}^{M-1} S_{ij}u_j^k + (2\theta-1)\tau\alpha v_i^k\sum_{j=1}^{M-1} S_{ij}u_j^k \\
&+ (2\theta-1)\tau\alpha u_i^k\sum_{j=1}^{M-1} S_{ij}v_j^k + (\theta-1)\tau\delta\frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{h^2},
\end{aligned}
$$

$$
\begin{aligned}
&v_i^{k+1} + \theta\tau\mu\frac{v_{i+1}^{k+1} - 2v_i^{k+1} + v_{i-1}^{k+1}}{h^2} + \theta\tau\xi v_i^{k+1}\sum_{j=1}^{M-1} S_{ij}v_j^k \\
&+ \theta\tau\xi v_i^k\frac{v_{i+1}^{k+1} - v_{i-1}^{k+1}}{2h} + \theta\tau\beta v_i^k\frac{u_{i+1}^{k+1} - u_{i-1}^{k+1}}{2h} + \theta\tau\beta v_i^{k+1}\times \\
&\sum_{j=1}^{M-1} S_{ij}u_j^k + \theta\tau\beta u_i^{k+1}\sum_{j=1}^{M-1} S_{ij}v_j^k + \theta\tau\beta u_i^k\frac{v_{i+1}^{k+1} - v_{i-1}^{k+1}}{2h} = \\
&v_i^k + (2\theta-1)\tau\xi v_i^k\sum_{j=1}^{M-1} S_{ij}v_j^k + (2\theta-1)\tau\beta v_i^k\sum_{j=1}^{M-1} S_{ij}u_j^k \\
&+ (2\theta-1)\tau\beta u_i^k\sum_{j=1}^{M-1} S_{ij}v_j^k + (\theta-1)\tau\mu\frac{v_{i+1}^k - 2v_i^k + v_{i-1}^k}{h^2}, \\
&\qquad\qquad 1 \le i \le M-1, 0 \le k \le N-1.
\end{aligned}
\tag{13}
$$

Remark 2. Substituting (10), (12) into (9), the fully discrete scheme of Burgers' equation can be expressed as follows

$$
\begin{aligned}
&u_i^{k+1} + \theta\tau(u_i^{k+1}\sum_{j=1}^{M-1} S_{ij}u_j^k + u_i^k\frac{u_{i+1}^{k+1} - u_{i-1}^{k+1}}{2h} - \\
&\varepsilon\frac{u_{i+1}^{k+1} - 2u_i^{k+1} + u_{i-1}^{k+1}}{h^2}) = u_i^k + (2\theta-1)\tau u_i^k\sum_{j=1}^{M-1} S_{ij}u_j^k \\
&+ (1-\theta)\tau\varepsilon\frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{h^2}, \\
&\qquad\qquad 1 \le i \le M-1, 0 \le k \le N-1.
\end{aligned}
\tag{14}
$$

III. Stability anslysis

A. The iterative format

We can rewrite Eq. (13) in the matrix form as

$$
\mathbf{A}\mathbf{Y}^{k+1} = \mathbf{B}\mathbf{Y}^k, \tag{15}
$$

where $\mathbf{Y}^k = (u_1^k, u_2^k, \cdots, u_{M-1}^k, v_1^k, v_2^k, \cdots, v_{M-1}^k)^T$,

$$
\mathbf{A} = \begin{pmatrix} \mathbf{A_1} & \mathbf{A_2} \\ \mathbf{A_3} & \mathbf{A_4} \end{pmatrix},
$$

$$
\mathbf{B} = \begin{pmatrix} \mathbf{B_1} & \mathbf{B_2} \\ \mathbf{B_3} & \mathbf{B_4} \end{pmatrix},
$$

