Generalized Additive Derivations on MV-algebras

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***Abstract*—In order to investigate the derivation theory on MV-algebras further more, we give the notion of τ-additive derivations as a generalization of additive derivations in [29]. Then we discuss some related properties and characterizations of τ-additive derivations, and obtain the condition for a τ-additive derivation to be an endomorphism. Furthermore, some properties of τ-additive derivations related with the fixed set $\Delta_d(A)$ are given.**

***Index Terms*—MV-algebra, τ-additive derivation, Homomorphism.**

I. Introduction

THe Łukasiewicz infinite-valued logic introduced for philosophical reasons by Jan Lukasiewicz is the most important and widely studied non-classical logic. MV-algebras (as algebras of type $(\oplus, *, 0)$ of signature (2, 1, 0)) have been introduced and studied by Chang in [1] as an algebraic counterpart of the Łukasiewicz infinite-valued propositional logic. The monograph [2] is entirely devoted to give self-contained proofs of all basic results concerning the infinite-valued propositional calculus of Łukasiewicz and its algebras, Chang's MV-alberas. By using special algebraic structures, some meaningful results are obtained in MV-algebras. It is worth noticing that the ideal theory is a powerful tool for studying MV-algebras. One of the reasons is that ideals are closely related to congruence relations, and quotient algebras can be constructed on the basis of congruence relations. Another reason is that the set of provable formulas in the corresponding reasoning system can be described by those fuzzy ideals of algebraic semantics. Hoo [3] showed that how to obtain results in MV-algebras by considering their fuzzy ideals. The concept of ideals in BL-algebras as the generalized form of ideals in MV-algebras, was introduced by Lele and Nganou from a purely algebraic point of view [4]. By using the set of complement elements, the relationships between ideals and deductive systems in BL-algebras were analyzed. Filters and ideals are dual notions in MV-algebras, while some papers investigated algebraic properties of MV-algebras via the filter theory [5]. It is noticed that the concepts of states and pseudo valuations which can measure the average truth-value of propositions, become very important in studying logical structures. Dvurečenskij and Zahir [6] defined a state as a $[0, 1]$-valued, finitely additive function attaining the value 1 on an EMV-algebra, and showed that states always exist and the extremal states are exactly state-morphisms. Yang and Zhu [7] gave the the concept of pseudo MV-valuations in MV-aglebras, and investigated some related characterizations of them.

From the analytic theory, the notion of derivations was first introduced in a prime ring as a map $d : R \to R$ satisfying the conditions $d(x + y) = d(x) + d(y)$ and $d(x \cdot y) = d(x) \cdot y + x \cdot d(y)$, for any $x, y \in R$ [8]. Inspired by the concept of derivations in prime rings, some researchers studied the derivation theory in rings and near rings, respectively ([9], [10]). In fact, the notion of derivations in the ring theory plays a significant role in algebraic geometry, so it has been investigated for the cases of lattices and algebras of fuzzy logic. The notion of left-right (resp. right-left) derivations of BCI-algebras was introduced by Jun and Xin [11], and some characterizations of a p-semisimple BCI-algebra were given by using the idea of regular derivations. Ciungu [12] defined two types of implicative derivations on pseudo-BCK algebras, and discussed some related properties of isotone implicative derivations. In 1975, Szász defined a derivation on a lattice $(L, \vee, \wedge)$ as a map $d : L \to L$ satisfying the conditions $d(x \vee y) = d(x) \vee d(y)$ and $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$, for any $x, y \in L$ [13]. Xin et al. [14] characterized modular lattices and distributive lattices by some particular derivations. As a generalization of derivations in [14], the notion of f-derivations on lattices was introduced, and some related properties were investigated in [15]. Continuing the work on derivations for lattices, Xin [16] established some characterizations of distributive lattices and modular lattices by the fixed set of isotone derivations. Torkzadeh and Abbasian [17] studied $(\odot, \vee)$-derivations on BL-algebras. Alsatayhi and Moussavi [18] gave the notions of (φ, ψ)-derivations of types 1 and 2 on BL-algebras via endomorphisms. In order to characterize some special types of residuated lattices in terms of derivations, He et al. gave the concepts of multiplicative derivations and implicative derivations [19]. Based on [19], Kondo considered some properties of multiplicative derivations and d-filters of commutative residuated lattices in [20]. Zhu et al. introduced the notion of a generalized derivation determined by a derivation for a residuated lattice in [21], and then presented the concept of derivations of state residuated lattices (L, τ) in [22]. A number of researchers have studied derivations in other algebraic systems, such as lattice implication algebras [23], residuated multilattices [24] and semihoops [25].

