Variable-coefficient Sub-equation Method for Conformable Fractional Partial Differential Equations

Qinghua Feng

Abstract—**In this work, we introduce a variable-coefficient sub-equation method for solving fractional partial differential equations with the fractional derivative defined by the conformable fractional derivative. By use of a nonlinear transformation and the properties of conformable fractional calculus, the fractional derivative can be converted into integer order derivative with respect to a new variable. With general solutions of two certain sub-equations, a series of exact solutions with variable coefficient function forms can be obtained subsequently with the aid of mathematical software. For illustrating the validity of this method, we apply it to the conformable fractional Bogoyavlenskii equations and the conformable fractional Jimbo-Miwa equation. As a result, some exact solutions of new forms are successfully obtained for them.**

Index Terms—**variable-coefficient sub-equation method; conformable fractional derivative; exact solutions; fractional differential equations; fractional Bogoyavlenskii equations; fractional Jimbo-Miwa equation**

I. Introduction

Recently, Fractional differential equations have been the focus of many studies due to their frequent appearance in various applications in physics, biology, engineering, signal processing, systems identification, control theory, finance and fractional dynamics [1-3]. In particular, fractional derivative is very useful in describing the memory and hereditary properties of materials and processes. One of its most important applications is to model the process of subdiffusion and superdiffusion of particles in physics, where the fractional diffusion equation is usually used for modeling this movement. To illustrate better the physical phenomena denoted by fractional differential equations, it is necessary to obtain analytical or numerical solutions for fractional differential equations. Many efficient methods have been proposed so far to obtain numerical solutions and exact solutions of fractional differential equations. For example, these methods include the coupled fractional reduced differential transform method [4], the Bernstein polynomials method [5], the residual power series method [6], the Jacobi elliptic function method [7], the finite difference method [8-10], the finite difference method [11,12], the $(\frac{G'}{G})$ method [13-16], the variational iterative method [17-20], the fractional Nikiforov-Uvarov method [21], the modified Kudryashov method [22-25], the exp method [26,27], the first integral method [28-30], the sub-equation method [31-34] and so on.

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We notice that most of the existing methods have been constructed to obtain exact solutions for fractional differential equations with constant coefficients, and almost none of the existing methods have been used to obtain exact solutions with variable coefficient function forms for fractional differential equations.

Motivated by the analysis above, in this paper, by use of two certain sub-equations, we develop a variable-coefficient sub-equation method for solving fractional partial differential equations, where the fractional derivative is defined in the sense of the conformable fractional derivative. Then we apply this method to seek exact solutions with variable coefficient function forms for some certain fractional partial differential equations.

The conformable fractional derivative is defined as below [35]

$$D^{\alpha}f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}.$$

We list some important properties for the conformable fractional derivative as follows:

$$(i).\ D_t^{\alpha}[af(t)+bg(t)]=aD^{\alpha}f(t)+bD^{\alpha}g(t).$$

$$(ii).\ D_t^{\alpha}(t^{\gamma})=\gamma t^{\gamma-\alpha}.$$

$$(iii).\ D_t^{\alpha}[f(t)g(t)]=f(t)D^{\alpha}g(t)+g(t)D^{\alpha}f(t).$$

$$(iv).\ D_t^{\alpha}C=0,\ \text{where } C \text{ is a constant.}$$

$$(v).\ D_t^{\alpha}f[g(t)]=f'_g[g(t)]D_t^{\alpha}g(t).$$

$$(vi).\ D_t^{\alpha}(\frac{f}{g})(t)=\frac{g(t)D^{\alpha}f(t)-f(t)D^{\alpha}g(t)}{g^2(t)}.$$

$$(vii).\ D_t^{\alpha}f(t)=t^{1-\alpha}f'(t).$$

Note that the properties above can be easily proved due to the definition of the conformable fractional derivative. So under a given transformation $T=\frac{ct^{\alpha}}{\alpha}$, by use of (ii) one can obtain $D_t^{\alpha}T=c$. Furthermore, by use of (v) one can deduce that $D_t^{\alpha}u=\frac{\partial u}{\partial T}D_t^{\alpha}T=c\frac{\partial u}{\partial T}$. So the fractional derivative can be converted into integer order case with respect to one new variable.

The next of this paper is organized as follows. In Section 2, we give the description of the variable-coefficient sub-equation method. Then in Section 3, we apply the method to solve the conformable fractional Bogoyavlenskii equations and the fractional Jimbo-Miwa equation. Some conclusions are presented at the end of the paper.

II. Description of the variable-coefficient sub-equation method

In this section, we give the description of the variable-coefficient sub-equation method for solving fractional partial differential equations.

Suppose that a fractional partial differential equation in the independent variables $t, x_1, x_2, ..., x_n$ is given by

$$P(u,\ D_t^{\alpha}u,\ u_{x_1}..., D_{x_i}^{\beta}u, ...) = 0, \quad (1)$$

where u is an unknown function, the orders of the fractional derivatives such as $\alpha,\ \beta \in (0, 1]$, P is a polynomial in u and its various partial derivatives including fractional derivatives. Without loss of generality, next we may assume the fractional partial derivatives are related to the variables $t,\ x_i$, while the other variables are related to integer order derivatives.