$$
\begin{aligned}
&\mathbf{A_1} = \mathbf{I} + \theta\tau\delta\mathbf{E} + \theta\tau\eta\mathbf{C_1} + \theta\tau\eta\mathbf{D_1} + \theta\tau\alpha\mathbf{D_2} + \theta\tau\alpha\mathbf{C_2}, \\
&\mathbf{A_2} = \theta\tau\alpha\mathbf{C_1} + \theta\tau\alpha\mathbf{D_1}, \\
&\mathbf{A_3} = \theta\tau\beta\mathbf{C_2} + \theta\tau\beta\mathbf{D_2}, \\
&\mathbf{A_4} = \mathbf{I} + \theta\tau\mu\mathbf{E} + \theta\tau\xi\mathbf{C_2} + \theta\tau\xi\mathbf{D_2} + \theta\tau\beta\mathbf{D_1} + \theta\tau\beta\mathbf{C_1}, \\
&\mathbf{B_1} = \mathbf{I} + (\theta-1)\tau\delta\mathbf{E} + (2\theta-1)\tau\eta\mathbf{C_1} + (2\theta-1)\tau\alpha\mathbf{C_2}, \\
&\mathbf{B_2} = (2\theta-1)\tau\alpha\mathbf{C_1}, \\
&\mathbf{B_3} = (2\theta-1)\tau\beta\mathbf{C_2}, \\
&\mathbf{B_4} = \mathbf{I} + (\theta-1)\tau\mu\mathbf{E} + (2\theta-1)\tau\xi\mathbf{C_2} + (2\theta-1)\tau\beta\mathbf{C_1},
\end{aligned}
\tag{16}
$$

here I is the unit matrix and

$$
\mathbf{C_1} = \begin{pmatrix} u_x(x_1,t_k) & & & \\ & u_x(x_2,t_k) & & \\ & & \ddots & \\ & & & u_x(x_{M-1},t_k) \end{pmatrix}_{(M-1)\times(M-1)},
$$

$$
\mathbf{C_2} = \begin{pmatrix} v_x(x_1,t_k) & & & \\ & v_x(x_2,t_k) & & \\ & & \ddots & \\ & & & v_x(x_{M-1},t_k) \end{pmatrix}_{(M-1)\times(M-1)},
$$

$$
\mathbf{D_1} = \frac{1}{2h}\begin{pmatrix} 0 & u_1^k & & & & \\ -u_2^k & 0 & u_2^k & & & \\ & -u_3^k & 0 & u_3^k & & \\ & & \ddots & \ddots & \ddots & \\ & & & -u_{M-2}^k & 0 & u_{M-2}^k \\ & & & & -u_{M-1}^k & 0 \end{pmatrix}_{(M-1)\times(M-1)},
$$

$$
\mathbf{D_2} = \frac{1}{2h}\begin{pmatrix} 0 & v_1^k & & & & \\ -v_2^k & 0 & v_2^k & & & \\ & -v_3^k & 0 & v_3^k & & \\ & & \ddots & \ddots & \ddots & \\ & & & -v_{M-2}^k & 0 & v_{M-2}^k \\ & & & & -v_{M-1}^k & 0 \end{pmatrix}_{(M-1)\times(M-1)},
$$

$$
\mathbf{E} = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}_{(M-1)\times(M-1)}.
$$

Remark 3. Simultaneously, scheme (14) can be rewritten as

$$
\mathbf{T}\mathbf{u}^{k+1} = \mathbf{P}\mathbf{u}^k, \tag{17}
$$

where $\mathbf{T} = \mathbf{I} + \theta\tau(\mathbf{C_1} + \mathbf{D_1} - v\mathbf{E})$, $\mathbf{P} = \mathbf{I} + (2\theta-1)\tau\mathbf{C_1} + (1-\theta)\tau\varepsilon\mathbf{E}$, $\mathbf{u}^k = (u_1^k, u_2^k, \cdots, u_{M-1}^k)^T$.

B. Stability

Let $\mathbf{H} = \mathbf{A}^{-1}\mathbf{B}$. The necessary condition for the stability of (15) is $\rho(\mathbf{H}) \le 1 + c\tau$, where c is a positive number [31]. Next, we will compute the spectral radius for a concrete example by numerical calculation.