In 2010, Alshehri introduced the notion of derivations for MV-algebras, and obtained some related properties [26]. Moreover, the notion of derivations on MV-algebras has been generalized to f-derivations and (f, g)-derivations [27]. Ghorbani et al. [28] presented the notions of $(\odot, \oplus)$-derivations and $(\ominus, \odot)$-derivations for MV-algebras, and studied the connection between these derivations. To give some representations of MV-algebras in terms of derivations, Wang et al. [29] introduced the notions of additive derivations (implicative derivations) and difference derivations in MV-algebras. As a supplement of the derivation theory, Yang and

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Zhu [30] presented the concept of τ–difference derivations on MV-algebras.

Motivated by notions of additive derivations in MV-algebras [29] and generalized derivations in BL-algebras [18], we introduced the notion of τ-additive derivations which is a generalization of additive derivations on MV-algebras. Some related properties and characterizations of τ-additive derivations are given. Usually, a τ-additive derivation d maybe not a homomorphism τ, however, we get some conditions for a τ-additive derivation d to be a homomorphism τ in the paper. Finally, we present some properties of τ-additive derivations related with the set $\Delta_d(A)$, and show that $d(A)$ forms an MV-algebra under some suitable operations.

II. Preliminaries

In the section, in order to facilitate our discussion, we summarize some definitions and results about MV-algebras, which will be used in the subsequent discussions.

An algebra $(A, \oplus, *, 0)$ of type $(2, 1, 0)$ is called an MV-algebra if it satisfies the following conditions: for any $x, y, z \in A$,

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(MV2) $x \oplus y = y \oplus x$,
(MV3) $x \oplus 0 = x$,
(MV4) $x^{**} = x$,
(MV5) $x \oplus 0^* = 0^*$,
(MV6) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

For any MV-algebra $(A, \oplus, *, 0)$, we define the constant 1 and the operations $\ominus$, $\vee$, $\wedge$, $\otimes$ and $\rightarrow$ as follows: for any $x, y \in A$,

(i) $1 = 0^*$,
(ii) $x \ominus y = (x^* \oplus y)^*$,
(iii) $x \vee y = (x \ominus y) \oplus y$,
(iv) $x \wedge y = y \ominus (y \ominus x)$,
(v) $x \otimes y = x \ominus y^*$,
(vi) $x \rightarrow y = x^* \oplus y$.

Lemma 2.1: [31] Let A be an MV-algebra. Then the following conditions are equivalent: for any $x, y \in A$,

(i) $x = x \wedge y$,
(ii) $x \ominus y = 0$,
(iii) there is $z \in A$ such that $y = x \oplus z$.

For any $x, y \in A$, $x \leq y$ iff x and y satisfy one of the above equivalent conditions (i)–(iii). It follows that $\leq$ is a partial order, called the natural order of A. And on each MV-algebra, the natural order determines a lattices structure. In fact, one can show that $(A, \vee, \wedge, 0, 1)$ forms a distributive lattice.

Proposition 2.2: [31] Let A be an MV-algebra. Then the following results are valid: for any $x, y, z \in A$,

(1) $1 \oplus x = 1$, $0 \oplus x = x$, $x \oplus x^* = 1$;
(2) $x \ominus y \leq z$ if and only if $x \leq y \oplus z$;
(3) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$;
(4) $y \otimes (x \oplus z) \leq x \oplus (y \otimes z)$;
(5) $x \ominus y \leq x \wedge y \leq x \vee y \leq x \oplus y$;
(6) $x \leq y$ if and only if $x \rightarrow y = 1$.

A subset D of an MV-algebra A is called a deductive system of A if it satisfies the following axioms:

(ds1) $1 \in D$,
(ds2) $x \in D$ and $x \rightarrow y \in D$ imply $y \in D$, for any $x, y \in A$.

Denote by $DS(A)$ the set of all deductive systems of A. If $X \subseteq A$, denote by

$$[X) = \bigcap\{D \in DS(A) | X \subseteq D\}.$$

Then one can prove that $[X)$ is the smallest deductive system of A containing X, and we call it the deductive system generated by X.

