Parameter Estimation for Squared Radial Ornstein-Uhlenbeck Process from Discrete Observation

Chao Wei, Dehe Li, Hejun Yao

Abstract—**This paper is concerned with parameter estimation problem for squared radial Ornstein-Uhlenbeck process driven by α-stable noises from discrete observation. Firstly, the existence and uniqueness of solutions to the stochastic differential equation is studied. Then, the contrast function is used to obtain the least squares estimator. The strong consistency and asymptotic distribution of the estimator are investigated. Finally, some numerical calculus and simulations are given to verify the effectiveness of estimator.**

Index Terms—**Existence and uniqueness of solutions; squared radial Ornstein-Uhlenbeck process; α-stable noises; consistency; asymptotic distribution.**

I. Introduction

Stochastic differential equations are important tools for studying random phenomena and widely used in the fields of physics, chemistry, medicine and finance ([2]–[4], [11], [16]). However, parameters in stochastic model are always unknown. In the past few decades, some popular methods have been put forward to estimate the parameters in Itô stochastic differential equations, such as maximum likelihood estimation ([1], [18]–[20]), least squares estimation ([10], [14], [15], [17]) and Bayes estimation ([5], [8], [9]). But, in fact, non-Gaussian noise can more accurately reflect the practical random perturbation. α-stable noise, as a kind of important non-Gaussian noise, has attracted wide attention in the research and practice in the fields of engineering, economy and society. From a practical point of view in parametric inference, it is more realistic and interesting to consider asymptotic estimation for stochastic differential equations driven by α-stable motions. Recently, a number of literatures have been devoted to the parameter estimation for the models with α-stable noises. When the coefficient of the α-stable motion term is constant, drift parameter estimation has been investigated by some authors ([12], [13]).

As we all know, parameter estimation for Ornstein-Uhlenbeck processes driven by α-stable motions has been studied by some authors ([6], [7], [21]). However, there are few literature about the parameter estimation problem for squared radial Ornstein-Uhlenbeck process driven by α-stable noises. This model has many appealing advantages. In particular, it is mean-reverting and remains non-negative. However, it is well-known that many financial processes exhibit discontinuous sample paths and heavy tailed properties (e.g. certain moments are infinite). These features cannot be captured by the squared radial Ornstein-Uhlenbeck process. Therefore, it is natural to replace the driving Brownian motion by an α-stable process. Since squared radial Ornstein-Uhlenbeck process has more complex drift coefficient than Ornstein-Uhlenbeck process, it is difficult to investigate the parameter estimation problem. Asymptotic properties of estimators such as consistency, asymptotic distribution of estimation errors, and hypothesis tests can reflect the effectiveness of estimators and estimation methods, which helps to obtain a more reasonable economic model structure and more accurately grasp the dynamics of related assets . Therefore, it is of great important to study the topics.

In this paper, we consider the parameter estimation problem for squared radial Ornstein-Uhlenbeck process with α-stable noises from discrete observations. The contrast function is introduced to obtain the least squares estimator. The strong consistency and asymptotic distribution of the estimator are proved by using ergodic theorem, Hölder inequality and Markov inequality. Some numerical calculus and simulations are given to verify the effectiveness of estimator.

This paper is organized as follows. In Section 2, the squared radial Ornstein-Uhlenbeck process driven by α-stable noises is introduced. In Section 3, the existence and uniqueness of solutions are proved, the contrast function is given, the explicit formula of the least squares estimator is obtained, the strong consistency and asymptotic distribution of the estimator are proved. In Section 4, some simulation results are made. The conclusion is given in Section 5.

II. Problem Formulation and Preliminaries

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a basic probability space equipped with a right continuous and increasing family of σ-algebras $\{\mathscr{F}_t\}_{t\geq 0}$ and $Z = \{Z_t, t \geq 0\}$ be a strictly symmetric α-stable Lévy motion.

A random variable η is said to have a stable distribution with index of stability $\alpha \in (0, 2]$, scale parameter $\sigma \in (0, \infty)$, skewness parameter $\beta \in [-1, 1]$ and location parameter $\mu \in (-\infty, \infty)$ if it has the following characteristic

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function:

$$\phi_\eta(u) = \begin{cases} -\sigma^\alpha |u|^\alpha (1 - i\beta sgn(u) \tan \frac{\alpha\pi}{2}) + i\mu u, if \alpha \neq 1, \\ -\sigma|u|(1 + i\beta \frac{2}{\pi} sgn(u) \log |u|) + i\mu u, if \alpha = 1. \end{cases}$$

We denote $\eta \sim S_\alpha(\sigma, \beta, \mu)$. When $\mu = 0$, we say η is strictly α-stable, if in addition $\beta = 0$, we call η symmetrical α-stable. Throughout this paper, it is assumed that α-stable motion is strictly symmetrical and $\alpha \in (1, 2)$.

In this paper, we investigate the parameter estimation problem for α-stable squared radial Ornstein-Uhlenbeck process described by the following stochastic differential equation:

$$\begin{cases} dX_t = (1 + 2\theta X_t)dt + 2\sqrt{X_t} dZ_t \\ X_0 = x_0, \end{cases} \quad (1)$$

where θ is an unknown parameter with $\theta < 0$ and Z is a strictly symmetric α-stable motion on $\mathbb{R}$ with the index $\alpha \in (1, 2)$.