We take $f(x) = sinx, g(x) = sinx$ in initial conditions and $\delta = -1, \mu = -1, \eta = -2, \xi = -2, \alpha = 1, \beta = 1, L = T = 2\pi$. By selecting $N = M = 5$ and $\theta = 1$, the iterative matrix $\mathbf{H}$ in the first iteration is

$$
\mathbf{H} = \begin{pmatrix}
0.169 & 0.536 & 0.168 & 0.007 & 0.263 & -0.388 & -0.118 & 0.009 \\
0.072 & 0.628 & 0.201 & 0.024 & 0.076 & -0.146 & -0.038 & 0.026 \\
0.024 & 0.201 & 0.628 & 0.072 & 0.026 & -0.038 & -0.146 & 0.076 \\
0.007 & 0.168 & 0.536 & 0.169 & 0.009 & -0.118 & -0.388 & 0.263 \\
0.263 & -0.388 & -0.118 & 0.009 & 0.169 & 0.536 & 0.168 & 0.007 \\
0.076 & -0.146 & -0.038 & 0.026 & 0.072 & 0.628 & 0.201 & 0.024 \\
0.026 & -0.038 & -0.146 & 0.076 & 0.024 & 0.201 & 0.628 & 0.072 \\
0.009 & -0.118 & -0.388 & 0.2626 & 0.007 & 0.1681 & 0.536 & 0.169
\end{pmatrix}
$$

It is noted that $\rho(H) = 1.0066 > 1$, so the scheme is not stable. Similarly, when we choose $\theta = 0$, the spectral radius of iterative matrix is $\rho(H) = 3.6646 > 1$. This means the numerical scheme is not stable. While $\theta = \frac{1}{2}$ is taken, the values of $\rho(H)$ are $0.7361, 0.7954, 0.9095, 0.9537, 0.9677$ in the first five steps, which ensured the stability of the iterations.

Remark 4. Let $\mathbf{G} = \mathbf{T}^{-1}\mathbf{P}$, denote $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ as the eigenvalues of $\mathbf{I}, \mathbf{C_1}, \mathbf{D_1}, \mathbf{E}$, respectively. It is obvious that $\lambda_1 = 1$ and λ_2 is a real number. Let $\lambda_3^R = Re(\lambda_3), \lambda_3^I = Im(\lambda_3), \lambda_4^R = Re(\lambda_4), \lambda_4^I = Im(\lambda_4)$. To ensure the stability of (17) , the following conditions must be met

$$\left|\frac{1 + (2\theta - 1)\tau\lambda_2 + (1-\theta)\tau\varepsilon\lambda_4^R + i(1-\theta)\tau\varepsilon\lambda_4^I}{1 + \theta\tau\lambda_2 + \theta\tau\lambda_3^R - \theta\tau\varepsilon\lambda_4^R + i(\theta\tau\lambda_3^I - \theta\tau\varepsilon\lambda_4^I)}\right| \leq 1,$$

which means

$$\begin{aligned} &(3\theta - 1)(\theta - 1)\tau^2\lambda_2^2 + (1 - 2\theta)\tau^2\varepsilon^2|\lambda_4|^2 - \theta^2\tau^2|\lambda_3|^2 \\ &\leq 2\theta^2\tau^2(\lambda_2\lambda_3^R - \varepsilon\lambda_3^R\lambda_4^R - \varepsilon\lambda_3^I\lambda_4^I) - (2\theta - 2)\tau\lambda_2 \\ &- 2\theta\tau(\varepsilon\lambda_4^R - \lambda_3^R) + (2\theta^2 - 6\theta + 2)\tau^2\varepsilon\lambda_2\lambda_4^R, \end{aligned} \tag{18}$$

where $|\lambda_3|$ and $|\lambda_4|$ represent the modulus of λ_3 and λ_4.

Through the above analysis, if (17) is stable, the eigenvalues must satisfy (18). In the case of $\theta = 0$, the scheme is stable under the condition on time step $\tau \leq \frac{2\lambda_2}{\lambda_2^2 + \varepsilon^2|\lambda_4|^2 - 2\varepsilon\lambda_2\lambda_4^R}$.

IV. Numerical Examples

In this section, we conduct some numerical examples to illustrate the effectiveness of the proposed method. The numerical solutions and maximum absolute errors are obtained. In particular, we will compare the results with those obtained by other methods. In the following, u and U are denoted the exact and the numerical solution. The maximum absolute error is defined as

$$e = \max_{0 \leq i \leq M, 0 \leq k \leq N} |u(x_i, t_k) - U(x_i, t_k)|.$$

Example 1. In the first example, we consider following Burgers' equation [8]:

$$\begin{aligned} &u_t + uu_x = \varepsilon u_{xx},\ 0 < x < 1,\ 0 < t \leq 1, \\ &u(x, 0) = \frac{2v\pi sin(\pi x)}{2 + cos(\pi x)}, \\ &u(0, t) = 0, u(1, t) = 0, \end{aligned}$$

which has the exact solution $u(x, t) = \frac{2\varepsilon\pi e^{-\pi^2\varepsilon t} sin(\pi x)}{2 + e^{-\pi^2\varepsilon t} cos(\pi x)}$.