Definition 2.3: [1] Let $f : A_1 \rightarrow A_2$ be a mapping, where $(A_1, \oplus_1, *_1, 0_1)$ and $(A_2, \oplus_2, *_2, 0_2)$ are two MV-algebras. Then f called a homomorphism iff it satisfies the following conditions:

(1) $f(0_1) = 0_2$,
(2) $f(x \oplus_1 y) = f(x) \oplus_2 f(y)$,
(3) $f(x^{*_1}) = f(x)^{*_2}$,

for any $x, y \in A_1$.

Definition 2.4: [29] Let A be an MV-algebra. A map $d : A \rightarrow A$ is called an additive derivation if it satisfies:

$$d(x \oplus y) = d(x) \oplus y,$$

for any $x, y \in A$.

III. τ-additive derivations on MV-algebras

In the section, we extend the notion of additive derivations on MV-algebras to the notion of τ-additive derivations, and investigate some characterizations of τ-additive derivations. In the following, unless mentioned otherwise, A is an MV-algebra.

Definition 3.1: Let τ be an endomorphism on A. A map $d : A \rightarrow A$ is called a τ-additive derivation on A if it satisfies:

$$d(x \oplus y) = d(x) \oplus \tau(y)$$

for any $x, y \in A$.

From the above definition, it is easy to see that

$$d(x \oplus y) = d(x) \oplus \tau(y) = \tau(x) \oplus d(y)$$

for any $x, y \in A$. If τ is the identity map on A, according to Definition 2.4, we get that the τ-additive derivation d is actually an ordinary additive derivation.

Example 3.2: Define a map $d : A \rightarrow A$ by

$$d(x) = x$$

for any $x \in A$. If τ is the identity map, then d is a τ-additive derivation on A, which is called an identity τ-additive derivation.

Example 3.3: Let $A = \{0, a, b, 1\}$, where $0 < a < 1$ and $0 < b < 1$. Then it is easy to check that $(A, \oplus, *, 0)$ is an MV-algebra, where the operation $\oplus$ on A is defined in Table I, and the operation $*$ on A is defined in Table II.

TABLE I
The Operation $\oplus$ on A

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

We define two maps $\tau : A \rightarrow A$ and $d : A \rightarrow A$ as follows

TABLE II
THE OPERATION $*$ ON A

$*$	0	a	b	1
	1	b	a	0

$$\tau(x)=\begin{cases}0, & x=0,\\ b, & x=a,\\ a, & x=b,\\ 1, & x=1.\end{cases}\qquad d(x)=\begin{cases}b, & x=0,\\ b, & x=a,\\ 1, & x=b,\\ 1, & x=1.\end{cases}$$

By routine calculations, τ is a homomorphism on A, and d is a τ-additive derivation on A.

Proposition 3.4: Let d be a τ-additive derivation on A Then the following properties hold: for any $x, y \in A$,

(1) $d(1) = 1$;
(2) if $x \le y$, then $d(x) \le d(y)$;
(3) $\tau(x) \le d(x)$;
(4) $d(x \oplus y) \le d(x) \oplus d(y)$;
(5) $d(x) \ominus d(y) \le d(x \ominus y)$.

Proof: (1) According to the definition of τ-additive derivations, we have

$$\begin{aligned} d(1) &= d(1 \oplus 1)\\ &= d(1) \oplus \tau(1)\\ &= d(1) \oplus 1\\ &= 1. \end{aligned}$$

(2) For any $x, y \in A$, if $x \le y$, then there exists $z \in A$ such that $y = x \oplus z$. It follows that

$$\begin{aligned} d(y) &= d(x \oplus z)\\ &= d(x) \oplus \tau(z)\\ &= d(z) \oplus \tau(x). \end{aligned}$$

Therefore, we get that $d(x) \le d(y)$ and $\tau(x) \le d(y)$ by Lemma 2.1.

(3) For any $x, y \in A$, if $x \le y$, then $\tau(x) \le d(y)$ follows the proof process of (2). Hence, we get $\tau(x) \le d(x)$ due to $x \le x$.

(4) According to (3), we have $\tau(y) \le d(y)$ for any $y \in A$. And so $d(x \oplus y) = d(x) \oplus \tau(y) \le d(x) \oplus d(y)$.