It is assumed that the process $\{X_t, t \geq 0\}$ can be observed at discrete point $\{t_i = ih, i = 0, 1, 2, ..., n\}$ with $h > 0$. We introduce the following contrast function:

$$\rho_n(\theta) = \frac{\sum_{i=1}^n |X_{t_i} - X_{t_{i-1}} - (1 + 2\theta X_{t_{i-1}})\Delta t_{i-1}|^2}{4X_{t_{i-1}}\Delta t_{i-1}}, \quad (2)$$

where $\Delta t_{i-1} = t_i - t_{i-1} = h$.

Then, we can obtain the estimator as follows

$$\widehat{\theta}_n = \frac{\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}}) - nh}{2h \sum_{i=1}^n X_{t_{i-1}}}. \quad (3)$$

Since the α-stable squared radial Ornstein-Uhlenbeck process can be written as

$$X_t = X_0 e^{2\theta t} - \frac{1}{2\theta}(1 - e^{2\theta t}) + 2\int_0^t e^{2\theta(t-s)} \sqrt{X_s} dZ_s. \quad (4)$$

The expression of $\widehat{\theta}_n$ can be changed as

$$\widehat{\theta}_n = \frac{e^{2\theta h} - 1}{2h} + \frac{n(\frac{1}{2\theta}e^{2\theta h} - \frac{1}{2\theta} - h)}{2h\sum_{i=1}^n X_{t_{i-1}}} + \frac{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} e^{2\theta(t_i - s)} dZ_s}{h\sum_{i=1}^n X_{t_{i-1}}}. \quad (5)$$

III. MAIN RESULTS AND PROOFS

The Lévy measure ν is defined as follows:

$$\nu(dz) = \frac{C_\alpha}{|z|^{\alpha+1}} dz, \quad (6)$$

where

$$C_\alpha = \frac{\alpha 2^{\alpha-1} \Gamma(\frac{(1+\alpha)}{2})}{\pi^{\frac{1}{2}} \Gamma(1 - \frac{\alpha}{2})}, \quad (7)$$

where $\Gamma(\cdot)$ is a Gamma function.

In the following theorem, the existence and uniqueness of solutions to the above stochastic differential equation are proved.

Theorem 1: The α-stable squared radial Ornstein-Uhlenbeck process has the unique global solution $x(t)$.

Proof: Let $k_0 > 0$ is large enough and $x_0 < k_0$. For every integer, let $k > k_0$. Define the stop time

$$\tau_k = \inf\{t \in [0, \tau_e) : x(t) \notin (\frac{1}{k}, k)\}, \quad (8)$$

where τ_e is the time of explosion.

Since the coefficients of Equation (1) satisfy the local Lipschitz conditions. Then, for the given initial value $x_0 \in R$, there exists a unique local solution $x(t)$ when $t \in [0, \tau_e)$. Let $\tau_\infty = \lim_{k\to\infty} \tau_k$ and $\tau_\infty < \tau_e$ a.s. If we get $\tau_\infty = \infty$ a.s., then, for every $t \geq 0$, $x(t) \in R$. For the constant $\gamma > 0$ and $0 < p < 1$, let

$$V_\gamma(x) = (x^2 + \gamma^2)^{\frac{p}{2}}, \quad x \in R. \quad (9)$$

For $0 \leq t \leq T$, according to Itô formula, it follows that

$$V_\gamma(x(t \wedge \tau_k)) = V_\gamma(x_0) + \int_0^{t\wedge\tau_k} LV_\gamma(x(s))ds + M_1(t \wedge \tau_k), \quad (10)$$

where $M_1(t \wedge \tau_k)$ is a local martingale and

$$LV_\gamma = px(x^2 + \gamma^2)^{\frac{p-2}{2}}(1 + 2\theta x) + \int_0^1 [V_\gamma(x + 2\sqrt{x}z) - V_\gamma(x) - V_\gamma^{'} 2\sqrt{x}z 1_{\{0\leq z\leq 1\}}] \frac{C_\alpha}{|z|^{\alpha+1}} dz. \quad (11)$$

According to Taylor formula, it can be checked that

$$\begin{aligned} &\int_0^1 [V_\gamma(x + 2\sqrt{x}z) - V_\gamma(x) - V_\gamma^{'} 2\sqrt{x}z] \frac{C_\alpha}{|z|^{\alpha+1}} dz \\ &\leq sgn(2\sqrt{x})|2\sqrt{x}|^\alpha \int_0^{2\sqrt{x}} [V_\gamma(x + 2\sqrt{x}z) - V_\gamma(x) - V_\gamma^{'} 2\sqrt{x}z] \\ &\times \frac{C_\alpha}{2^\alpha |\sqrt{x}z|^{\alpha+1}} d(\sqrt{x}z) \\ &= sgn(2\sqrt{x})|x|^{\frac{\alpha}{2}} \int_0^{2\sqrt{x}} V_\gamma^{''}(\xi) \frac{C_\alpha}{2|\sqrt{x}z|^{\alpha+1}} d(\sqrt{x}z) \end{aligned}$$

,

where $\xi \in (x, x + 2\sqrt{x}z)$ and

$$V_\gamma^{''}(\xi) = p(\xi^2 + \gamma^2)^{\frac{p-2}{2}}((p-1)\xi^2 + \gamma^2). \quad (12)$$

From the basic inequality, it is easy to check that

$$-\xi^2 - \gamma^2 \leq (p-1)\xi^2 + \gamma^2 \leq \xi^2 + \gamma^2. \quad (13)$$