Following the numerical scheme described in Section II, we take $M = 80, \theta = \frac{1}{2}, h = 1/M, \tau = h^2$. Fig. 1 and 2 plot the numerical solution and the absolute error. The maximum absolute errors for different M and θ are shown in Table I. Moreover, we compare the numerical results with those of [8] at $u(x, 0.001)$.

Example 2. In this example, we consider the following Burgers' equation [8, 32–34]:

$$\begin{aligned} &u_t + uu_x = u_{xx},\ 0 < x < 1,\ 0 < t \leq 1, \\ &u(x, 0) = \varphi(x), \\ &u(0, t) = 0, u(1, t) = 0. \end{aligned}$$

The initial value $\varphi(x)$ have two cases: $\varphi(x) = sin(\pi x)$ and $\varphi(x) = 4x(1 - x)$. We choose the step size $h = 0.0125$ and $\tau = 0.00001$. The numerical results are demonstrated in Table III and IV. Besides, the numerical solutions are illustrated in Fig. 3 and 4 when t takes different values.

Example 3. We discuss coupled Burgers' equation [35]:

$$\begin{aligned} &u_t - u_{xx} - 2uu_x + (uv)_x = 0,\ 0 < x < 2\pi,\ 0 < t \leq T, \\ &v_t - v_{xx} - 2vv_x + (uv)_x = 0,\ 0 < x < 2\pi,\ 0 < t \leq T, \\ &u(x, 0) = sin(x - \pi),\ v(x, 0) = sin(x - \pi), \\ &u(0, t) = 0,\ u(2\pi, t) = 0, \\ &v(0, t) = 0,\ v(2\pi, t) = 0. \end{aligned}$$

The exact solution is $u(x, t) = v(x, t) = e^{-t} sin(x - \pi)$. The maximum absolute errors are compared with those of [35] in Table V for $N = 1000$ and $\theta = \frac{1}{2}$. We plot the numerical results in Fig. 5 and 6. Besides, TableVI shows the maximum absolute errors for different θ.

TABLE I
Maximum absolute errors with $v = 0.01, h = 1/M, \tau = h^2$

M	$\theta = 0$	$\theta = \frac{1}{2}$	$\theta = 1$
10	4.1310×10^{-4}	3.0734×10^{-4}	2.2844×10^{-4}
20	7.9113×10^{-5}	6.9289×10^{-5}	5.9420×10^{-5}
40	1.7494×10^{-5}	1.5789×10^{-5}	1.4986×10^{-5}
80	4.0220×10^{-6}	3.8340×10^{-6}	3.7569×10^{-6}
160	9.6329×10^{-7}	9.4059×10^{-7}	9.3960×10^{-7}