(5) Using (2) and (4), we obtain that

$$\begin{aligned} d(x \ominus y) \oplus d(y) &\ge d((x \ominus y) \oplus y)\\ &= d(x \vee y)\\ &\ge d(x), \end{aligned}$$

which means that $d(x \ominus y) \oplus d(y) \ge d(x)$, and therefore $d(x) \ominus d(y) \le d(x \ominus y)$. ■

Theorem 3.5: Let $d : A \to A$ be a map. Then the following statements are equivalent:

(1) d is a τ-additive derivation on A;
(2) $d(x) = d(0) \oplus \tau(x)$, for any $x \in A$;
(3) $d(x \to y) = \tau(x) \to d(y)$, for any $x, y \in A$.

Proof: (1) $\Rightarrow$ (2) Assume that d is a τ-additive derivation on A, then $d(x) = d(0 \oplus x) = d(0) \oplus \tau(x)$ for any $x \in A$.

(2) $\Rightarrow$ (3) For any $x, y \in A$,

$$\begin{aligned} d(x \to y) &= d(0) \oplus \tau(x \to y)\\ &= d(0) \oplus \tau(x^* \oplus y)\\ &= d(0) \oplus (\tau(x)^* \oplus \tau(y))\\ &= \tau(x)^* \oplus (d(0) \oplus \tau(y))\\ &= \tau(x)^* \oplus d(y)\\ &= \tau(x) \to d(y). \end{aligned}$$

(3) $\Rightarrow$ (1) By hypothesis, we have

$$\begin{aligned} d(x \oplus y) &= d(x^* \to y)\\ &= \tau(x^*) \to d(y)\\ &= \tau(x)^* \to d(y)\\ &= \tau(x) \oplus d(y) \end{aligned}$$

for any $x, y \in A$. By the commutativity of $\oplus$, we can get

$$d(x \oplus y) = d(x) \oplus \tau(y).$$

Hence, d is a τ-additive derivation on A. ■

Proposition 3.6: Let d be a τ-additive derivation on A. Then for any $x, y \in A$,

(1) $d(x \wedge y) = d(x) \wedge d(y)$;
(2) $d(x \otimes y) \ge \tau(x) \otimes d(y)$.

Proof: (1) Using Theorem 3.5 and Proposition 2.2 (3), we get that

$$\begin{aligned} d(x \wedge y) &= d(0) \otimes \tau(x \wedge y)\\ &= d(0) \oplus (\tau(x) \wedge \tau(y))\\ &= (d(0) \oplus \tau(x)) \wedge ((d(0) \oplus \tau(y))\\ &= d(x) \wedge d(y) \end{aligned}$$

for any $x, y \in A$.

(2) For any $x, y \in A$, we have

$$\begin{aligned} d(x \otimes y) &= d(0) \oplus \tau(x \otimes y)\\ &= d(0) \oplus (\tau(x) \otimes \tau(y))\\ &\ge \tau(x) \otimes (d(0) \oplus \tau(y))\\ &= \tau(x) \otimes d(y) \end{aligned}$$

by Theorem 3.5 and Proposition 2.2 (4). ■

An MV-algebra A is a Boolean algebra if it satisfies the additional equation $x \oplus x = x$ (or $x \otimes x = x$) for any $x \in A$, and denote by

$$B(A) = \{x \in A | x \oplus x = x\}$$

the set of all idempotent elements of A.

Proposition 3.7: [1] Let A be an MV-algebra. Then following conditions are equivalent: for any $x, y \in A$,

(i) $x \in B(A)$;
(ii) $x \oplus x = x$;
(iii) $x \otimes x = x$;
(iv) $x \oplus y = x \vee y$;
(v) $x \otimes y = x \wedge y$.

Proposition 3.8: Let d be a τ-additive derivation on A, and $d(0) \in B(A)$. Then for any $x, y \in A$,

(1) $d(x) = d(x) \vee \tau(x)$;
(2) $d(x \oplus y) = d(x) \oplus d(y)$;
(3) $d(x \vee y) = d(x) \vee d(y)$;
(4) $d(x \otimes y) \ge d(x) \otimes d(y)$.

Proof: (1) Since $d(0) \in B(A)$, then we have

$$d(x) = d(0) \oplus \tau(x) = d(x) \vee \tau(x)$$

by Proposition 3.7.