Step 1. For those variables involving fractional derivatives, fulfil corresponding nonlinear transformations so that the fractional partial derivatives can be converted into integer order partial derivatives with respect to new variables.

Take the expressions $D_t^{\alpha}u$ and $D_{x_i}^{\beta}u$ for example, one can use two nonlinear transformations $T = c\frac{t^{\alpha}}{\alpha}$ and $X_i = k\frac{x_i^{\beta}}{\beta}$, and denote $u(t, x_1, ..., x_i, ..., x_n) = \widetilde{u}(T, x_1, ..., X_i, ..., x_n)$. Then due to the properties (ii) and (v) one can obtain that

$$\begin{cases} D_t^{\alpha}u = \frac{\partial \widetilde{u}}{\partial T}D_t^{\alpha}T = \frac{\partial \widetilde{u}}{\partial T}c = c\widetilde{u}_T, \\ D_{x_i}^{\beta}u = \frac{\partial \widetilde{u}}{\partial X_i}D_{x_i}^{\beta}X_i = \frac{\partial \widetilde{u}}{\partial X_i}k = k\widetilde{u}_{X_i}. \end{cases}$$

So the original fractional partial differential equation can be converted into another partial differential equation of integer order as follows

$$\widetilde{P}(\widetilde{u},\ \widetilde{u}_T,\ \widetilde{u}_{x_1}, ..., \widetilde{u}_{X_i}, ...) = 0. \quad (2)$$

Step 2. Suppose that

$$\widetilde{u}(T, x_1, ..., X_i, ..., x_n) = U(\xi), \quad \xi = \xi(T, x_1, ..., X_i, ..., x_n).$$

Then Eq. (2) can be turned into the following form

$$\widetilde{\widetilde{P}}(U,\ U',\ U'', ...) = 0, \quad (3)$$

where ξ will be determined later. And in Eq. (3), the highest order derivatives and nonlinear terms for U as well as various derivatives for ξ are involved.

Step 3. Suppose that the solution of (3) can be expressed by a polynomial in $(\frac{\phi'}{\phi})$ as follows:

$$U(\xi) = \sum_{i=0}^{m} a_i(T, x_1, ..., X_i, ..., x_n)(\frac{\phi'}{\phi})^i, \quad (4)$$

where $a_m(T, x_1, ..., X_i, ..., x_n),\ a_{m-1}(T, x_1, ..., X_i, ..., x_n),\ ...,\ a_0(T, x_1, ..., X_i, ..., x_n)$ are all unknown functions to be determined later with $a_m(T, x_1, ..., X_i, ..., x_n) \neq 0$, and $\phi = \phi(\xi)$ satisfies some certain sub-equation with the following form

$$F(\phi, \phi', \phi'', ...) = 0 \quad (5)$$

whose solutions are known. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (3).

Step 4. Substituting (4) into (2)-(3) and using the relation between $\phi'(\xi)$ and $\phi(\xi)$ deduced by (5), collecting all terms with the same order of $\phi(\xi)$ together, the left-hand side of (3) is converted to another polynomial in $\phi(\xi)$. Equating each coefficient of this polynomial to zero, yields a set of partial differential equations for $a_m(T, x_1, ..., X_i, ..., x_n),\ a_{m-1}(T, x_1, ..., X_i, ..., x_n)...,$

$a_0(T, x_1, ..., X_i, ..., x_n),\ \xi(T, x_1, ..., X_i, ..., x_n)$.

Step 5. Solving the equations yielded in Step 4, and by using the solutions of Eq. (5), together with the nonlinear transformations introduced in Step 1, one can obtain exact solutions for Eq. (1).

Remark 1. The most prominent characters of the present method different from other methods in [3-24] lies in two aspects. One is the transformation of ξ is under-determined, and the other is the coefficients in Eq. (4) are variable coefficient functions, which may help to seek exact solutions with more general forms.

Remark 2. If we take Eq. (5) for some different forms such as the Riccati equation, Bernoulli equation, Jacobi elliptic equation and so on, then different exact solutions for Eq. (1) can be obtained.

Remark 3. As the partial differential equations yielded in Step 4 are usually over-determined, we may choose some special forms of $a_m,\ a_{m-1},\ ...,\ a_0$ as did in the following.

III. Application of the variable-coefficient sub-equation method to some fractional partial differential equations

A. Conformable fractional Bogoyavlenskii equations

First we consider the conformable time fractional Bogoyavlenskii equations with the following forms:

$$\begin{cases} 4D_t^{\alpha}u + u_{xxy} - 4u^2u_y - 4u_xv = 0,\ 0 < \alpha \leq 1. \\ v_x + uu_y = 0, \end{cases} \quad (6)$$

The fractional Bogoyavlenskii equations were derived in [36] as a member of a (2+1) Schwarzian breaking soliton hierarchy. In [29], Eslami etc. solved Eqs. (6) by use of the first integral method, and obtained a series of exact solutions for the equations.