Then, we obtain

$$|(p-1)\xi^2 + \gamma^2| \leq \xi^2 + \gamma^2. \quad (14)$$

Substituting (10) into (8), it follows that

$$V_\gamma^{''}(\xi) \leq p(\xi^2 + \gamma^2)^{\frac{p}{2}-1} \leq p\xi^{p-2}. \quad (15)$$

Then, it can be checked that

$$\begin{aligned} &\int_0^1 [V_\gamma(x + 2\sqrt{x}z) - V_\gamma(x) - V_\gamma^{'} 2\sqrt{x}z] \frac{C_\alpha}{|z|^{\alpha+1}} dz \\ &\leq sgn(2\sqrt{x})|2\sqrt{x}|^\alpha \int_0^{2\sqrt{x}} 2xz^2 p||x| - |2\sqrt{x}z||^{p-2} \\ &\frac{C_\alpha}{|2\sqrt{x}z|^{\alpha+1}} d(\sqrt{x}z) \\ &\leq sgn(2\sqrt{x}) \frac{C_\alpha p|\varphi(x)|^{p-2}}{2(2-\alpha)} 4x, \end{aligned} \quad (16)$$

where $\varphi(x) = \min(||x| - |2\sqrt{xz}||)$.

When $|x| = |2\sqrt{xz}|$, we obtain

$$\begin{aligned}
& sgn(2\sqrt{x})|2\sqrt{x}|^\alpha \int_0^{2\sqrt{x}} 2xz^2 V_\gamma^{''}(\xi) \\
& \frac{C_\alpha}{|2\sqrt{xz}|^{\alpha+1}} d(\sqrt{xz}) \\
& \le sgn(2\sqrt{x})|2\sqrt{x}|^\alpha \\
& \int_0^{2\sqrt{x}} 2xz^2 p(\xi^2+\gamma^2)^{\frac{p-2}{2}} \frac{C_\alpha}{|2\sqrt{xz}|^{\alpha+1}} d(\sqrt{xz}) \\
& \le sgn(2\sqrt{x})|2\sqrt{x}|^\alpha \\
& \int_0^{2\sqrt{x}} 2xz^2 p(||x|-|2\sqrt{xz}||+\gamma^2)^{\frac{p-2}{2}} \\
& \times \frac{C_\alpha}{|2\sqrt{xz}|^{\alpha+1}} d(\sqrt{xz}) \\
& \le \frac{C_\alpha p|\gamma|^{p-2}}{2(2-\alpha)} 4x.
\end{aligned}$$

Since $0 < p < 1$, it is easy to check that

$$\lim_{\gamma\to\infty} \frac{C_\alpha p|\gamma|^{p-2}}{2(2-\alpha)} 4x = 0. \tag{17}$$

Therefore, when γ is big enough, $\int_0^1 [V_\gamma(x+2\sqrt{xz}) - V_\gamma(x) - V_\gamma^{'} 2\sqrt{xz}] \frac{C_\alpha}{|z|^{\alpha+1}} dz$ is convergent.

Let $f(\gamma) = V_\gamma(x+2\sqrt{xz}) - V_\gamma(x)$, then we obtain

$$f'(\gamma) = p\gamma[((x+2\sqrt{xz})^2+\gamma^2)^{\frac{p-2}{2}} - (x^2+\gamma^2)^{\frac{p-2}{2}}].$$

Since $|x+2\sqrt{xz}| > x$, it is easy to check that $f'(\gamma) < 0$, then $f(\gamma) < f(0)$.

Thus,

$$\begin{aligned}
& |f(\gamma)| \\
& = |V_\gamma(x+2\sqrt{xz}) - V_\gamma(x)| < |f(0)| \\
& = |(x+2\sqrt{xz})^p - x^p| < |2\sqrt{xz}|^p.
\end{aligned} \tag{18}$$

Therefore,

$$\begin{aligned}
& \int_1^{+\infty} [V_\gamma(x+2\sqrt{xz}) - V_\gamma(x)] \frac{C_\alpha}{|z|^{\alpha+1}} dz \\
& \le sgn(2\sqrt{x})|2\sqrt{x}|^\alpha \int_{2\sqrt{x}}^{+\infty} |V_\gamma(x+2\sqrt{xz}) - V_\gamma(x)| \\
& \frac{C_\alpha}{|2\sqrt{xz}|^{\alpha+1}} d(2\sqrt{xz}) \\
& \le sgn(2\sqrt{x})|2\sqrt{x}|^\alpha \int_{2\sqrt{x}}^{+\infty} |x+2\sqrt{xz}|^p \frac{C_\alpha}{|2\sqrt{xz}|^{\alpha+1}} \\
& d(2\sqrt{xz}) \\
& = sgn(2\sqrt{x})|2\sqrt{x}|^{\alpha+p} \frac{C_\alpha}{\alpha-p}.
\end{aligned} \tag{19}$$