(2) For any $x, y \in A$, we get

$$\begin{aligned}d(x \oplus y) &= d(0) \oplus \tau(x \oplus y)\\ &= (d(0) \oplus d(0)) \oplus (\tau(x) \oplus \tau(y))\\ &= (d(0) \oplus \tau(x)) \oplus (d(0) \oplus \tau(y))\\ &= d(x) \oplus d(y).\end{aligned}$$

(3) According to (1) and Proposition 3.7, we obtain that

$$\begin{aligned}d(x \vee y) &= d(0) \oplus \tau(x \vee y)\\ &= d(0) \vee \tau(x \vee y)\\ &= (d(0) \vee d(0)) \vee (\tau(x) \vee \tau(y))\\ &= (d(0) \vee \tau(x)) \vee (d(0) \vee \tau(y))\\ &= d(x) \vee d(y).\end{aligned}$$

(4) Applying Proposition 2.2 (4) twice, we have

$$\begin{aligned}d(x \otimes y) &= d(0) \oplus \tau(x \otimes y)\\ &= (d(0) \oplus d(0)) \oplus (\tau(x) \otimes \tau(y))\\ &= d(0) \oplus (d(0) \oplus (\tau(x) \otimes \tau(y)))\\ &\geq d(0) \oplus (\tau(y) \otimes (d(0) \oplus \tau(x)))\\ &\geq (d(0) \oplus \tau(x)) \otimes (d(0) \oplus \tau(y))\\ &= d(x) \otimes d(y).\end{aligned}$$

■

Proposition 3.9: Let $\tau : A \to A$ be a homomorphism. For a fixed element $a \in A$, if a map $d_a^{\oplus} : A \to A$ is defined by

$$d_a^{\oplus}(x) = \tau(x) \oplus a$$

for any $x \in A$, then $d_a^{\oplus}$ is a τ-additive derivation on A.

Proof: For any $x, y \in A$,

$$\begin{aligned}d_a^{\oplus}(x \oplus y) &= \tau(x \oplus y) \oplus a\\ &= (\tau(x) \oplus \tau(y)) \oplus a\\ &= (\tau(x) \oplus a) \oplus \tau(y)\\ &= d_a^{\oplus}(x) \oplus \tau(y).\end{aligned}$$

According to Definition 3.1, we get that $d_a^{\oplus}$ is a τ-additive derivation on A. ■

Proposition 3.10: Let $\tau : A \to A$ be a homomorphism. For a fixed element $a \in A$, if a map $d_a^{\to} : A \to A$ is defined by

$$d_a^{\to}(x) = a \to \tau(x)$$

for any $x \in A$, then $d_a^{\to}$ is a τ-additive derivation on A.

Proof: For any $x, y \in A$,

$$\begin{aligned}d_a^{\to}(x \to y) &= a \to \tau(x \to y)\\ &= a \to (\tau(x) \to \tau(y))\\ &= \tau(x) \to (a \to \tau(y))\\ &= \tau(x) \to d_a^{\to}(y).\end{aligned}$$

Using Theorem 3.5, we get that $d_a^{\to}$ is a τ-additive derivation on A. ■

Theorem 3.11: Let the endomorphism τ on A be an identity map and d be a τ-additive derivation on A. If $\emptyset \neq X \subseteq A$, then

$$[X) = \{a \in A | d(x_1) \to (\cdots \to (d(x_n) \to a) \cdots) = 1 \ for\ some\ x_1, \cdots, x_n \in X\ and\ n \in N\}.$$

Proof: For any $D \in DS(A)$, we have that $d(D) \subseteq D$. In fact, suppose that $y \in d(D)$, that is, there exists $x \in D$ such that $y = d(x)$. Noticing that τ is an identity map and d is a τ-additive derivation on A, we get $x = \tau(x) \leq d(x)$ by Proposition 3.4 (3). And so $x \to y = x \to d(x) = 1 \in D$, hence $y \in D$, so that $d(D) \subseteq D$. Denote

$$\begin{aligned}X^* =&\{a \in A | d(x_1) \to (\cdots \to (d(x_n) \to a) \cdots) = 1,\\ &x_1, \cdots, x_n \in X, n \in N\}.\end{aligned}$$

Obviously, $1 \in X^*$. Assume $a, b \in A$ such that $a, a \to b \in X^*$, then there exist $m, n \in N$ and $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n \in X$ such that

$$d(x_1) \to (\cdots \to (d(x_m) \to (a \to b)) \cdots) = 1$$

and

$$d(y_1) \to (\cdots \to (d(y_n) \to a) \cdots) = 1.$$

Then we get inductively:

$$\begin{aligned}&a \to (d(x_1) \to (\cdots \to (d(x_m) \to b) \cdots))\\ =&d(x_1) \to (\cdots \to (d(x_m) \to (a \to b)) \cdots)\\ =&1\end{aligned}$$