Now we use the method introduced in Section 2 to solve them. To this end, let $T = c\frac{t^{\alpha}}{\alpha}$ and $u(x, y, t) = \widetilde{u}(x, y, T),\ v(x, y, t) = \widetilde{v}(x, y, T)$. Then $D_t^{\alpha}u = c\widetilde{u}_T$, and Eqs. (6) are converted into the following forms

$$\begin{cases} 4c\widetilde{u}_T + \widetilde{u}_{xxy} - 4\widetilde{u}^2\widetilde{u}_y - 4\widetilde{u}_x\widetilde{v} = 0, \\ \widetilde{v}_x + \widetilde{u}\widetilde{u}_y = 0. \end{cases} \quad (7)$$

Assume that $\widetilde{u}(x, y, T) = U(\xi),\ \widetilde{v}(x, y, T) = V(\xi)$, where

$\xi = \xi(x, y, T)$. Then Eqs. (7) are converted into

$$\begin{cases} 4c\xi_T U' + (\xi_x^2 \xi_y U''' + 2\xi_x \xi_{xy} U'' + \xi_{xx}\xi_y U'' \\ \quad +\xi_{xxy} U') - 4\xi_y U^2 U' - 4\xi_x U' V = 0, \\ \xi_x V' + \xi_y U U' = 0. \end{cases} \tag{8}$$

Suppose that the solutions of Eqs. (8) can be expressed by a polynomial in $(\frac{\phi'}{\phi})$ as follows:

$$U(\xi) = \sum_{i=0}^{m} a_i(y,T)(\frac{\phi'}{\phi})^i, \; V(\xi) = \sum_{i=0}^{n} b_i(y,T)(\frac{\phi'}{\phi})^i, \tag{9}$$

where $a_i(y, T), b_i(y, T)$ are under-determined functions, and $\phi = \phi(\xi)$ satisfies Eq. (5). Balancing the order of U''' and $U'V$, V' and UU' in Eqs. (8), we can obtain $m + 3 = m + 1 + n$, $n + 1 = m + m + 1 \Rightarrow \; m = 1$, $n = 2$. So one has

$$\begin{cases} U(\xi) = a_1(y,T)(\frac{\phi'}{\phi}) + a_0(y,T), \\ V(\xi) = b_2(y,T)(\frac{\phi'}{\phi})^2 + b_1(y,T)(\frac{\phi'}{\phi}) + b_0(y,T). \end{cases} \tag{10}$$

Next we will discuss the process of finding exact solutions in two cases, in which ϕ satisfies two certain sub-equations.

Case 1: $\phi = \phi(\xi)$ satisfies the following Bernoulli equation

$$\phi' + \lambda\phi = \mu\phi^3. \tag{11}$$

Substituting (10) into (7)-(8), using Eq. (11) and collecting all the terms with the same power of ϕ together, equating each coefficient to zero, yields a set of under-determined partial differential equations for $a_0(y, T)$, $a_1(y, T)$, $b_0(y, T)$, $b_1(y, T)$, $b_2(y, T)$ and $\xi(x, y, T)$. Solving these equations, yields that

$$b_2(y,T) = \frac{2C_1}{\lambda} F'_y(y,T), \; b_1(y,T) = 2C_1 F'_y(y,T),$$

$$b_0(y,T) = \frac{c\lambda}{C_1} F'_T(y,T),$$

$$a_1(y,T) = \frac{2C_1}{\lambda}, \; a_0(y,T) = C_1,$$

$$\xi(x,y,T) = \frac{C_1}{\lambda} x + F(y,T) = \frac{C_1}{\lambda} x + F(y, c\frac{t^\alpha}{\alpha}),$$

where C_1 is an arbitrary nonzero constant, and $F(y, T)$ is an arbitrary function with respect to the variables y and T.

On the general solutions of Eq. (11), one has

$$\begin{cases} \phi(\xi) = \pm \dfrac{1}{\sqrt{\frac{\mu}{\lambda} + Ae^{2\lambda\xi}}}, \\ \dfrac{\phi'}{\phi} = -\dfrac{A\lambda e^{2\lambda\xi}}{\frac{\mu}{\lambda} + Ae^{2\lambda\xi}}, \end{cases} \tag{12}$$

where λ, μ, A are arbitrary constants with $\lambda \neq 0$, and $\mu^2 + A^2 \neq 0$.

Substituting the result above into Eqs. (10), and combining with (12), one can obtain the following exact solutions for the fractional Bogoyavlenskii equations

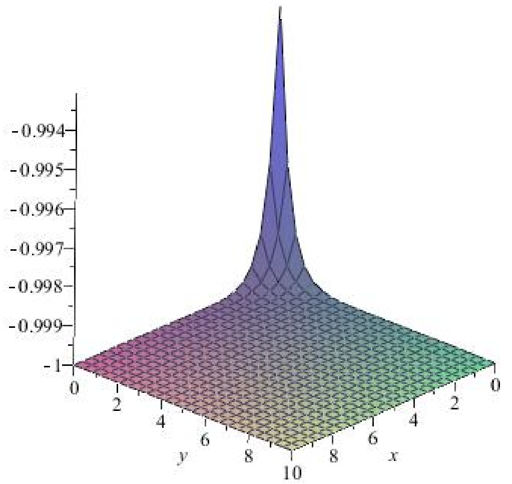

Fig. 1. The solitary wave solution u_2 with $c = \lambda = C_1 = 1$, $\alpha = 0.5$, $t = 2$