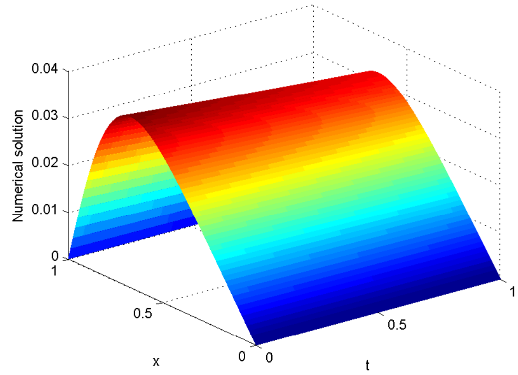

Fig. 1. Numerical solution when $M = 80, \theta = \frac{1}{2}$

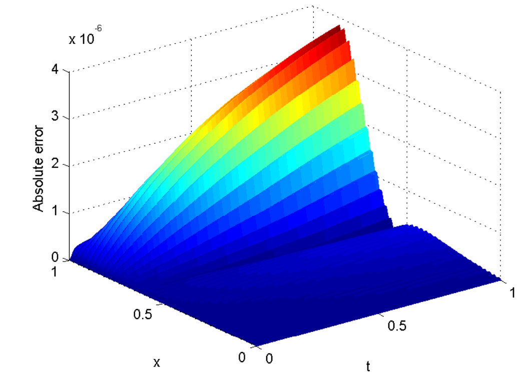

Fig. 2. Absolute error when $M = 80, \theta = \frac{1}{2}$

TABLE II
SOLUTIONS WITH $v = 1$, $h = 1/40$, $L = 1$, $T = 0.001$, $\theta = 1/2$ AT POINT $(x, 0.001)$ FOR EXAMPLE 1

x	$\tau = 0.0001$		$\tau = 0.00005$		$\tau = 0.000025$		
	[8]	Present method	[8]	Present method	[8]	Present method	Exact solution
0.1	0.653589	0.6535551453	0.653553	0.6535551422	0.653545	0.6535551418	0.6535444868
0.2	1.305611	1.3055327866	1.305550	1.3055327876	1.305534	1.3055327887	1.3055335298
0.3	1.949485	1.9493471176	1.949389	1.9493471225	1.949365	1.9493471250	1.9493635654
0.4	2.566103	2.5658880089	2.565964	2.5658880184	2.565930	2.5658880227	2.5659249142
0.5	3.110992	3.1106788779	3.110803	3.1106788951	3.110756	3.1106789026	3.1107388846
0.6	3.493222	3.4927929863	3.492978	3.4927930217	3.492917	3.4927930367	3.4928657149
0.7	3.550079	3.5495549472	3.549794	3.5495550209	3.549723	3.5495550499	3.5495951291
0.8	3.050702	3.0501943654	3.050432	3.0501944772	3.050365	3.0501945156	3.0501344786
0.9	1.817077	1.8167470798	1.816924	1.8167471802	1.816886	1.8167472066	1.8166603703

TABLE III
EXACT SOLUTIONS AND NUMERICAL SOLUTIONS WHEN $\varphi(x) = sin(\pi x)$

x	t	Present method			HC[32]	RHC [32]	RPA[33]	Exact solution
		$\theta = 0$	$\theta = \frac{1}{2}$	$\theta = 1$				
0.25	0.1	0.253650	0.253665	0.253679	0.252942	0.264126	0.252875	0.253637
0.25	0.15	0.156616	0.156628	0.156640	0.156059	0.165683	0.155447	0.156601
0.25	0.2	0.096455	0.096465	0.096475	0.095889	0.101617	0.094289	0.096440
0.25	0.25	0.059229	0.059236	0.059243	0.056174	0.059113	0.055674	0.059217
0.5	0.1	0.371605	0.371625	0.371644	0.376474	0.393354	0.373085	0.371579
0.5	0.15	0.226850	0.226867	0.226884	0.235875	0.251788	0.228940	0.226825
0.5	0.2	0.138495	0.138509	0.138523	0.153645	0.163931	0.142127	0.138473
0.5	0.25	0.084554	0.084565	0.084575	0.112810	0.120967	0.091944	0.084537
0.75	0.1	0.272610	0.272623	0.272635	0.271517	0.285579	0.272368	0.272580
0.75	0.15	0.164392	0.164404	0.164416	0.162739	0.176957	0.163628	0.164369
0.75	0.2	0.099452	0.099462	0.099472	0.098431	0.111020	0.098656	0.099434
0.75	0.25	0.060359	0.060367	0.060375	0.057394	0.068569	0.059343	0.060346