When $x \in [0,1)$, substituting (12) and (15) into (7), we get

$$\begin{aligned}
& LV_\gamma(x) \\
& \le px(x^2+\gamma^2)^{\frac{p-2}{2}}(1+2\theta x) \\
& + sgn(2\sqrt{x}) \frac{C_\alpha p|\varphi(x)|^{p-2}}{2(2-\alpha)} 4x + sgn(2\sqrt{x})|2\sqrt{x}|^p \\
& \frac{C_\alpha}{\alpha-p} \\
& = px(x^2+\gamma^2)^{\frac{p-2}{2}}(1+2\theta x) \\
& + sgn(2\sqrt{x}) \frac{C_\alpha}{2(2-\alpha)} |2\sqrt{x}|^p (\frac{|2\sqrt{x}|}{|\varphi(x)|})^{2-p} \\
& + sgn(2\sqrt{x}) \frac{C_\alpha}{\alpha-p} |2\sqrt{x}|^p \\
& = px(x^2+\gamma^2)^{\frac{p-2}{2}}(1+2\theta x) \\
& + sgn(2\sqrt{x}) C_\alpha 2^p x^{\frac{p}{2}} (\frac{2^{1-p}}{2-\alpha}(\frac{|\sqrt{x}|}{|\varphi(x)|})^{2-p} + \frac{1}{\alpha-p}).
\end{aligned}$$

It is assumed that $C_\alpha(\frac{2^{1-p}}{2-\alpha}(\frac{|\sqrt{x}|}{|\varphi(x)|})^{2-p} + \frac{1}{\alpha-p}) \le K(x^2+\gamma^2)^{\frac{p-2}{2}}$ and $px(1+2\theta x) + K_1 2^p x^{\frac{p}{2}} \le K_2(x^2+\gamma^2)$.

Then,

$$\begin{aligned}
& LV_\gamma(x) \\
& \le px(x^2+\gamma^2)^{\frac{p-2}{2}}(1+2\theta x) \\
& + K sgn(2\sqrt{x}) 2^p x^{\frac{p}{2}} (x^2+\gamma^2)^{\frac{p-2}{2}} \\
& \le (x^2+\gamma^2)^{\frac{p-2}{2}} (px(1+2\theta x) + K_1 2^p x^{\frac{p}{2}}) \\
& \le (x^2+\gamma^2)^{\frac{p-2}{2}} K_2(x^2+\gamma^2) \\
& = K_2 V_\gamma(x).
\end{aligned}$$

When $x = 0$, $V_\gamma(x) = V_\gamma(0) = \gamma^p$. Then, $V_\gamma^{'}(x) = 0$ and

$$LV_\gamma(x) \le K_2\gamma^p.$$

According to Itô formula, as $x(t\wedge\tau_k) \in R$, $0 \le t \le T$, it follows that

$$\mathbb{E}V_\gamma(x(T\wedge\tau_k)) = V_\gamma(x_0) + \mathbb{E}\int_0^{t\wedge\tau_k} LV_\gamma(x(s))ds. \tag{20}$$

From (16), we obtain

$$\begin{aligned}
& \mathbb{E}V_\gamma(x(T\wedge\tau_k)) \\
& \le V_\gamma(x_0) + K_2\gamma^p \mathbb{E}(T\wedge\tau_k) \\
& \le V_\gamma(x_0) + K_2\gamma^p T.
\end{aligned} \tag{21}$$

Note that for all $\delta \in \{\tau_k \le T\}$, there exists a constant k large enough satisfying $x(\tau_k,\delta) \ge k$ or $x(\tau_k,\delta) \le \frac{1}{k}$.

Hence,

$$V_\gamma(x(\tau_k,\delta)) \ge (k^2+\gamma^2)^{\frac{p}{2}} \wedge (\frac{1}{k^2}+\gamma^2)^{\frac{p}{2}}. \tag{22}$$

Therefore,

$$\begin{aligned}
& (k^2+\gamma^2)^{\frac{p}{2}} \wedge (\frac{1}{k^2}+\gamma^2)^{\frac{p}{2}} P(\tau_k \le T) \\
& \le \mathbb{E}(V_\gamma(\tau_k,\delta) 1_{\tau_k \le T}) \\
& \le V_\gamma(x_0) + K_2\gamma^p T.
\end{aligned} \tag{23}$$

Let $k \to \infty$, we obtain

$$P(\tau_\infty \le T) = 0. \tag{24}$$

Then,

$$P(\tau_\infty = \infty) = 1. \tag{25}$$

The proof is complete. ■

In the following theorem, the strong consistency of the least square estimator $\widehat{\theta}_n$ is proved.

Theorem 2: When $h \to 0$ and $nh \to \infty$,

$$\widehat{\theta}_n \stackrel{a.s.}{\to} \theta. \tag{26}$$

Proof: It can be checked that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} |X_{t_{i-1}}| = \mathbb{E}|X_\infty| = -\frac{1}{2\theta}, a.s. \tag{27}$$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} |X_{t_{i-1}}|^p = \mathbb{E}|X_\infty|^p = \infty, \quad p \geq \alpha, a.s. \tag{28}$$

and

$$\lim_{n\to\infty} \sup \frac{|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} e^{2\theta(t_i-s)} dZ_s|}{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} |\sqrt{X_s} e^{2\theta(t_i-s)}|^\alpha ds} = 0, a.s. \tag{29}$$