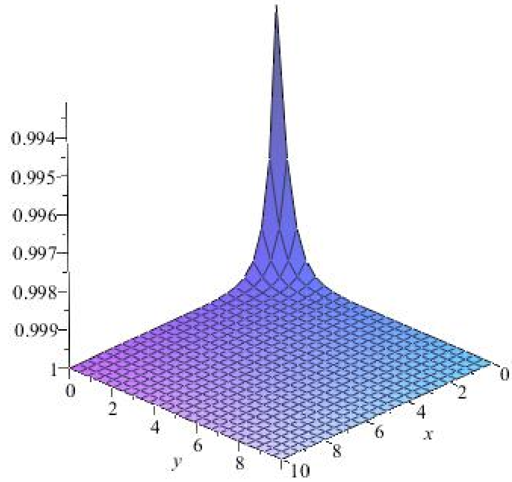

Fig. 2. The solitary wave solution v_2 with $c = \lambda = C_1 = 1$, $\alpha = 0.5$, $t = 2$

$$\begin{cases} u_1(x,y,t) = C_1 + \dfrac{2C_1}{\lambda}(-\dfrac{A\lambda e^{2\lambda\xi}}{\frac{\mu}{\lambda} + Ae^{2\lambda\xi}}), \\ v_1(x,y,t) = \dfrac{c\lambda}{C_1} F'_T(y, c\frac{t^\alpha}{\alpha}) + 2C_1 F'_y(y, c\frac{t^\alpha}{\alpha}) \\ (-\dfrac{A\lambda e^{2\lambda\xi}}{\frac{\mu}{\lambda} + Ae^{2\lambda\xi}}) + \dfrac{2C_1}{\lambda} F'_y(y, c\frac{t^\alpha}{\alpha})(-\dfrac{A\lambda e^{2\lambda\xi}}{\frac{\mu}{\lambda} + Ae^{2\lambda\xi}})^2, \end{cases} \tag{13}$$

where $\xi = \frac{C_1}{\lambda} x + F(y, c\frac{t^\alpha}{\alpha})$.

Especially, if we set $\mu = \lambda A$ in Eq. (13), then we obtain the following solitary wave solutions:

$$\begin{cases} u_2(x,y,t) = -C_1 \tanh(\lambda\xi), \\ v_2(x,y,t) = \dfrac{c\lambda}{C_1} F'_T(y, c\frac{t^\alpha}{\alpha}) - C_1 \lambda F'_y(y, c\frac{t^\alpha}{\alpha}) \\ \quad [(1 + \tanh(\lambda\xi))] + \dfrac{C_1\lambda}{2}[1 + \tanh(\lambda\xi)]^2. \end{cases}$$

If we take $F(y, T) = y + T$, then the solutions u_2, v_2 are demonstrated in Figs. 1-2.

Case 2: $\phi = \phi(\xi)$ satisfies the following Riccati equation

$$\phi'(\xi) = a + \phi^2(\xi). \tag{14}$$

Substituting (10) into (7)-(8), using Eq. (14) and collecting all the terms with the same power of ϕ together, equating each coefficient to zero, yields a set of under-determined partial differential equations. Solving these equations, yields that

$$b_2(y,T) = \frac{C_1}{2}F_y'(y,T),\ b_1(y,T) = 0,$$

$$b_0(y,T) = \frac{1}{C_1}(C_1^2\sigma F_y'(y,T) - cF_T'(y,T)),$$

$$a_1(y,T) = C_1,\ a_0(y,T) = 0,$$

$$\xi(x,y,T) = C_1 x + F(y,T) = C_1 x + F(y, c\frac{t^\alpha}{\alpha}),$$

where C_1 is an arbitrary nonzero constant, and $F(y,T)$ is an arbitrary function.

On the other hand, for Eq. (14), the following solutions are known to us.

$$\begin{cases} \phi_1(\xi) = -\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\xi + c_0),\ \sigma < 0, \\ \phi_2(\xi) = -\sqrt{-\sigma}\coth(\sqrt{-\sigma}\xi + c_0),\ \sigma < 0, \\ \phi_3(\xi) = \sqrt{\sigma}\tan(\sqrt{\sigma}\xi + c_0),\ \sigma > 0, \\ \phi_4(\xi) = -\sqrt{\sigma}\cot(\sqrt{\sigma}\xi + c_0),\ \sigma > 0, \\ \phi_{5,\ 6}(\xi) = \sqrt{\sigma}[\tan(2\sqrt{\sigma}\xi + c_0) \pm \sec(2\sqrt{\sigma}\xi + c_0)],\ \sigma > 0, \\ \phi_7(\xi) = -\dfrac{1}{\xi + c_0},\ \sigma = 0, \end{cases} \tag{15}$$

where c_0 is a constant.

By a combination of the result above and (15), together with the expression of ξ, one can obtain the following exact solutions for the fractional Bogoyavlenskii equations.