TABLE IV
EXACT SOLUTIONS AND NUMERICAL SOLUTIONS WHEN $\varphi(x) = 4x(1 - x)$

x	t	Present method			[8]	EFDM[34]	EEFDM[34]	Exact solution
		$\theta = 0$	$\theta = \frac{1}{2}$	$\theta = 1$				
0.25	0.01	0.66007	0.66008	0.66009	0.66008	0.65915	0.66007	0.66006
0.25	0.05	0.42629	0.42631	0.42632	0.42631	0.42582	0.42629	0.42629
0.25	0.10	0.26149	0.26151	0.26152	0.26151	0.26121	0.26149	0.26148
0.25	0.15	0.16149	0.16150	0.16152	0.16148	0.16132	0.16148	0.16148
0.25	0.25	0.06110	0.06111	0.06111	0.06111	0.06103	0.06109	0.06109
0.5	0.01	0.91972	0.91972	0.91972	0.91973	0.91890	0.91972	0.91972
0.5	0.05	0.62811	0.62812	0.62814	0.62812	0.62745	0.62809	0.62808
0.5	0.10	0.38345	0.38347	0.38349	0.38347	0.38304	0.38343	0.38342
0.5	0.15	0.23408	0.23410	0.23412	0.23410	0.23382	0.23406	0.23406
0.5	0.25	0.08725	0.08726	0.08727	0.08726	0.08715	0.08724	0.08723
0.75	0.01	0.68365	0.68365	0.68366	0.68667	0.68304	0.68364	0.68364
0.75	0.05	0.46528	0.46529	0.46530	0.46529	0.46481	0.46526	0.46525
0.75	0.10	0.28160	0.28161	0.28163	0.28162	0.28129	0.28158	0.28157
0.75	0.15	0.16976	0.16977	0.16979	0.16977	0.16957	0.16974	0.16974
0.75	0.25	0.06230	0.06231	0.06232	0.06231	0.06223	0.06229	0.06229

TABLE V
THE MAXIMUM ABSOLUTE ERROR

M	$t = 0.1$		$t = 0.5$	
	[35]	Present method	[35]	Present method
32	2.9104×10^{-4}	2.9038×10^{-4}	9.7478×10^{-4}	9.7384×10^{-4}
64	7.2704×10^{-5}	7.2655×10^{-5}	2.4361×10^{-4}	2.4354×10^{-4}
128	1.8178×10^{-5}	1.8168×10^{-5}	6.0896×10^{-5}	6.0887×10^{-5}
256	4.5497×10^{-5}	4.5421×10^{-6}	1.5223×10^{-5}	1.5217×10^{-5}
512	1.1430×10^{-6}	1.1355×10^{-6}	3.8052×10^{-5}	3.7996×10^{-6}

V. CONCLUSIONS

This paper propose a mixed numerical method to solve Burgers' and coupled Burgers' equation by combining finite difference method with sinc collocation method. Detailed