Moreover,

$$\begin{aligned} &|\frac{n(\frac{1}{2\theta}e^{2\theta h} - \frac{1}{2\theta} - h)}{2h\sum_{i=1}^n X_{t_{i-1}}} \\ &+\frac{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} e^{2\theta(t_i-s)} dZ_s}{h\sum_{i=1}^n X_{t_{i-1}}}| \\ \leq\ & \frac{n|\frac{1}{2\theta}e^{2\theta h} - \frac{1}{2\theta} - h|}{2h\sum_{i=1}^n |X_{t_{i-1}}|} \\ &+\frac{|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} e^{2\theta(t_i-s)} dZ_s|}{h\sum_{i=1}^n |X_{t_{i-1}}|}. \end{aligned}$$

$$\frac{n|\frac{1}{2\theta}e^{2\theta h} - \frac{1}{2\theta} - h|}{2h\sum_{i=1}^n |X_{t_{i-1}}|} = \frac{|\frac{1}{2\theta}e^{2\theta h} - \frac{1}{2\theta} - h|}{2h}\frac{1}{\frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}}|}.$$

When $h \to 0$, it is easy to check that

$$\frac{|\frac{1}{2\theta}e^{2\theta h} - \frac{1}{2\theta} - h|}{2h} \to 0. \tag{30}$$

According to (27) and (28), it follows that

$$\frac{n|\frac{1}{2\theta}e^{2\theta h} - \frac{1}{2\theta} - h|}{2h\sum_{i=1}^n |X_{t_{i-1}}|} \stackrel{a.s.}{\to} 0. \tag{31}$$

When $h \to 0$ and $nh \to \infty$, by Hölder inequality and (27), (28), (29), we obtain

$$\begin{aligned} &\frac{|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} e^{2\theta(t_i-s)} dZ_s|}{h\sum_{i=1}^n X_{t_{i-1}}} \\ &= \frac{|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} e^{2\theta(t_i-s)} dZ_s|}{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} |\sqrt{X_s} e^{2\theta(t_i-s)}|^\alpha ds} \\ &\frac{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} |\sqrt{X_s} e^{2\theta(t_i-s)}|^\alpha ds}{h\sum_{i=1}^n X_{t_{i-1}}} \\ &\leq \sup_n \frac{|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} e^{2\theta(t_i-s)} dZ_s|}{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} |\sqrt{X_s} e^{2\theta(t_i-s)}|^\alpha ds} \\ &\frac{(n)^{\frac{\alpha}{2}}(\sum_{i=1}^n |\int_{t_{i-1}}^{t_i} \sqrt{X_s} e^{2\theta(t_i-s)} dZ_s|^2)^{1-\frac{\alpha}{2}}}{h\sum_{i=1}^n X_{t_{i-1}}} \\ &\leq \sup_n \frac{|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} e^{2\theta(t_i-s)} dZ_s|}{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} |\sqrt{X_s} e^{2\theta(t_i-s)}|^\alpha ds} \\ &\frac{(\frac{1}{nh}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} |\sqrt{X_s} e^{2\theta(t_i-s)}|^2 ds)^{1-\frac{\alpha}{2}}}{h^{\frac{\alpha}{2}}(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})} \\ &\stackrel{a.s.}{\to} 0. \end{aligned}$$

Thus, we have

$$|\frac{n(\frac{1}{2\theta}e^{2\theta h} - \frac{1}{2\theta} - h)}{2h\sum_{i=1}^n X_{t_{i-1}}} + \frac{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} e^{2\theta(t_i-s)} dZ_s}{h\sum_{i=1}^n X_{t_{i-1}}}| \stackrel{a.s.}{\to} 0. \tag{32}$$

When $h \to 0$, it is obvious that

$$\frac{e^{2\theta h} - 1}{2h} \to \theta. \tag{33}$$

Therefore, from (32) and (33), it follows that

$$\widehat{\theta}_n \stackrel{a.s.}{\to} \theta. \tag{34}$$

The proof is complete. ■

Let

$$C_\alpha = (\int_0^\infty x^{-\alpha} \sin x dx)^{-1} = [\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2}]^{-1},$$

and $\sigma_1 = (C_{\alpha/2})^{-2/\alpha}$, $\sigma_2 = (C_\alpha)^{-1/\alpha}$.

Theorem 3: When $h \to 0$, $n \to \infty$, $nh^{1+\alpha}/\log n \to 0$, $nh^{2\alpha-1}\log n \to \infty$ and $nh^{2-\alpha/2+\rho} \to \infty$ for some $\rho > 0$ small enough,

$$(\frac{nh}{\log n})^{\frac{1}{\alpha}}(\widehat{\theta}_n - \theta) \stackrel{d}{\to} \frac{2\theta(\alpha\theta)^{-\frac{1}{\alpha}}\widetilde{Y}}{Y_0}, \tag{35}$$

where $Y_0 \sim S_{\alpha/2}(\sigma_1, 1, 0)$, $\widetilde{Y} \sim S_\alpha(\sigma_2, 0, 0)$.

Proof: From (25), we obtain

$$\begin{aligned} &(\frac{nh}{\log n})^{\frac{1}{\alpha}}(\widehat{\theta}_n - \theta) \\ &= (\frac{nh}{\log n})^{\frac{1}{\alpha}}(\frac{e^{2\theta h} - 1}{2h} - \theta) \\ &+(\frac{nh}{\log n})^{\frac{1}{\alpha}}\frac{n(\frac{1}{2\theta}e^{2\theta h} - \frac{1}{2\theta} - h)}{2h\sum_{i=1}^n X_{t_{i-1}}} \\ &+(\frac{nh}{\log n})^{\frac{1}{\alpha}}\frac{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sqrt{X_s} e^{2\theta(t_i-s)} dZ_s}{h\sum_{i=1}^n X_{t_{i-1}}}. \end{aligned}$$