When $\sigma < 0$:

$$\begin{cases} u_3(x,y,t) = \sqrt{-\sigma}C_1 \\ \quad \{\dfrac{sech^2[\sqrt{-\sigma}(\frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha})) + c_0]}{\tanh[\sqrt{-\sigma}(\frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha})) + c_0]}\}, \\ v_3(x,y,t) = -\dfrac{C_1\sigma}{2}F_y'(y, c\frac{t^\alpha}{\alpha}) \\ \quad \{\dfrac{sech^2[\sqrt{-\sigma}(\frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha})) + c_0]}{\tanh[\sqrt{-\sigma}(\frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha})) + c_0]}\}^2 \\ \quad + \dfrac{1}{C_1}(C_1^2\sigma F_y'(y, c\frac{t^\alpha}{\alpha}) - cF_T'(y, c\frac{t^\alpha}{\alpha})). \end{cases} \tag{16}$$

$$\begin{cases} u_4(x,y,t) = -\sqrt{-\sigma}C_1 \\ \quad \{\dfrac{csch^2[\sqrt{-\sigma}(\frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha})) + c_0]}{\coth[\sqrt{-\sigma}(\frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha})) + c_0]}\}, \\ v_4(x,y,t) = -\dfrac{C_1\sigma}{2}F_y'(y, c\frac{t^\alpha}{\alpha}) \\ \quad \{\dfrac{csch^2[\sqrt{-\sigma}(\frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha})) + c_0]}{\coth[\sqrt{-\sigma}(\frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha})) + c_0]}\}^2 \\ \quad + \dfrac{1}{C_1}(C_1^2\sigma F_y'(y, c\frac{t^\alpha}{\alpha}) - cF_T'(y, c\frac{t^\alpha}{\alpha})). \end{cases} \tag{17}$$

When $\sigma > 0$:

$$\begin{cases} u_5(x,y,t) = \sqrt{\sigma}C_1 \\ \quad \{\dfrac{\sec^2[\sqrt{\sigma}(\frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha})) + c_0]}{\tan[\sqrt{\sigma}(\frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha})) + c_0]}\}, \\ v_5(x,y,t) = \dfrac{C_1\sigma}{2}F_y'(y, c\frac{t^\alpha}{\alpha}) \\ \quad \{\dfrac{\sec^2[\sqrt{\sigma}(\frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha})) + c_0]}{\tan[\sqrt{\sigma}(\frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha})) + c_0]}\}^2 \\ \quad + \dfrac{1}{C_1}(C_1^2\sigma F_y'(y, c\frac{t^\alpha}{\alpha}) - cF_T'(y, c\frac{t^\alpha}{\alpha})). \end{cases} \tag{18}$$

$$\begin{cases} u_6(x,y,t) = -\sqrt{\sigma}C_1 \\ \quad \{\dfrac{\csc^2[\sqrt{\sigma}(\frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha})) + c_0]}{\cot[\sqrt{\sigma}(\frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha})) + c_0]}\}, \\ v_6(x,y,t) = \dfrac{C_1\sigma}{2}F_y'(y, c\frac{t^\alpha}{\alpha}) \\ \quad \{\dfrac{\csc^2[\sqrt{\sigma}(\frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha})) + c_0]}{\cot[\sqrt{\sigma}(\frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha})) + c_0]}\}^2 \\ \quad + \dfrac{1}{C_1}(C_1^2\sigma F_y'(y, c\frac{t^\alpha}{\alpha}) - cF_T'(y, c\frac{t^\alpha}{\alpha})). \end{cases} \tag{19}$$

$$\begin{cases} u_{7,8}(x,y,t) = 2\sqrt{\sigma}C_1 \\ \quad \{\dfrac{\sec^2[2\sqrt{\sigma}\xi + c_0] \pm \sec(2\sqrt{\sigma}\xi + c_0)\tan(2\sqrt{\sigma}\xi + c_0)}{\tan(2\sqrt{\sigma}\xi + c_0) \pm \sec(2\sqrt{\sigma}\xi + c_0)}\}, \\ v_{7,8}(x,y,t) = 2\sigma^2 C_1 F_y'(y, c\frac{t^\alpha}{\alpha}) \\ \quad \{\dfrac{\sec^2[2\sqrt{\sigma}\xi + c_0] \pm \sec(2\sqrt{\sigma}\xi + c_0)\tan(2\sqrt{\sigma}\xi + c_0)}{\tan(2\sqrt{\sigma}\xi + c_0) \pm \sec(2\sqrt{\sigma}\xi + c_0)}\}^2 \\ \quad\quad + \dfrac{1}{C_1}(C_1^2\sigma F_y'(y, c\frac{t^\alpha}{\alpha}) - cF_T'(y, c\frac{t^\alpha}{\alpha})), \end{cases} \tag{20}$$

where $\xi = \frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha})$.