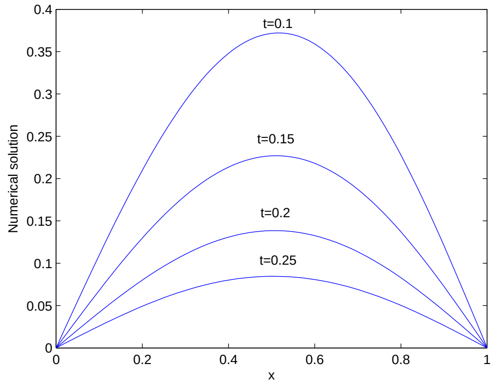

Fig. 3. Numerical solution when $\varphi(x) = sin(\pi x), \theta = \frac{1}{2}$

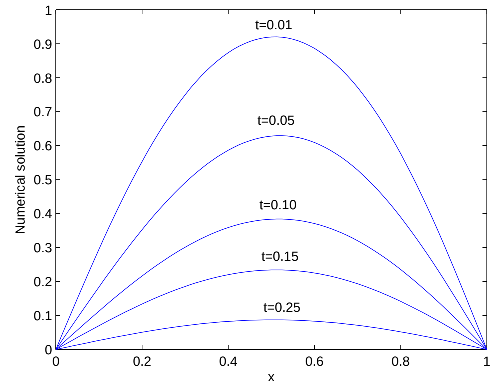

Fig. 4. Numerical solution when $\varphi(x) = 4x(1-x), \theta = \frac{1}{2}$

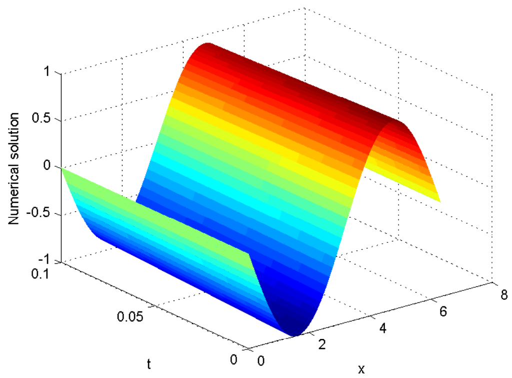

Fig. 5. Numerical solution of u and v

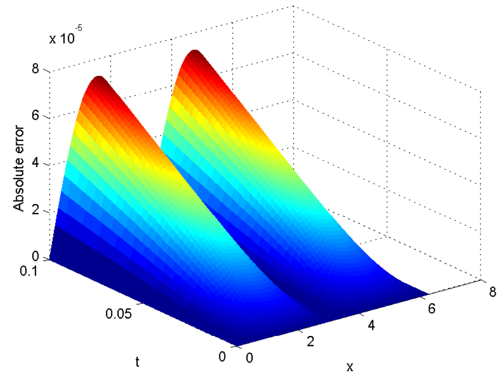

Fig. 6. Absolute error of u and v

TABLE VI
MAXIMUM ABSOLUTE ERRORS OF u AND v

t	$\theta = 0$	$\theta = \frac{1}{2}$	$\theta = 1$
0.5	3.1948×10^{-5}	1.5629×10^{-5}	6.3114×10^{-5}
1	6.0779×10^{-5}	2.9734×10^{-5}	1.2008×10^{-4}
1.5	8.6720×10^{-5}	4.2427×10^{-5}	1.7134×10^{-4}
2	1.0999×10^{-4}	5.3810×10^{-5}	2.1731×10^{-4}

stability analysis is given for the fully discrete scheme. The results show the stability of the schemes is related to matrix eigenvalues, weight parameter θ and time step size τ. All of the given examples reveal that the proposed method can be used to obtain numerical solution of Burgers' and coupled Burgers' equation.

REFERENCES

[1] A. Veksler and Y. Zarmi, "Wave interactions and the analysis of the perturbed Burgers equation," *Physica D Nonlinear Phenomena*, vol. 211, no. 1-2, pp. 57–73, 2005.

[2] J. D. Cole, "On a quasi-linear parabolic equation occurring in aerodynamics," *Quarterly of Applied Mathematics*, vol. 9, no. 3, pp. 225–236, 1951.

[3] J. M. Burgers, "A mathematical model illustrating the theory of turbulence," *Advances in Applied Mechanics*, vol. 1, pp. 171–199, 1948.

[4] E. Hopf, "The partial differential equation $u_t + uu_x = \mu u_{xx}$," *Communications on Pure and Applied Mathematics*, vol. 3, no. 3, pp. 201–230, 1950.

[5] B. S. Kerner, S. L. Klenov, and P. Konhäuser, "Asymptotic theory of traffic jams," *Physical Review E*, vol. 56, no. 4, pp. 4200–4216, 1997.

[6] H. Bateman, "Some recent researches on the motion of fluids," *Monthly Weather Review*, vol. 43, no. 4, pp. 163–170, 1915.

[7] J. M. Burgers, "A mathematical model illustrating the theory of turbulence," *Advances in Applied Mechanics*, vol. 1, pp. 171–199, 1948.

[8] A. Asaithambi, "Numerical solution of the Burgers' equation by automatic differentiation," *Applied Mathematics and Computation*, vol. 216, no. 9, pp. 2700–2708, 2010.

[9] S. E. Esipov, "Coupled Burgers equations: a model of polydispersive sedimentation," *Physical Review E*, vol. 52, no. 4, pp. 3711–3718, 1995.

[10] D. Kaya, "An explicit solution of coupled viscous Burgers' equation by the decomposition method," *International Journal of Mathematics and Mathematical Sciences*, vol. 27, no. 11, pp. 675–680, 2001.

[11] V. K. Srivastava, M. Tamsir, M. K. Awasthi, and S. Singh, "One-dimensional coupled Burgers' equation and its numerical solution by an implicit logarithmic finite-difference method," *AIP Advances*, vol. 4, 037119, 2014.

[12] J. Nee and J. Duan, "Limit set of trajectories of the coupled viscous Burgers' equations," *Applied Mathematics Letters*, vol. 11, no. 1, pp. 57–61, 1998.

[13] D. Deng and T. Pan, "A fourth-order singly diagonally implicit Runge-Kutta method for solving one-dimensional Burgers' equation," *IAENG International Journal of Applied Mathematics*, vol. 45, no. 4, pp. 327–333, 2015.

[14] M. Sari and G. Gürarslan, "A sixth-order compact finite difference scheme to the numerical solutions of Burgers' equation," *Applied Mathematics and Computation*, vol. 208, no. 2, pp. 475–483, 2009.