It is easy to check that

$$|(\frac{nh}{\log n})^{\frac{1}{\alpha}}(\frac{e^{2\theta h} - 1}{2h} - \theta)| \leq \frac{2\theta^2 n^{\frac{1}{\alpha}} h^{1+\frac{1}{\alpha}}}{(\log n)^{\frac{1}{\alpha}}} \to 0. \tag{36}$$

According to (27) and (28),

$$
\begin{aligned}
&(\frac{nh}{\log n})^{\frac{1}{\alpha}}\frac{n(\frac{1}{2\theta}e^{2\theta h}-\frac{1}{2\theta}-h)}{2h\sum_{i=1}^{n}X_{t_{i-1}}}\\
&=(\frac{nh^{1+\alpha}}{\log n})^{\frac{1}{\alpha}}\frac{(\frac{1}{2\theta}e^{2\theta h}-\frac{1}{2\theta}-h)}{2h^2\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}\stackrel{a.s.}{\rightarrow}0.
\end{aligned}
$$

$$
\begin{aligned}
&(\frac{nh}{\log n})^{\frac{1}{\alpha}}\frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sqrt{X_s}e^{2\theta(t_i-s)}dZ_s}{h\sum_{i=1}^{n}X_{t_{i-1}}}\\
&=\frac{(n\log n)^{-\frac{1}{\alpha}}h^{-\frac{1}{\alpha}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sqrt{X_s}e^{2\theta(t_i-s)}dZ_s}{n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n}X_{t_{i-1}}}.
\end{aligned}
$$

From (24), we find

$$
\begin{aligned}
&X_{t_i}\\
&=-\frac{1}{2\theta}+(x_0+\frac{1}{2\theta})e^{2\theta ih}\\
&+2\sum_{k=1}^{i}e^{2\theta ih}\int_{t_{k-1}}^{t_k}e^{-2\theta s}\sqrt{X_s}dZ_s. \qquad (37)
\end{aligned}
$$

By the property of the α-stable stochastic integral, we obtain that

$$
V_{k-1}=\int_{t_{k-1}}^{t_k}e^{-2\theta s}\sqrt{X_s}dZ_s
$$

has the same distribution as $Z_{\tau_{k-1}}$ for all $k=1,2,...,n$, where

$$
\tau_{k-1}=\int_{t_{k-1}}^{t_k}|e^{-2\theta s}\sqrt{X_s}|^{\alpha}ds.
$$

Let $U_{k-1}=\frac{V_{k-1}}{(\tau_{k-1})^{\frac{1}{\alpha}}}$, then, $U_k, k=1,2,...,n$ are an i.i.d random sequence with same stable distribution $S_{\alpha}(1,0,0)$.

Then, (37) can be written as

$$
X_{t_i}=-\frac{1}{2\theta}+(x_0+\frac{1}{2\theta})e^{2\theta ih}+2\sum_{k=1}^{i}e^{2\theta ih}U_{k-1}(\tau_{k-1})^{\frac{1}{\alpha}}. \qquad (38)
$$

As

$$
n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n}X_{t_{i-1}}^2=n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}x_0^2+n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}X_{t_i}^2.
$$

It is obvious that

$$
n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}x_0^2\stackrel{P}{\rightarrow}0.
$$

According to (38), we obtain that

$$
\begin{aligned}
&n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}X_{t_i}^2\\
&=n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}(-\frac{1}{2\theta}+(x_0+\frac{1}{2\theta})e^{2\theta ih}\\
&+2\sum_{k=1}^{i}e^{2\theta ih}U_{k-1}(\tau_{k-1})^{\frac{1}{\alpha}})^2\\
&=n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}(x_0^2e^{4\theta ih}+\frac{1}{4\theta^2}(e^{2\theta ih}-1)^2\\
&+4e^{4\theta ih}(\sum_{k=1}^{i}U_{k-1}(\tau_{k-1})^{\frac{1}{\alpha}})^2\\
&+\frac{x_0e^{2\theta ih}}{\theta}(e^{2\theta ih}-1)+4x_0e^{4\theta ih}\sum_{k=1}^{i}U_{k-1}(\tau_{k-1})^{\frac{1}{\alpha}}\\
&+\frac{2}{\theta}(e^{2\theta ih}-1)e^{2\theta ih}\sum_{k=1}^{i}U_{k-1}(\tau_{k-1})^{\frac{1}{\alpha}}\\
&=n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}x_0^2e^{4\theta ih}\\
&+n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}\frac{1}{4\theta^2}(e^{2\theta ih}-1)^2\\
&+n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}\frac{x_0e^{2\theta ih}}{\theta}(e^{2\theta ih}-1)\\
&+n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}4x_0e^{4\theta ih}\sum_{k=1}^{i}U_{k-1}(\tau_{k-1})^{\frac{1}{\alpha}}\\
&+n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}\frac{2}{\theta}(e^{2\theta ih}-1)e^{2\theta ih}\sum_{k=1}^{i}U_{k-1}(\tau_{k-1})^{\frac{1}{\alpha}}\\
&+n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}4e^{4\theta ih}\sum_{k=1}^{i}U_{k-1}^2(\tau_{k-1})^{\frac{2}{\alpha}}\\
&+n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}8e^{4\theta ih}\sum_{k\neq j}^{i}U_{k-1}U_{j-1}(\tau_{k-1})^{\frac{1}{\alpha}}(\tau_{j-1})^{\frac{1}{\alpha}}.
\end{aligned}
$$