When $\sigma = 0$:

$$\begin{cases} u_9(x,y,t) = -\dfrac{C_1}{\frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha}) + c_0}, \\ v_9(x,y,t) = \dfrac{C_1 F_y'(y, c\frac{t^\alpha}{\alpha})}{2[\frac{C_1}{\lambda}x + F(y, c\frac{t^\alpha}{\alpha}) + c_0]^2} \\ \quad + \dfrac{1}{C_1}(C_1^2\sigma F_y'(y, c\frac{t^\alpha}{\alpha}) - cF_T'(y, c\frac{t^\alpha}{\alpha})), \end{cases} \tag{21}$$

Remark 4. If we take $F(y,T)$ for linear functions such as $F(y,T) = k_1 y + T$, then the transformation denoted by ξ becomes $\xi(x,y,T) = \frac{C_1}{\lambda}x + k_1 y + T = \frac{C_1}{\lambda}x + k_1 y + c\frac{t^\alpha}{\alpha}$, which has been used by many authors in the existing papers. So the solutions obtained here are of more general forms than most of the existing results, and are different from the solutions in [29].

B. Conformable fractional Jimbo-Miwa equation

We consider the conformable time fractional Jimbo-Miwa (JM) equation [22,37] of the form:

$$u_{xxxy} + pu_y u_{xx} + qu_x u_{xy} + rD_t^\alpha u_y - su_{xz} = 0,\ 0 < \alpha \le 1, \tag{22}$$

where $p,\ q,\ r,\ s$ are arbitrary nonzero constants.

The integer ordered form of the JM equation is a member of the KP-hierarchy and is not capable of passing the integrability tests. In [22,37], the authors obtained some exact solutions for the fractional JM equation by use of the Kudryashov method. Now we solve this equation by use of the method mentioned above.

Let $T = c\frac{t^\alpha}{\alpha}$ and $u(x,y,z,t) = \widetilde{u}(x,y,z,T)$. Then $D_t^\alpha u = c\widetilde{u}_T$, and Eq. (22) is converted into the following form

$$\widetilde{u}_{xxxy} + p\widetilde{u}_y\widetilde{u}_{xx} + q\widetilde{u}_x\widetilde{u}_{xy} + rc\widetilde{u}_{yT} - s\widetilde{u}_{xz} = 0. \quad (23)$$

Assume that $\widetilde{u}(x,y,z,T) = U(\xi)$, where $\xi = \xi(x,y,z,T)$. Then Eq. (23) is converted into

$$\xi_x^3\xi_y U'''' + (3\xi_x^2\xi_{xy} + 3\xi_x\xi_{xx}\xi_y)U'''$$
$$+[3\xi_{xx}\xi_{xy} + 3\xi_x\xi_{xxy} + \xi_y\xi_{xxx} - s\xi_x\xi_z + rc\xi_y\xi_T]U''$$
$$+\xi_{xxxy}U' + (p+q)\xi_y\xi_x^2U'U'' + (p\xi_y\xi_{xx} + q\xi_x\xi_{xy})(U')^2 = 0. \quad (24)$$

Suppose that the solutions of Eq. (24) can be expressed by a polynomial in $(\frac{\phi'}{\phi})$ as follows:

$$U(\xi) = \sum_{i=0}^{m} a_i(y,z,T)(\frac{\phi'}{\phi})^i, \quad (25)$$

where $a_i(y,z,T)$ are under-determined functions, and $\phi = \phi(\xi)$ satisfies Eq. (5). By balancing the order of U'''' and $U'U''$ in Eq. (24) one can obtain $m = 1$. So one has

$$U(\xi) = a_1(y,z,T)(\frac{\phi'}{\phi}) + a_0(y,z,T). \quad (26)$$

Similar to above, next we will process the computation in two cases.

Case 1: If $\phi = \phi(\xi)$ satisfies the Bernoulli equation denoted by Eq. (11), then substituting (26) into (23)-(24), using Eq. (11) and collecting all the terms with the same power of ϕ together, equating each coefficient to zero, yields a set of under-determined partial differential equations for $a_0(y,z,T)$, $a_1(y,z,T)$, $\xi(x,y,z,T)$. Solving these equations, yields several families of results as follows, where C_1, C_2 are arbitrary constants, and F_i, $i = 1,2,...,5$ are arbitrary functions with respect to their variables respectively.

Family 1.

$$a_0(y,z,T) = \int \frac{-24s(F_1(y,z))_z}{pC_1(p+q)}dy + F_2(z,T),$$
$$a_1(y,z,T) = C_1,$$
$$\xi(x,y,z,T) = -\frac{1}{24}C_1(p+q)x + \frac{C_1^3\lambda^2(p+q)^3}{3456rc}T + F_1(y,z).$$

Family 2.

$$a_0(y,z,T) = -\frac{12F_1''(z)y^2s}{(p+q)C_1p} +$$
$$[\frac{C_1p\lambda^2F_1'(z)}{6(p+q)} + \frac{C_1\lambda^2F_1'(z)q^2}{6(p+q)p} + \frac{C_1\lambda^2F_1'(z)q}{3(p+q)} - \frac{24F_2'(z)s}{(p+q)C_1p}]y$$
$$+F_3(z,T),$$
$$a_1(y,z,T) = C_1,$$
$$\xi(x,y,z,T) = \frac{-24rcF_1(z) + C_1s(p+q)T}{C_1s(p+q)} + F_1'(z)y$$
$$+F_2(z) - \frac{p+q}{24}C_1x.$$