[15] S. Kutluay and Y. Ucar, "Numerical solutions of the coupled Burgers' equation by the Galerkin quadratic B-spline finite element method," *Mathematical Methods in the Applied Sciences*, vol. 36, no. 17, pp. 2403–2415, 2013.

[16] V. K. Srivastava, M. K. Awasthi, and M. Tamsir, "A fully implicit finite-difference solution to one dimensional coupled nonlinear Burgers' equations," *International Journal of Mathematical, Computational Science and Engineering*, vol. 7, no. 4, pp. 283–287, 2013.

[17] Ö. Oruç, F. Bulut, and A. Esen, "Chebyshev wavelet method for numerical solutions of coupled Burgers' equation," *Hacettepe Journal of Mathematics and Statistics*, vol. 48, no. 1, pp. 1–16, 2018.

[18] M. Norouzi and H. Saberi Najafi, "The solution of coupled nonlinear Burgers' equations using interval finite-difference method," *International Journal of Industrial Mathematics*, vol. 9, pp. 215–224, 2017.

[19] D. Deng and J. Xie, "High-order exponential time differencing methods for solving one-dimensional Burgers' equation," *IAENG International Journal of Computer Science*, vol. 43, no. 2, pp. 167–175, 2016.

[20] F. Stenger, "A "Sinc-Galerkin" method of solution of boundary value problems," *Mathematics of Computation*, vol. 33, no. 145, pp. 85–109, 1979.

[21] M. Dehghan and F. Emami-Naeini, "The Sinc-collocation and Sinc-Galerkin methods for solving the two-dimensional schrödinger equation with nonhomogeneous boundary conditions," *Applied Mathematical Modelling*, vol. 37, no. 22, pp. 9379–9397, 2013.

[22] G.A. Zakeri and M. Navab, "Sinc collocation approximation of non-smooth solution of a nonlinear weakly singular Volterra integral equation," *Journal of Computational Physics*, vol. 229, no. 18, pp. 6548–6557, 2010.

[23] M. Dehghan and A. Saadatmandi, "The numerical solution of a nonlinear system of second-order boundary value problems using the sinc-collocation method," *Mathematical and Computer Modelling*, vol. 46, no. 11-12, pp. 1434–1441, 2007.

[24] R. E. Mickens, *Nonstandard finite difference models of differential equations*. World Scientific, 1994.

[25] D. Kong, Y. Xu, and Z. Zheng, "A hybrid numerical method for the KdV equation by finite difference and sinc collocation method," *Applied Mathematics and Computation*, vol. 355, pp. 61–72, 2019.

[26] M. Dehghan and A. Saadatmandi, "Chebyshev finite difference method for Fredholm integro-differential equation," *International Journal of Computer Mathematics*, vol. 85, no. 1, pp. 123–130, 2008.

[27] K. Al-Khaled, N. Haynes, W. Schiesser, and M. Usman, "Eventual periodicity of the forced oscillations for a Korteweg-de Vries type equation on a bounded domain using a sinc collocation method," *Journal of Computational and Applied Mathematics*, vol. 330, pp. 417–428, 2018.

[28] J. Lund and K. L. Bowers, *Sinc methods for quadrature and differential equations*. SIAM, 1992.

[29] F. Stenger, *Numerical methods based on sinc and analytic functions*. Springer-Verlag, New York, 1993.

[30] G. D. Smith, *Numerical solution of partial differential equations: finite difference methods*. Oxford University Press, 1978.

[31] J. C. Strikwerda, *Finite difference schemes and partial differential equations*. SIAM, 2004.

[32] M. Gülsu and T. Öziş, "Numerical solution of Burgers' equation with restrictive Taylor approximation," *Applied Mathematics and Computation*, vol. 171, no. 2, pp. 1192–1200, 2005.

[33] M. Gülsu, "A finite difference approach for solution of Burgers' equation," *Applied Mathematics and Computation*, vol. 175, no. 2, pp. 1245–1255, 2006.

[34] S. Kutluay, A. R. Bahadir, and A. Özdeş, "Numerical solution of one-dimensional Burgers equation: explicit and exact-explicit finite difference methods," *Journal of Computational and Applied Mathematics*, vol. 103, no. 2, pp. 251–261, 1999.

[35] R. C. Mittal and G. Arora, "Numerical solution of the coupled viscous Burgers' equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 3, pp. 1304–1313, 2011.