It is easy to see that

$$
n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}x_0^2e^{4\theta ih}\leq\frac{1}{2\theta}(nh)^{-2/\alpha}x_0^2\stackrel{P}{\rightarrow}0.
$$

Thus, we can obtain that

$$
n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}\frac{1}{4\theta^2}(e^{2\theta ih}-1)^2\stackrel{P}{\rightarrow}0,
$$

and

$$
n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}\frac{x_0e^{2\theta ih}}{\theta}(e^{2\theta ih}-1)\stackrel{P}{\rightarrow}0.
$$

By the Markov inequality, for any $\varepsilon>0$, it can be checked that

$$P(|n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}4x_0e^{4\theta ih}\sum_{k=1}^{i}U_{k-1}(\tau_{k-1})^{\frac{1}{\alpha}}|>\varepsilon)$$
$$\le \varepsilon^{-1}\mathbb{E}|n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}4x_0e^{4\theta ih}\sum_{k=1}^{i}U_{k-1}(\tau_{k-1})^{\frac{1}{\alpha}}|$$
$$\le 4\varepsilon^{-1}n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}e^{4\theta ih}\sum_{k=1}^{i}\mathbb{E}|x_0|\mathbb{E}|U_{k-1}(\tau_{k-1})^{\frac{1}{\alpha}}|$$
$$\le C\varepsilon^{-1}(nh)^{-\frac{2}{\alpha}}\frac{h}{e^{-4\theta h}-1}\to 0.$$

By using the same methods, it follows that

$$n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}\frac{2}{\theta}(e^{2\theta ih}-1)e^{2\theta ih}\sum_{k=1}^{i}U_{k-1}(\tau_{k-1})^{\frac{1}{\alpha}}\xrightarrow{P}0.$$

and

$$n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}8e^{4\theta ih}\sum_{k\neq j}^{i}U_{k-1}U_{j-1}(\tau_{k-1})^{\frac{1}{\alpha}}(\tau_{j-1})^{\frac{1}{\alpha}}\xrightarrow{P}0.$$

With the result that $n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n-1}4e^{4\theta ih}\sum_{k=1}^{i}U_{k-1}^2(\tau_{k-1})^{\frac{2}{\alpha}}\xrightarrow{d}\frac{C_\alpha^{\frac{2}{\alpha}}}{2\theta}$, we obtain that

$$n^{-\frac{2}{\alpha}}h^{1-\frac{2}{\alpha}}\sum_{i=1}^{n}X_{t_{i-1}}^2\xrightarrow{d}\frac{C_\alpha^{\frac{2}{\alpha}}}{2\theta}.$$

Furthermore,

$$(n\log n)^{-\frac{1}{\alpha}}h^{-\frac{1}{\alpha}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sqrt{X_s}e^{-2\theta s}dZ_s$$
$$=(n\log n)^{-\frac{1}{\alpha}}h^{-\frac{1}{\alpha}}\sum_{i=0}^{n-1}X_{t_i}(\tau_i)^{\frac{1}{\alpha}}U_i$$
$$=(n\log nh)^{-\frac{1}{\alpha}}\sum_{i=0}^{n-1}(\tau_i)^{\frac{1}{\alpha}}U_i(-\frac{1}{2\theta}+(x_0+\frac{1}{2\theta})e^{2\theta ih}$$
$$+2\sum_{k=1}^{i}e^{2\theta ih}U_{k-1}(\tau_{k-1})^{\frac{1}{\alpha}})$$
$$=-\frac{1}{2\theta}(n\log nh)^{-\frac{1}{\alpha}}\sum_{i=0}^{n-1}(\tau_i)^{\frac{1}{\alpha}}U_i$$
$$+(n\log nh)^{-\frac{1}{\alpha}}\sum_{i=0}^{n-1}(\tau_i)^{\frac{1}{\alpha}}U_i(x_0+\frac{1}{2\theta})e^{2\theta ih}$$
$$+2(n\log nh)^{-\frac{1}{\alpha}}\sum_{i=0}^{n-1}(\tau_i)^{\frac{1}{\alpha}}U_i\exp^{2\theta ih}\sum_{k=1}^{i}U_{k-1}(\tau_{k-1})^{\frac{1}{\alpha}}.$$

By the Markov inequality, for any $\varepsilon>0$, it follows that

$$P(|(n\log nh)^{-\frac{1}{\alpha}}\sum_{i=0}^{n-1}(\tau_i)^{\frac{1}{\alpha}}U_i(x_0+\frac{1}{2\theta})e^{2\theta ih}|>\varepsilon)$$
$$\le \varepsilon^{-1}\mathbb{E}|(n\log nh)^{-\frac{1}{\alpha}}\sum_{i=0}^{n-1}(\tau_i)^{\frac{1}{\alpha}}U_i(x_0+\frac{1}{2\theta})e^{2\theta ih}|$$
$$\le \varepsilon^{-1}(n\log nh)^{-\frac{1}{\alpha}}\sum_{i=0}^{n-1}e^{2\theta ih}\mathbb{E}|(\tau_i)^{\frac{1}{\alpha}}U_i|\mathbb{E}|(x_0+\frac{1}{2\theta})|$$
$$\le C\varepsilon^{-1}(n\log nh^{\alpha})^{-\frac{1}{\alpha}}\to 0.$$