Family 3.

$$a_0(y,z,T) = \int \frac{1}{6C_1^2p(p+q)^2}[(C_1^3\lambda^2p^3 + 3C_1^3\lambda^2qp^2$$
$$+3C_1^3\lambda^2q^2p + C_1^3\lambda^2q^3 - 3456C_2cr)(F_1(y,z))_y$$
$$-144C_1s(p+q)(F_1(y,z))_z]dy + F_2(z,T),$$
$$a_1(y,z,T) = C_1,$$
$$\xi(x,y,z,T) = F_1(y,z) + (\frac{1}{24}(-p-q))xC_1 + C_2T.$$

Family 4.

$$a_0(y,z,T) = \int \frac{1}{6(pF_1(y,z)(p+q))}[(p^2\lambda^2 + 2p\lambda^2q + \lambda^2q^2)$$
$$(F_2(y,z))_y(F_1(y,z))^2(3\lambda p^2 + 3\lambda qp)F_1(y,z)(F_1(y,z))_y$$
$$-144s(F_2(y,z))_z]dy + F_3(z,T),$$
$$a_1(y,z,T) = F_1(y,z), \xi(x,y,z,T) = F_2(y,z).$$

Family 5.

$$a_0(y,z,T) = \int \frac{1}{6(pC_1(p+q))}[(p^2\lambda^2C_1^2 + 2\lambda^2qC_1^2p$$
$$+C_1^2\lambda^2q^2)(F_1(y,z))_y - 144s(F_1(y,z))_z]dy + F_2(z,T),$$
$$a_1(y,z,T) = C_1, \xi(x,y,z,T) = F_1(y,z) - \frac{p+q}{24}C_1x.$$

Family 6.

$$a_0(y,z,T) = F_4(y,z) + F_3(z,T),$$
$$a_1(y,z,T) = F_1(y,z), \xi(x,y,z,T) = F_2(y,z).$$

Family 7.

$$a_0(y,z,T) = F_5(y,z) + F_4(z,T),$$
$$a_1(y,z,T) = F_2(y,z) + F_1(z,T), \xi(x,y,z,T) = F_3(z).$$

Family 8.

$$a_0(y,z,T) = F_4(y,z) + F_3(z,T),$$
$$a_1(y,z,T) = F_1(z,T), \xi(x,y,z,T) = F_2(z,T).$$

Family 9.

$$a_0(y,z,T) = F_4(z)y + F_3(z,T),$$
$$a_1(y,z,T) = F_1(z,T), \xi(x,y,z,T) = F_2(z,T).$$

Family 10.

$$a_0(y,z,T) = F_3(z,T),$$
$$a_1(y,z,T) = F_1(z,T), \xi(x,y,z,T) = F_2(T).$$

Family 11.

$$a_0(y,z,T) = F_3(z,T),$$
$$a_1(y,z,T) = F_1(z,T), \xi(x,y,z,T) = F_2(z,T).$$

Family 12.

$$a_0(y,z,T) = F_3(z,T),$$

$$a_1(y,z,T)=F_1(T),\xi(x,y,z,T)=F_2(x,T).$$

Substituting the results above into Eq. (26), and combining with (12), one can obtain a series of exact solutions for the fractional JM equation. Take Families 1 and 5 for example, one has the following solutions.

$$u_1(x,y,z,t)=\int\frac{-24s(F_1(y,z))_z}{pC_1(p+q)}dy$$
$$+F_2(z,c\frac{t^\alpha}{\alpha})+C_1(-\frac{A\lambda e^{2\lambda\xi}}{\frac{\mu}{\lambda}+Ae^{2\lambda\xi}}), \quad (27)$$

where $\xi=-\frac{1}{24}C_1(p+q)x+\frac{C_1^3\lambda^2(p+q)^3}{3456r\alpha}t^\alpha+F_1(y,z)$.

$$u_2(x,y,z,t)=\int\frac{1}{6(pC_1(p+q))}$$
$$[(p^2\lambda^2C_1^2+2\lambda^2qC_1^2p+C_1^2\lambda^2q^2)(F_1(y,z))_y$$
$$-144s(F_1(y,z))_z]dy$$
$$+F_2(z,T)+C_1(-\frac{A\lambda e^{2\lambda\xi}}{\frac{\mu}{\lambda}+Ae^{2\lambda\xi}}), \quad (28)$$

where $\xi=F_1(y,z)-\frac{p+q}{24}C_1x$.

Especially, if we set $\mu=\lambda A$ in Eq. (27), then we obtain the following solitary wave solution:

$$u_3(x,y,z,t)=\int\frac{-24s(F_1(y,z))_z}{pC_1(p+q)}dy$$
$$+F_2(z,c\frac{t^\alpha}{\alpha})-\frac{C_1\lambda}{2}[(1+\tanh(\lambda\xi))].$$

Case 2: If $\phi=\phi(\xi)$ satisfies the Riccati equation denoted by Eq. (14), then substituting (26) into (23)-(24), using Eq. (14) and collecting all the terms with the same power of ϕ together, equating each coefficient to zero, yields a set of under-determined partial differential equations. Solving these equations, yields several families of results as follows, where $F_i,\ i=1,2,...,5$ are arbitrary functions with respect to their variables respectively.