It follows that $a \leq (d(x_1) \to (\cdots \to (d(x_m) \to b) \cdots))$, and we get inductively:

$$\begin{aligned}1 &= d(y_1) \to (\cdots \to (d(y_n) \to a) \cdots)\\ &\leq d(y_1) \to (\cdots \to (d(y_n) \to (d(x_1) \to (\cdots \to (d(x_m)\\ &\qquad \to b) \cdots))) \cdots).\end{aligned}$$

So that $d(y_1) \to (\cdots \to (d(y_n) \to (d(x_1) \to (\cdots (d(x_m) \to b) \cdots))) \cdots) = 1$. Therefore, $b \in X^*$, that is, $X^* \in DS(A)$. Let $X' \in DS(A)$ such that $X \subseteq X'$, and let $a \in X^*$. Then there exist $n \in N$ and $x_1, x_2, \cdots, x_n \in X$ such that

$$d(x_1) \to (\cdots \to (d(x_n) \to a) \cdots) = 1.$$

From $x_i \in X \subseteq X'$, $X' \in DS(A)$ and $d(X') \subseteq X'$, we have $d(x_i) \in X'$, hence $a \in X'$. It follows that $X^* \subseteq X'$, thus X^* is the smallest deductive system of A containing X. We conclude that $X^* = [X)$. ■

In general, an endomorphism τ is a τ-additive derivation on an MV-algebra, but a τ-additive derivation d maybe not an endomorphism. In the following, we derive a set of conditions for a τ-additive derivation d to be an endomorphism.

Proposition 3.12: Let d be a τ-additive derivation on A. Then

(1) $d(0) = 0$;

(2) $d = \tau$;

(3) $d(x \to y) = d(x) \to \tau(y)$ for any $x, y \in A$.

Proof: (1) $\Rightarrow$ (2) Suppose that $d(0) = 0$, then

$$\begin{aligned}d(x) &= d(0 \oplus x)\\ &= d(0) \oplus \tau(x)\\ &= 0 \oplus \tau(x)\\ &= \tau(x)\end{aligned}$$

for any $x \in A$, and so $d = \tau$.

(2) $\Rightarrow$ (3) It is obviously.

(3) $\Rightarrow$ (1) Since $1 = d(1) = d(0 \to 0) = d(0) \to \tau(0) = d(0) \to 0$, then $d(0) \leq 0$. By Proposition 3.4 (3), we have

$$0 = \tau(0) \leq d(0),$$

hence $d(0) = 0$. ■

Let f and g be two maps on A. We define $f \cdot g$ by

$$(f \cdot g)(x) = f(g(x))$$

for any $x \in A$. If $f = f^2 := f \cdot f$, then f is called an idempotent map.

Proposition 3.13: Let d_1 and d_2 be two τ-additive derivations on A. If τ is an idempotent homomorphism, then $d_1 \cdot d_2$ is also a τ-additive derivation on A.

Proof: Noticing that d_1 and d_2 are τ-additive derivations on A, and τ is an idempotent homomorphism, we get that

$$\begin{aligned}(d_1 \cdot d_2)(x \oplus y) &= d_1(d_2(x) \oplus \tau(y)) \\ &= d_1(d_2(x)) \oplus \tau^2(y) \\ &= (d_1 \cdot d_2)(x) \oplus \tau(y)\end{aligned}$$

for any $x, y \in A$. Therefore $d_1 \cdot d_2$ is a τ-additive derivation on A. ■

For a τ-additive derivation d on A, we denote the fixed set $\Delta_d(A)$ with respect to the endomorphism τ by

$$\Delta_d(A) = \{x \in A | d(x) = \tau(x)\}.$$

If τ is the identity map and d is idempotent, then clearly $d(x) \in \Delta_d(A)$ for any $x \in A$.

Proposition 3.14: Let d be a τ-additive derivation on A.