Thus, we can also obtain that

$$\frac{1}{2\theta}(n\log nh)^{-\frac{1}{\alpha}}\sum_{i=0}^{n-1}(\tau_i)^{\frac{1}{\alpha}}U_i\xrightarrow{P}0.$$

Together with the results that

$$2(n\log nh)^{-\frac{1}{\alpha}}\sum_{i=0}^{n-1}(\tau_i)^{\frac{1}{\alpha}}U_ie^{2\theta ih}\sum_{k=1}^{i}U_{k-1}(\tau_{k-1})^{\frac{1}{\alpha}}$$
$$\xrightarrow{d}(C_\alpha)^{\frac{2}{\alpha}}\widetilde{Y},$$

we can derive that

$$(\frac{nh}{\log n})^{\frac{1}{\alpha}}(\widehat{\theta}_n-\theta)\xrightarrow{d}\frac{2\theta(\alpha\theta)^{-\frac{1}{\alpha}}\widetilde{Y}}{Y_0}. \tag{39}$$

The proof is complete. ■

Remark 1: For the conditions in Theorem 2, h can take $h=cn^{-\lambda}$ with

$$\lambda\in\begin{cases}([\frac{1}{1+\alpha},\frac{1}{2\alpha-1}] & if\alpha\in(\frac{6}{5},2)\\ [\frac{1}{1+\alpha},\frac{1}{2-\alpha/2+\rho}] & if\alpha\in(1,\frac{6}{5}).\end{cases}$$

The choice $\lambda=\frac{1}{1+\alpha}$ leads to the optimal convergence rate in Theorem 2.

IV. Simulation

In this experiment, we generate a discrete sample $(X_{t_i})_{i=0,1,\dots,n}$ and compute $\widehat{\theta}_n$ from the sample. In Table 1 and 2, we let $x_0=1$ and $\alpha=1.8$. In Table 3 and 4, we let $x_0=0.5$ and $\alpha=1.2$. In Table 5, we let $x_0=0.06$, $\alpha=0.8$. For every given true value of the parameter-θ, the size of the sample is represented as"Size n" and given in the first column of the table. In Table 1 and 3, $h=0.1$, the size is increasing from 1000 to 5000. In Table 2 and 4, $h=0.01$, the size is increasing from 10000 to 50000. In Table 5, $h=0.05$, the size is increasing from 10000 to 50000. The tables list the value of least squares estimator "$\widehat{\theta}_n$" and the absolute errors (AE)"$|\theta-\widehat{\theta}_n|$".

Two tables illustrate that when n is large enough and h is small enough, the obtained estimators are very close to the true parameter value. If we let n converge to the infinity and h converge to zero, the estimator will converge to the true value.

V. Conclusion

The aim of this paper is to estimate the parameter of squared radial Ornstein-Uhlenbeck process driven by α-stable noises from discrete observation. The existence and uniqueness of solutions to the stochastic differential equation has been studied. The contrast function has been introduced to obtain the least squares estimator. The strong consistency and asymptotic distribution of the estimator have been discussed by using ergodic theorem, Hölder inequality and Markov inequality. Some numerical calculus and simulations have been given to verify the effectiveness of estimator. Further research topics will include parameter estimation for partially observed stochastic differential equation driven by α-stable noises.

TABLE I
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF θ

True	Aver		AE
θ	Size n	$\widehat{\theta}_n$	$\lvert\theta-\widehat{\theta}_n\rvert$
1	1000	1.3641	0.3641
	2000	1.2530	0.2530
	3000	1.1836	0.1836
	5000	1.0127	0.0127
2	1000	2.4135	0.4135
	2000	2.2821	0.2821
	3000	2.1731	0.1731
	5000	2.0386	0.0356

TABLE II
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF θ

True	Aver		AE
θ	Size n	$\widehat{\theta}_n$	$\lvert\theta-\widehat{\theta}_n\rvert$
1	10000	1.2175	0.2175
	20000	1.1458	0.1458
	30000	1.0618	0.0618
	50000	1.0082	0.0082
2	10000	2.2637	0.2637
	20000	2.1715	0.1715
	30000	2.0593	0.0593
	50000	2.0071	0.0071

TABLE III
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF θ

True	Aver		AE
θ	Size n	$\widehat{\theta}_n$	$\lvert\theta-\widehat{\theta}_n\rvert$
1	1000	1.4131	0.4131
	2000	1.2983	0.2983
	3000	1.1769	0.1769
	5000	1.0315	0.0315
2	1000	2.4926	0.4926
	2000	2.3164	0.3164
	3000	2.2035	0.2035
	5000	2.0871	0.0871

TABLE IV
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF θ

True	Aver		AE
θ	Size n	$\widehat{\theta}_n$	$\lvert\theta-\widehat{\theta}_n\rvert$
1	10000	1.2356	0.2356
	20000	1.1249	0.1249
	30000	1.0725	0.0725
	50000	1.0091	0.0091
2	10000	2.2826	0.2826
	20000	2.1532	0.1532
	30000	2.0638	0.0638
	50000	2.0053	0.0053

TABLE V
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF θ

True	Aver		AE
θ	Size n	$\widehat{\theta}_n$	$\lvert\theta-\widehat{\theta}_n\rvert$
1	10000	1.1976	0.1976
	20000	1.0854	0.0854
	30000	1.0073	0.0073
	50000	1.0006	0.0006
2	10000	2.1849	0.1849
	20000	2.0732	0.0732
	30000	2.0081	0.0081
	50000	2.0005	0.0005

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