Family 1.

$$a_0(y,z,T)=F_3(z,T),a_1(y,z,T)=F_2(z,T),$$
$$\xi(x,y,z,T)=F_1(z,T).$$

Family 2.

$$a_0(y,z,T)=F_3(z,T),a_1(y,z,T)=F_2(T),$$
$$\xi(x,y,z,T)=F_1(x,T).$$

Family 3.

$$a_0(y,z,T)=F_5(z)*y+F_4(z,T),a_1(y,z,T)=F_2(z,T),$$
$$\xi(x,y,z,T)=F_1(z,T).$$

Family 4.

$$a_0(y,z,T)=F_4(y,z)+F_3(z,T),a_1(y,z,T)=F_2(z,T),$$
$$\xi(x,y,z,T)=F_1(z,T).$$

Family 5.

$$a_0(y,z,T)=F_5(y,z)+F_4(z,T),$$
$$a_1(y,z,T)=F_3(y,z)+F_2(z,T),\ \xi(x,y,z,T)=F_1(z).$$

Family 6.

$$a_0(y,z,T)=F_4(y,z)+F_3(z,T),a_1(y,z,T)=F_2(y,z),$$
$$\xi(x,y,z,T)=F_1(y,z).$$

By a combination of the results above and (15), one can obtain abundant exact solutions for the fractional JM equation. Take Family 1 for example, one has the following solutions.

When $\sigma<0$:

$$u_4(x,y,z,t)=F_3(z,c\frac{t^\alpha}{\alpha})$$
$$+F_2(z,c\frac{t^\alpha}{\alpha})\sqrt{-\sigma}\{\frac{sech^2[\sqrt{-\sigma}F_1(z,c\frac{t^\alpha}{\alpha})+c_0]}{\tanh[\sqrt{-\sigma}F_1(z,c\frac{t^\alpha}{\alpha})+c_0]}\}. \quad (29)$$

$$u_5(x,y,z,t)=F_3(z,c\frac{t^\alpha}{\alpha})$$
$$-F_2(z,c\frac{t^\alpha}{\alpha})\sqrt{-\sigma}\{\frac{csch^2[\sqrt{-\sigma}F_1(z,c\frac{t^\alpha}{\alpha})+c_0]}{\coth[\sqrt{-\sigma}F_1(z,c\frac{t^\alpha}{\alpha})+c_0]}\}. \quad (30)$$

When $\sigma>0$:

$$u_6(x,y,z,t)=F_3(z,c\frac{t^\alpha}{\alpha})$$
$$+F_2(z,c\frac{t^\alpha}{\alpha})\sqrt{\sigma}\{\frac{\sec^2[\sqrt{\sigma}F_1(z,c\frac{t^\alpha}{\alpha})+c_0]}{\tan[\sqrt{\sigma}F_1(z,c\frac{t^\alpha}{\alpha})+c_0]}\}. \quad (31)$$

$$u_7(x,y,z,t)=F_3(z,c\frac{t^\alpha}{\alpha})$$
$$-F_2(z,c\frac{t^\alpha}{\alpha})\sqrt{\sigma}\{\frac{\csc^2[\sqrt{\sigma}F_1(z,c\frac{t^\alpha}{\alpha})+c_0]}{\cot[\sqrt{\sigma}F_1(z,c\frac{t^\alpha}{\alpha})+c_0]}\}. \quad (32)$$

$$u_{8,9}(x,y,z,t)=F_3(z,c\frac{t^\alpha}{\alpha})+2\sqrt{\sigma}F_2(z,c\frac{t^\alpha}{\alpha})$$
$$\{\frac{\sec^2[2\sqrt{\sigma}\xi+c_0]\pm\sec(2\sqrt{\sigma}\xi+c_0)\tan(2\sqrt{\sigma}\xi+c_0)}{\tan(2\sqrt{\sigma}\xi+c_0)\pm\sec(2\sqrt{\sigma}\xi+c_0)}\}, \quad (33)$$

where $\xi=F_1(z,T)=F_1(z,c\frac{t^\alpha}{\alpha})$.

When $\sigma=0$:

$$u_{10}(x,y,z,t)=F_3(z,c\frac{t^\alpha}{\alpha})-\frac{F_2(z,c\frac{t^\alpha}{\alpha})}{\frac{C_1}{\lambda}x+F(y,c\frac{t^\alpha}{\alpha})+c_0}. \quad (34)$$

Remark 5. The established solutions above for the fractional JM equation are new exact solutions to our best knowledge.

IV. Conclusions

We have proposed a variable-coefficient sub-equation method for solving fractional partial differential equations, and applied it to find exact solutions for the fractional Bogoyavlenskii equations and the fractional JM equation. Some exact solutions with new forms and variable coefficient functions for them have been successfully found, which may provide some references for the research in related physical phenomena. This method can be supposed to be applied to solve other types of fractional partial differential equations, which is expected to further research.

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