(1) If τ is the identity map on A, then
$$\Delta_d(A) \subseteq \Delta_{d^2}(A).$$
(2) If $x \in A$ and $y \in \Delta_d(A)$, then
$$d(x \oplus y) = d(x) \oplus d(y).$$
(3) If $x \in \Delta_d(A)$ and $y \in \Delta_d(A)$, then
$$x \oplus y \in \Delta_d(A) \text{ and } x \wedge y \in \Delta_d(A).$$

Proof: (1) If τ is the identity map on A, then we have $x = d(x)$ for any $x \in \Delta_d(M)$. It follows that

$$d^2(x) = d(d(x)) = d(x) = x,$$

and so $x \in \Delta_{d^2}(A)$, therefore $\Delta_d(A) \subseteq \Delta_{d^2}(A)$.

(2) If $x \in A$ and $y \in \Delta_d(A)$, then $d(y) = \tau(y)$, and therefore $d(x \oplus y) = d(x) \oplus \tau(y) = d(x) \oplus d(y)$.

(3) If $x \in \Delta_d(A)$ and $y \in \Delta_d(A)$, then $d(x) = \tau(x)$ and $d(y) = \tau(y)$. Using (2) we have

$$\begin{aligned}d(x \oplus y) &= d(x) \oplus d(y) \\ &= \tau(x) \oplus \tau(y) \\ &= \tau(x \oplus y),\end{aligned}$$

and so $x \oplus y \in \Delta_d(A)$.

From Proposition 3.6 (1), it follows that

$$\begin{aligned}d(x \wedge y) &= d(x) \wedge d(y) \\ &= \tau(x) \wedge \tau(y) \\ &= \tau(x \wedge y),\end{aligned}$$

hence $x \wedge y \in \Delta_d(A)$. ■

Theorem 3.15: Let d_1 and d_2 be two idempotent τ-additive derivations on A such that $d_1 \cdot d_2 = d_2 \cdot d_1$, and τ be the identity map on A. Then the following asserts are equivalent:

(1) $d_1 = d_2$;
(2) $d_1(A) = d_2(A)$;
(3) $\Delta_{d_1}(A) = \Delta_{d_2}(A)$.

Proof: (1) $\Rightarrow$ (2) Obviously.

(2) $\Rightarrow$ (3) Assume that $d_1(A) = d_2(A)$. For any $x \in \Delta_{d_1}(A)$, we have

$$x = d_1(x) \in d_1(A) = d_2(A).$$

Then there exists $y \in A$ such that $x = d_2(y)$. And

$$d_2(x) = d_2(d_2(y)) = d_2(y) = x,$$

so $x \in \Delta_{d_2}(A)$, it follows that $\Delta_{d_1}(A) \subseteq \Delta_{d_2}(A)$. Similarly, we can prove $\Delta_{d_2}(A) \subseteq \Delta_{d_1}(A)$. Thus $\Delta_{d_1}(A) = \Delta_{d_2}(A)$.

(3) $\Rightarrow$ (1) Suppose that $\Delta_{d_1}(A) = \Delta_{d_2}(A)$. For any $x \in A$, since $d_1(x) \in \Delta_{d_1}(A) = \Delta_{d_2}(A)$, we get that

$$d_2(d_1(x)) = d_1(x).$$

Similarly, $d_1(d_2(x)) = d_2(x)$. Then

$$\begin{aligned}d_1(x) &= d_2(d_1(x)) \\ &= (d_2 \cdot d_1)(x) \\ &= (d_1 \cdot d_2)(x) \\ &= d_1(d_2(x)) \\ &= d_2(x),\end{aligned}$$

thus, $d_1 = d_2$. ■

Theorem 3.16: Let d be a τ-additive derivation on A such that $d(0) \in B(A)$. Then $(d(A), +, \star, d(0))$ is an MV-algebra, where

$$d(x) + d(y) = d(x \oplus y),$$
$$d(x)^\star = d(x^*)$$

for any $x, y \in A$.

Proof: According to Proposition 3.8, we obtain that

$$d(x) + d(y) = d(x \oplus y) = d(x) \oplus d(y).$$

It is easy to check that $d(A)$ satisfies the conditions $(MV1)$–$(MV6)$, hence $(d(A), +, \star, d(0))$ is an MV-algebra. ■

IV. Conclusion

The notion of derivations is helpful for studying structures in algebraic systems. In the paper, we have generalized the notion of derivations to the notion of additive derivations based on endomorphisms, and introduced the notion of τ-additive derivations on MV-algebras. Some related properties and characterizations of τ-additive derivations are discussed. Some conditions for a τ-additive derivation d to be a homomorphism are also obtained. The results we obtained are complete and further generalize the known ones about additive derivations for MV-algebras.

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